Problem Set for Exercise Session No.7

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)
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1 Lie Bracket

Answer the following questions:
(1) Let us consider the following vector fields on $\mathbb{R}^3$:

$$X = x\partial_x - y\partial_y + z\partial_z\,, \qquad Y = x\partial_y\,, \qquad Z = y\partial_x + \frac{1+yz}{x}\partial_z\,.$$

Compute Lie brackets of these vector fields.

(2) Let us consider a differentiable manifold M of class C^∞ and a set $\mathfrak{X}(M)$ of differentiable vector fields on it. For $X, Y, Z \in \mathfrak{X}(M)$, $f, g \in C^\infty(M)$ and $a, b \in \mathbb{R}$, prove the following properties of the Lie bracket:

$$\begin{aligned}
&[aX + bY, Z] = a[X, Z] + b[Y, Z]\,,\\
&[Z, aX + bY] = a[Z, X] + b[Z, Y]\,,\\
&[X, Y] = -[Y, X]\,,\\
&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\,,\\
&[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X\,.
\end{aligned}$$

2 f-related Vector Field

Answer the following questions:

(1) Let us consider a C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, x^2 + y)$ for $(x, y) \in \mathbb{R}^2$ and a C^∞ vector field V on $\mathbb{R}^2$ defined by

$$V = -y\partial_x + x\partial_y\,.$$

Compute the f-related vector field of V on $\mathbb{R}^2$.

(2) Construct a C^∞ vector field on $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ which vanishes at the north pole $(0, 0, 1)$ only. (Use the stereographic coordinate constructed in Problem Set No.5.)

3 Left-invariant Vector Field

Answer the following questions:

1. We consider a Lie group $GL(n,\mathbb{R})$ and the left translation ℓ_g by $g \in GL(n,\mathbb{R})$. We also denote by X an element of $T_{I_n}GL(n,\mathbb{R})$. Here I_n the unit element of $GL(n,\mathbb{R})$ (that is, the $n \times n$ unit matrix). Prove the following relation:

$$(d\ell_g)_{I_n}(X) = gX\ .$$

2. Let X be an element of $T_{I_n}GL(n,\mathbb{R})$. Construct the left-invariant vector field $\tilde{X}$ which becomes X at $g = I_n$.

   Note: A Lie group $GL(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and then we can naturally introduce a coordinate x_{ij} (where $i,j = 1,2,\cdots,n$), and denote an element of $X \in T_{I_n}GL(n,\mathbb{R})$ as

$$X = \sum_{i,j} A_{ij} \left(\frac{\partial}{\partial x_{ij}} \right) \Bigg|_{I_n} .$$

   By using this, gX appearing above is defined by

$$gX = \sum_{i,j,k} g_{ik} A_{kj} \left(\frac{\partial}{\partial x_{ij}} \right) \Bigg|_{g} .$$

Note on Revision

Janurary 23 2015
Explanation added in Problem 3. A typo corrected.