Problem Set for Exercise Session No.8

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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1 Integral Curve

Let us consider a vector field $X = -y\partial_x + x\partial_y$ on $\mathbb{R}^2$. Find the corresponding integral curve $c(t)$ starting with (x_0, y_0) at $t = 0$.

2 Some Property of Exponential Map of Matrix

Prove some identities related to the exponential of $n \times n$ matrices:

1. Let us consider a $n \times n$ matrix A and a parameter t. Show
$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A\,.$$

2. Let us consider two $n \times n$ matrices A, B satisfying $[A, B] = 0$. Show
$$e^{A+B} = e^A e^B\,.$$

3. Let us consider two $n \times n$ matrices A, B satisfying $[A, B] = B$ and a parameter t. For these matrices, show
$$e^{tA}Be^{-tA} = e^t B\,.$$

4. Let us consider two $n \times n$ matrices A, B satisfying $[A, B] = C$ and $[A, C] = B$ for a square matrix C and a parameter t. For these matrices, show
$$e^{tA}Be^{-tA} = (\cosh t)\,B + (\sinh t)\,C\,.$$

5. For a $n \times n$ matrix A which is diagonalizable, show $\det(\exp(A)) = \exp(\mathrm{tr}A)$.

6. For a general $n \times n$ matrix A, show $\det(\exp(A)) = \exp(\mathrm{tr}A)$.

3 Lie Group and Lie Algebra

(1) Let us consider a Lie group $GL(n, \mathbb{R})$ and denote by $X = \sum_{i,j} A_{ij}(\partial/\partial x_{ij})|_{I_n}$ an element of $T_{I_n}GL(n, \mathbb{R})$. We also denote by $\tilde{X}$ the left-invariant vector field corresponding

to X as derived in Problem Set No.7. Show that the integral curve $c(t)$ for $\tilde{X}$ satisfying $c(t=0)=I_n$ is given by

$$c(t)=\exp(tA)=\sum_{n=0}^{\infty}\frac{1}{n!}t^nA^n\,.$$

Here A is an $n\times n$ matrix whose (i,j)-component is given by A_{ij}.
(2) Let us consider a Lie group $SO(n,\mathbb{R})$ defined by

$$SO(n,\mathbb{R})=\{O\in GL(n,\mathbb{R})|O^TO=I_n \text{ and } \det O=1\}\,.$$

Here I_n is the $n\times n$ unit matrix and O^T is the transpose of O.

1. Let us consider the one-parameter subgroup $c(t)=\exp(tA)$ of $SO(n,\mathbb{R})$ (here A is a real $n\times n$ matrix). Show that A satisfies $A+A^T=0$.

2. What is the dimension of of the Lie algebra $\mathfrak{so}(n,\mathbb{R})$?

3. Let us consider the case $n=3$. In this case, confirm that we can write A in general as

$$A=a_1A_1+a_2A_2+a_3A_3\,,$$

where $a_1,a_2,a_3\in\mathbb{R}$ and A_1,A_2,A_3 are defined by

$$A_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\,,\qquad A_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\,,\qquad A_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\,.$$

Compute the commutator of A_i's to show

$$[A_i,A_j]=A_iA_j-A_jA_i=\sum_k\epsilon_{ijk}A_k\,,$$

where ϵ_{ijk} is the totally anti-symmetric tensor satisfying $\epsilon_{123}=1$.

(3) Let us consider a Lie group $Sp(2n,\mathbb{R})$ defined by

$$Sp(2n,\mathbb{R})=\{M\in GL(2n,\mathbb{R})|M^TJM=J\}\,.$$

Here $2n\times 2n$ matrix J is defined by

$$J=\begin{pmatrix}0&I_n\\-I_n&0\end{pmatrix}\,.$$

Carry out the same analysis as Problem (2)-1 and (2)-2 for $Sp(2n,\mathbb{R})$. (That is, consider the one-parameter subgroup $c(t)=\exp(tA)$ (here A is a $2n\times 2n$ real matrix) and determine the constraint on A. Compute the dimension of the Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$.)

Note on Revision

Janurary 23 2015
Some correction in Problem 3.