Problem Set for Exercise Session No.1

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

13th November, 2014, at Room 235A

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1 Some Basics

(1) Answer the following questions:

1. Show that the multiplication table of the group of order 2 is determined uniquely. Confirm that this group is Abelian (=commutative).
2. Show that the multiplication table of the group of order 3 is determined uniquely. Confirm that this group is Abelian.
3. Let us consider a group of order r denoted as $G = \{g_1, g_2, \cdots, g_r\}$. We can compute $g_1 g, g_2 g, \cdots, g_r g$ for a given $g \in G$. Show that $\{g_1 g, g_2 g, \cdots, g_r g\}$ contains all the elements of G and each element of G appears one and only one time.

   We can also show that the above statements hold for $gg_1, gg_2, \cdots, gg_r$ for a given $g \in G$. These results mean that, in each row and column of a multiplication table for a group G, each element of G appears one and only one time.
4. Show that there are two non-isomorphic groups (i.e. two different multiplication tables) of order 4. Confirm that both of them are Abelian.

(2) Let us consider a set S of maps from $\mathbb{N}$ to $\mathbb{N}$ (here $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$). We define a multiplication of two elements $f_1, f_2 \in S$ by the composition of the maps, $(f_1 \cdot f_2)(n) = f_1(f_2(n))$ for $n \in \mathbb{N}$. The map $id(n) = n$ for $\forall n \in \mathbb{N}$ satisfies $(f \cdot id)(n) = (id \cdot f)(n) = f(n)$ for $n \in \mathbb{N}$ and thus is the left and right unit. Let us now consider an element $g \in S$ defined by

$$g(n) = \begin{cases} n-1 & (n \geq 2)\,, \\ 1 & (n = 1)\,. \end{cases}$$

Show there exists a right inverse of g in S but not a left inverse.

2 Dihedral Group D_3: Symmetry of Equilateral Triangle

Let us consider the following transformations which map an equilateral triangle to itself:

- Rotation. We call $2\pi/3$ (counter-clockwise) rotation as c_3. Since $-2\pi/3$ rotation is its inverse, we can denote it as c_3^{-1}.
- Reflections with respect to the three axes given in Fig.1 where we labeled the vertices by $1, 2$ and 3. We call the reflection with respect to the axis (i) as σ_i $(i = 1, 2, 3)$.

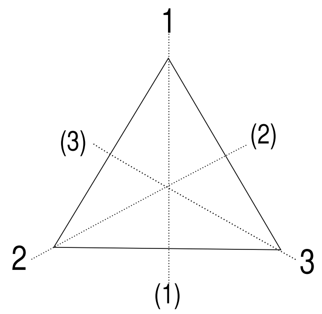

Figure 1: Equilateral triangle and axes for reflections

We also denote the identity transformation (i.e. no transformation) as e. Then $D_3 = \{e, c_3, c_3^{-1}, \sigma_1, \sigma_2, \sigma_3\}$ forms a group under the multiplication $g_1 \cdot g_2$ ($g_1, g_2 \in D_3$) defined as 'first act on the triangle the transformation g_2 and then g_1'. Answer the following questions:

1. Write down the multiplication table of D_3.
2. List up the nontrivial subgroup(s) of D_3 (hint: there are four non-trivial subgroups other than $\{e\}$ and G).
3. Decompose D_3 into the left cosets of a nontrivial subgroup of G (it is enough to write down one example of the decomposition).
4. List up the nontrivial normal subgroup(s) of D_3.
5. List up the conjugacy class(es) of D_3.

3 Permutation Group

(1) Let us consider the permutation group S_3 (the group formed by permutations of three elements $(1,2,3)$). We denote the permutation $(1,2,3) \to (p_1, p_2, p_3)$ (here $\{p_1, p_2, p_3\} = \{1,2,3\}$) as

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

The product $\pi_1 \cdot \pi_2$ of two elements $\pi_1, \pi_2 \in S_3$ is defined as 'first do the permutation corresponding to π_2 and then π_1'. Answer the following questions:

1. What is the order of S_3?
2. Let us consider the following two permutations:

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

   Compute $\pi_1 \cdot \pi_2$.

3. Show that S_3 can be generated by

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

4. Explain that S_3 is isomorphic to D_3.

(2) Prove the following theorem:
Cayley's Theorem
A group of order n is isomorphic to a subgroup of the permutation group S_n or S_n itself.

Hint: For a group $G = \{g_1, g_2, \cdots, g_n\}$ and $g \in G$, from the result of Problem 1 (1)-3, we have $\{gg_1, gg_2, \cdots, gg_n\} = \{g_1, g_2, \cdots, g_n\}$. Thus we can define a map

$$\pi : g \mapsto \pi(g) = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ gg_1 & gg_2 & \cdots & gg_n \end{pmatrix}.$$

Show that π is an isomorphic map from G to $H = \{\pi(g) | g \in G\}$ and H is a subgroup of S_n or S_n itself.

(3) For a positive integer n, a partition $[\lambda_1, \lambda_2, \cdots, \lambda_r]$ (here $r \geq 1$) is defined by integers λ_i $(i = 1, 2, \cdots, r)$ satisfying

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = n, , \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0\,.$$

Let us now consider the permutation group S_n. In S_n, there is a special type of elements called cycles. A cycle $(p_1 p_2 \cdots p_\lambda)$ is defined as the cyclic permutation of $p_1, p_2, \cdots, p_\lambda$. For example, $(123 \cdots \lambda_1)$ is

$$(123 \cdots \lambda) = \begin{pmatrix} 1 & 2 & \cdots & \lambda - 1 & \lambda & \lambda + 1 & \cdots & n \\ 2 & 3 & \cdots & \lambda & 1 & \lambda + 1 & \cdots & n \end{pmatrix}.$$

Since $\pi \in S_n$ can be decomposed into a product of cycles where each element in $\{1, 2, \cdots, n\}$ appears one and only one time, by using the corresponding partition of n, we can write this decomposition as

$$\pi = (p_1^{(1)} p_2^{(1)} \cdots p_{\lambda_1}^{(1)})(p_1^{(2)} p_2^{(2)} \cdots p_{\lambda_2}^{(2)}) \cdots (p_1^{(r)} p_2^{(r)} \cdots p_{\lambda_r}^{(r)})\,.$$

For $\sigma \in S_n$, prove that $\sigma\pi\sigma^{-1}$ is of the form

$$\sigma\pi\sigma^{-1} = (q_1^{(1)} q_2^{(1)} \cdots q_{\lambda_1}^{(1)})(q_1^{(2)} q_2^{(2)} \cdots q_{\lambda_2}^{(2)}) \cdots (q_1^{(r)} q_2^{(r)} \cdots q_{\lambda_r}^{(r)})\,.$$

Here $\{q_1^{(i)}, q_2^{(i)}, \cdots, q_{\lambda_i}^{(i)}\}$ $(i = 1, 2, \cdots, r)$ are all different.

From this result, it follows that the number of the conjugacy classes of S_n is equal to the number of the partition of n.

Note on Revision

December 23 2014

- Very minor revision and typos corrected in Problem 1 and 3.
- In Problem 2, the definition of the right/left coset is changed to make it consistent with the lecture.