Problem Set for Exercise Session No.3

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)
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1 More on Representations

Answer the following question:
(1) Let us consider the dihedral group D_3.

1. Write down the matrix representations of the elements in D_3 for the regular representation.

2. Let us recall the three dimensional representation ρ of D_3 in the Problem Set No.2. The matrix representation of the elements are

$$M(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M(c_3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M(c_3^{-1}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$M(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Now we consider the tensor product representation of two ρ's. Write down the matrix representations of the elements in D_3 for this tensor product representation.

(2) Let us consider a finite group G and its representation $\rho : G \to GL(n, \mathbb{C})$.

1. Show that $\bar{\rho} : G \to GL(n, \mathbb{C})$ defined by $\bar{\rho}(g) = (\rho(g^{-1}))^T$ (for $g \in G$) is a representation of G (that is, confirm that $\bar{\rho}$ is a homomorphism).

2. Show that $\rho^* : G \to GL(n, \mathbb{C})$ defined by $\rho^*(g) = (\rho(g))^*$ (for $g \in G$) is a representation of G (that is, confirm that $\bar{\rho}$ is a homomorphism). Here $(\cdots)^*$ means the complex conjugate of $(\cdots)$.

2 Symmetry in Quantum Mechanics

Here we consider a quantum mechanical system with Hamiltonian H (recall that $H = H^\dagger$ where $(\cdots)^\dagger$ represents the Hermitian conjugate of $(\cdots)$).

1. Let us denote by $|\psi\rangle$ and $|\phi\rangle$ eigenstates of H with eigenvalues E and E' ($E \neq E'$), respectively. Show that these states are orthogonal, $\langle\psi|\phi\rangle = 0$.

2. Let us collect invertible and linear operators commuting with the Hamiltonian to form a set. We notice that the identity operator id obviously commuting with any operators is included in this set. The inverse of the operator A in this set is denoted as A^{-1} and satisfies $AA^{-1} = A^{-1}A = id$. For operators A and B in this set, confirm

$$(1) \quad ABH = HAB\,, \qquad (2) \quad A^{-1}H = HA^{-1}\,.$$

   This result indicates that these operators form a group. In the following we consider its subgroup and denote as G. We assume that G is a finite group.

3. For an eigenstate $|\psi\rangle$ of the Hamiltonian with the eigenvalue E, confirm that $A|\psi\rangle$ (for $A \in G$) is also an eigenstate of the Hamiltonian with the eigenvalue E.

   From this, we obtain the following result: We denote eigenstates of the Hamiltonian H with eigenvalue E as $|\psi_a\rangle$ $(a = 1, 2, \cdots, n)$. Then, under the action of $A \in G$, the eigenstates transform as

$$A|\psi_a\rangle = \sum_{b=1}^{n} |\psi_b\rangle M_{ba}(A)\,.$$

   It is straightforward to see that the $n \times n$ matrix $M(A) = (M_{ab}(A))$ is a matrix representation of $A \in G$. Therefore the eigenstates of the Hamiltonian with the same eigenvalue form a basis for a representation of G. Since G is a finite group, $M(A)$ is a unitary matrix.

4. Now we label the (non-isomorphic) irreducible representations of G by α and denote them by $\rho^{(\alpha)}$. Let us suppose that the eigenstates $|\psi_a^{(\alpha)}\rangle$ $(a = 1, 2, \cdots, n_\alpha)$ of the Hamiltonian with the same eigenvalue form an orthonormal basis for the irreducible representation $\rho^{(\alpha)}$, $\langle\psi_a^{(\alpha)}|\psi_b^{(\beta)}\rangle = \delta_{ab}\delta_{\alpha\beta}$. Then the matrix representation of $A \in G$ in the irreducible representation $\rho^{(\alpha)}$ is given by a $n_\alpha \times n_\alpha$ matrix $M^{(\alpha)}(A) = (M_{ab}^{(\alpha)}(A))$.

   Prove the orthogonality of the irreducible representations:

$$\sum_{A\in G} (M_{ac}^{(\alpha)}(A))^* M_{bd}^{(\beta)}(A) = \frac{|G|}{n_\alpha}\delta_{\alpha\beta}\delta_{ab}\delta_{cd}\,.$$

   Here $(M_{ac}^{(\alpha)}(A))^*$ means the complex conjugate of $M_{ac}^{(\alpha)}(A)$ and $|G|$ is the order of G.
   (hint: use Schur's lemma.)

5. Let us consider an operator O which commutes with any element of G. Show that the matrix element $\langle\psi_a^{(\alpha)}|O|\psi_b^{(\beta)}\rangle$ can be written as

$$\langle\psi_a^{(\alpha)}|O|\psi_b^{(\beta)}\rangle = C^{(\alpha)}\delta_{ab}\delta_{\alpha\beta}\,,$$

   where $C^{(\alpha)}$ is a constant which depends only on the representation α. If needed, one can use the fact that the identity operator $id \in G$ can be written now as $id = \sum_\alpha \sum_a |\psi_a^{(\alpha)}\rangle\langle\psi_a^{(\alpha)}|$.
   (hint: use the Schur's lemma or the orthogonality relation derived in the previous problem.)

Note on Revision

December 23 2014
Typos corrected.