Homework 2

$\boxed{I}$ $Z = \begin{cases} e^{aN} & \text{with proba } 1-e^{-cN} \\ e^{bN} & e^{-cN} \end{cases}$

$$\mathbb{E}[Z] = e^{aN}(1-e^{-cN}) + e^{bN}e^{-cN} = e^{aN} + e^{(b-c)N} - e^{(a-c)N}$$

$$\frac{1}{N}\ln \mathbb{E}[Z] \xrightarrow[N\to\infty]{} \max(a, b-c)$$

$$\mathbb{E}[\ln Z] = aN(1-e^{-cN}) + bN e^{-cN}$$

$$\frac{1}{N}\mathbb{E}[\ln Z] \xrightarrow[N\to\infty]{} a$$

when $b-c>a$, difference due to the rare events that dominate $\mathbb{E}[Z]$, $e^{bN}\times e^{-cN} \gg e^{aN}(1-e^{-cN})$

contribution to the average from rare events — typical value of $Z$

$$\mathbb{E}[Z^n] = e^{aNn}(1-e^{-cN}) + e^{bNn}e^{-cN}$$

$n\to 0$ first, $\mathbb{E}[Z^n] = (1+aNn)(1-e^{-cN}) + (1+bNn)e^{-cN} + O(n^2)$

$$= 1 + n\left(aN(1-e^{-cN}) + bNe^{-cN}\right) + O(n^2)$$

$$\frac{\mathbb{E}[Z^n]-1}{n} \xrightarrow[n\to 0]{} aN(1-e^{-aN}) + bNe^{-cN} = \mathbb{E}[\ln Z] \quad \text{ok}$$

$N\to\infty$ first, $\mathbb{E}[Z^n] \sim e^{N\max(an,\, bn-c)}$

$$\lim_{N\to\infty}\frac{1}{N}\frac{1}{n}\ln\mathbb{E}[Z^n] = \frac{1}{n}\max(an, bn-c) \xrightarrow[n\to 0]{} a$$

$Z^n$ with $n\to 0$ less sensitive to large deviations as $n=1$

$\boxed{II}$ $Z = \sum_\sigma e^{\beta h\sigma} = e^{\beta h} + e^{-\beta h}$ $\qquad \ln Z = \ln\left((1+e^{2\beta h})e^{-\beta h}\right) = -\beta h + \ln(1+e^{2\beta h})$

$$\mathbb{E}[\ln Z] = \int\frac{dh}{\sqrt{2\pi}}e^{-\frac{1}{2}h^2}\ln(1+e^{2\beta h}) \quad \text{as } \mathbb{E}[h]=0$$

$$Z^n = \sum_{k=0}^{n}\binom{n}{k}e^{\beta h k}e^{-\beta h(n-k)} = \sum_{k=0}^{n}\binom{n}{k}e^{h\beta(2k-n)}$$

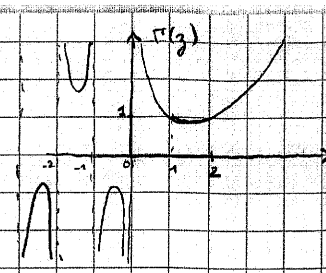
$\mathbb{E}[e^{ah}] = e^{\frac{1}{2}a^2\mathbb{E}[h^2]}$ for $h$ centered gaussian

$$\mathbb{E}[Z^n] = \sum_{k=0}^{n}\binom{n}{k}e^{\frac{1}{2}\beta^2(2k-n)^2}$$

$$\mathbb{E}[Z^n]-1 = \sum_{k=1}^{n}\frac{n!}{(n-k)!\,k!}e^{\frac{1}{2}\beta^2(2k-n)^2} + e^{\frac{1}{2}\beta^2n^2} - 1$$

$$= \sum_{k=1}^{n}\frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}e^{\frac{1}{2}\beta^2(2k-n)^2} + e^{\frac{1}{2}\beta^2n^2} - 1 \quad \text{with } \Gamma(p+1)=p! \text{ for } p\in\mathbb{N}$$

as $\Gamma(z) = \infty$ for $z = 0, -1, -2, \dots$




$$\mathbb{E}\left[Z^n\right] - 1 = \sum_{k=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-k)\,\Gamma(k+1)} \, e^{\frac{1}{2}\beta^2(2k-n)^2} + e^{\frac{1}{2}\beta^2 n^2} - 1$$

if $\ell = 1, 2, \dots$ $\quad \Gamma(\ell+n) \xrightarrow[n\to 0]{} \Gamma(\ell) = (\ell-1)!$

if $\ell = 0, -1, -2, \dots$ $\quad \Gamma(\ell+n) = \frac{1}{\ell+n}\,\Gamma(\ell+n+1) = \frac{1}{(\ell+n)(\ell+n+1)\cdots(n-1)\,n}\,\underbrace{\Gamma(n+1)}_{\xrightarrow[n\to 0]{} 1}$

$$= \frac{(-1)^{-\ell}}{(-\ell)(-\ell-1)\cdots(-\ell-(-\ell-1))} \, \frac{1}{n}\,(1+o(1))$$

$$= \frac{(-1)^{-\ell}}{(-\ell)!} \, \frac{1}{n}\,(1+o(1))$$

$$\frac{\Gamma(n+1)}{\Gamma(k+1)\,\Gamma(1-k+n)} \sim \frac{1}{k!}\,\frac{(k-1)!}{(-1)^{k-1}}\,n = \frac{(-1)^{k-1}}{k}\,n$$

in $e^{\frac{1}{2}\beta^2(2k-n)^2}$ one can take $n = 0$, $\quad e^{\frac{1}{2}\beta^2 n^2} - 1 = O(n^2)$

$$\frac{1}{n}\left(\mathbb{E}\left[Z^n\right] - 1\right) \xrightarrow[n\to 0]{} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, e^{2\beta^2 k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int \frac{dh}{\sqrt{2\pi}}\, e^{-\frac{1}{2}h^2}\, e^{2\beta h k}$$

$$= \int \frac{dh}{\sqrt{2\pi}}\, e^{-\frac{1}{2}h^2} \sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k-1}\left(e^{2\beta h}\right)^k$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^k}{k}$$

$$= \int \frac{dh}{\sqrt{2\pi}}\, e^{-\frac{1}{2}h^2}\, \ln\left(1 + e^{2\beta h}\right)$$