P. Marios Petropoulos

General Relativity

Test projects

Ecole polytechnique – Ecole normale supérieure *2019-2020*

Contents

1 Test 2014-2015

- *The proposed exercises do not require long technical developments.*
- *A few questions refer to general understanding, irrespective of the problem at hand. Try to keep the answers concise.*

1.1 The wormhole geometry

The purpose of this exercise is to analyse the geometric properties of a four-dimensional space–time $\mathcal{M}$ equipped with the metric ($c = 1$)

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \left(b^2 + r^2\right)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right),} \tag{1.1}$$

where b is a constant, $r \in \mathbb{R}$, $\vartheta \in [0, \pi[$ and $\varphi \in [0, 2\pi[$. We will not be interested in the nature of the sources required for making this geometry a solution of Einstein's equations.

1. Which are the isometries of (1.1)? Give their number and their qualitative features – do not display explicitly the associated Killing fields.
2. Consider now the two-dimensional space-like surface $\mathcal{S}$ defined by $t = t_0$ and $\vartheta = \pi/2$.
   (a) Why studying this surface can provide a faithful characterization of the whole space–time?
   (b) • Write the metric of $\mathcal{S}$, $\mathrm{d}\Sigma^2$.
   • In the natural basis at hand, determine the Christoffel symbols, the components of the corresponding Riemann and Ricci tensors, as well as the scalar curvature.
   • How many independent components does the curvature have?
3. In order to acquire a better picture of the geometry of $\mathcal{S}$, it is desirable to embed it inside three-dimensional Euclidean space E_3.
   (a) • Is every two-dimensional space-like surface embeddable in E_3?
   • Are there specific relationships that must hold between curvatures?

- Do you know any (counter-)example?

(b) Consider E_3 in cylindrical coordinates with Euclidean metric

$$\mathrm{d}s^2 = \mathrm{d}\varrho^2 + \varrho^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2,$$

$\varrho \in \mathbb{R}_+$, $\varphi \in [0, 2\pi[$ and $z \in \mathbb{R}$.

- Determine the coordinate transformation $\varrho = \varrho(r)$ as well as the function $z = z(r)$ that defines an embedded surface $\mathcal{S}$, for which the induced metric is the above $\mathrm{d}\Sigma^2$.
- Recast this surface as $\varrho = \varrho(z)$ and plot this function, showing the regions with positive and negative r. What happens for vanishing b?
- Draw a picture of the surface $\mathcal{S}$ inside E_3 (including the angle φ). Exhibit its asymptotically flat regions. Justify the name of *wormhole geometry* connecting two mirror universes given to (1.1).

4. We now turn to geodesic motion in the wormhole geometry (1.1).

(a) Enumerate the conserved quantities.

(b) Justify why radial (*i.e.* at constant ϑ and φ) geodesics exist.

(c) A point-like traveller starting from $r = R$ with radial initial velocity $u^r = -U < 0$ falls freely and radially ($u^\vartheta = u^\varphi = 0$). What is the proper-time lapse $\Delta\tau$ needed for going through the wormhole throat and reaching the mirror point $r = -R$?

1.2 *Isometries and conformal isometries*

1. Show that for any conformal Killing vector field ξ with scale factor Ω

$$\boxed{\nabla_\lambda \nabla_\mu \xi_\nu = R^\rho{}_{\lambda\mu\nu}\xi_\rho + g_{\mu\nu}\partial_\lambda\Omega + g_{\lambda\nu}\partial_\mu\Omega - g_{\mu\lambda}\partial_\nu\Omega.}$$

2. Conclude that the maximal number of conformal and plain Killing[1] vector fields is $\frac{1}{2}(n+2)(n+1)$.

[1] A plain Killing is a particular case of a conformal Killing.

1.3 *Solving Einstein–Maxwell equations*

1. The aim of the present is to prove that ($c = 1$)

$$\boxed{\begin{aligned} \mathrm{d}s^2 &= -\left(1+\lambda^2 z^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{1+\lambda^2 z^2} + \left(1-\lambda^2 y^2\right)\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{1-\lambda^2 y^2} \\ \mathrm{A} &= p\,x\,\mathrm{d}y + q\,t\,\mathrm{d}z \end{aligned}}$$

solve Einstein–Maxwell vacuum equations[2] for appropriate values of the constants p and q in terms of the arbitrary constant λ. The use of *Cartan formalism in orthonormal frame* is highly recommended.

[2] Vacuum means without matter sources, charges or currents.

(a) Define the orthonormal coframe and determine the spin connection as well as the curvature two-form.

(b) Extract the Riemann tensor components in the frame at hand and determine the Ricci.

(c) Compute the Maxwell field F and its Hodge-dual $*F$.

(d) Check Maxwell's equations (written in the most convenient form).

(e) Compute the electromagnetic energy–momentum tensor.

(f) Impose Einstein's equations ($c = G = 1$) and conclude.

2. Bonus. Prove that for a general Maxwell field, the energy–momentum tensor components read:

$$\boxed{T_{ab} = \frac{1}{8\pi}\left(F_{ac}\,F_b{}^c + *F_{ac} * F_b{}^c\right).} \tag{1.2}$$

2 Test 2015-2016

- *The proposed exercises do not require long technical developments. Sometimes a few words are enough to justify an expression.*
- *A few questions refer to general understanding, irrespective of the specific problem. Try to keep the answers concise.*
- *Reminder:*

$$\nabla_\nu A^\nu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}A^\mu\right),$$
$$\nabla_\mu\nabla_\nu w^\rho - \nabla_\nu\nabla_\mu w^\rho = R^\rho{}_{\lambda\mu\nu}w^\lambda.$$

- *We set here* $c = 1$.

2.1 Raychaudhury equation for null geodesic congruences

Raychaudhury equation describes the focusing properties of geodesic congruences. It has important implications in the context of singularity theorems of Hawking, Penrose... We will here study the case of light, and thus consider a null congruence

$$\mathbf{l} = \ell^\mu\,\partial_\mu, \quad \ell^\mu\ell_\mu = 0.$$

This is assumed to be geodesic (auto-parallel), not necessarily affinely parameterized though:

$$\ell^\nu\,\nabla_\nu\ell^\mu = \kappa(\lambda)\,\ell^\mu,$$

where λ is the parameter along the lines – affine iff $\kappa = 0$. The aim of the exercise is to determine the variation of the congruence expansion Θ (defined below) with λ, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Theta = \ell^\nu\,\nabla_\nu\Theta.$$

The central object in this study is the rank-2 tensor $\mathbf{B} = \nabla\mathbf{l}$, with components

$$\boxed{B_{\mu\nu} = \ell_{\mu;\nu} = \nabla_\nu\ell_\mu} \tag{2.1}$$

(beware of the index positions – there is no symmetry). As usual, $g_{\mu\nu}$ are the components of the metric, and the connection is Levi–Civita.

2.1.1 The orthogonal spatial surface

1. The vector $\mathbf{l}$ does not allow to define a normal projector as $h^{\nu}_{\mu} = \delta^{\nu}_{\mu} + \ell_{\mu}\ell^{\nu}$. Why?

2. We thus need to introduce a complementary vector $\mathbf{n} = n^{\mu}\,\partial_{\mu}$ such that

$$\mathbf{n}\cdot\mathbf{n} = 0, \quad \mathbf{n}\cdot\mathbf{l} = -1, \tag{2.2}$$

and define the symmetric tensor of components

$$\boxed{\begin{aligned} s_{\mu\nu} &= g_{\mu\nu} + \ell_{\mu}n_{\nu} + n_{\mu}\ell_{\nu} \\ &\Updownarrow \\ s^{\nu}_{\mu} &= \delta^{\nu}_{\mu} + \ell_{\mu}n^{\nu} + n_{\mu}\ell^{\nu} \quad \text{with} \quad s = s^{\mu}_{\mu}. \end{aligned}} \tag{2.3}$$

(a) Show that

$$\begin{cases} s_{\mu\nu}\ell^{\nu} = s_{\mu\nu}n^{\nu} = 0, \\ s^{\nu}_{\mu} = s_{\mu\rho}s^{\rho\nu} = s^{\rho}_{\mu}\,s^{\nu}_{\rho}, \\ s = s_{\mu\nu}s^{\mu\nu} = 2. \end{cases}$$

(b) Interpret s^{ν}_{μ} and u^{ν}_{μ} defined as

$$u^{\nu}_{\mu} = -\ell_{\mu}n^{\nu} - n_{\mu}\ell^{\nu}.$$

<u>Hint:</u> for the latter act on $\mathbf{l}$ and $\mathbf{n}$, compute its square and its trace, add it up to s^{ν}_{μ}.

2.1.2 The variation along $\boldsymbol{\lambda}$

1. Show that

$$\begin{cases} \ell^{\mu}B_{\mu\nu} = 0, \\ \ell^{\nu}B_{\mu\nu} = \kappa\,\ell_{\mu}. \end{cases}$$

2. From the congruence variation $B_{\mu\nu}$ given in Eq. (2.1) one defines

$$\boxed{b_{\mu\nu} = s^{\rho}_{\mu}\,s^{\sigma}_{\nu}\,B_{\rho\sigma}.}$$

(a) Show that it obeys

$$\begin{cases} s^{\lambda}_{\mu}b_{\lambda\nu} = s^{\lambda}_{\nu}b_{\mu\lambda} = b_{\mu\nu}, \\ b_{\mu\nu}\ell^{\nu} = \ell^{\mu}b_{\mu\nu} = b_{\mu\nu}n^{\nu} = n^{\mu}b_{\mu\nu} = 0. \end{cases}$$

What do we learn on $b_{\mu\nu}$ from these properties?

(b) Show its explicit relationship with $B_{\rho\sigma}$:

$$b_{\mu\nu} = B_{\mu\nu} + \ell_{\mu}n^{\rho}B_{\rho\nu} + B_{\mu\sigma}n^{\sigma}\ell_{\nu} + \ell_{\mu}\ell_{\nu}n^{\rho}n^{\sigma}B_{\rho\sigma}.$$

(c) Compute its square (beware of the index positions):

$$b^{\mu\nu}b_{\nu\mu} = B^{\mu\nu}B_{\nu\mu} - \kappa^2.$$

3. The spatial transverse variation $b_{\mu\nu}$ can be decomposed into the trace (expansion), the symmetric (shear) and the antisymmetric (vorticity) pieces:

$$b_{\mu\nu} = \frac{1}{2}\Theta s_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{2.4}$$

where by definition

$$\Theta = s^{\mu\nu} b_{\mu\nu}.$$

One can easily prove that *the vorticity ω vanishes iff* $\mathbf{l}$ *is hypersurface-orthogonal*.

(a) Show that the expansion is given by

$$\Theta = g^{\mu\nu} B_{\mu\nu} - \kappa = \nabla_\nu \ell^\nu - \kappa. \tag{2.5}$$

(b) Using its definition via (2.4), show that the shear is traceless:

$$\begin{cases} g^{\mu\nu}\sigma_{\mu\nu} = 0, \\ s^{\mu\nu}\sigma_{\mu\nu} = 0. \end{cases}$$

(c) Show that the square is decomposed as

$$b^{\mu\nu} b_{\nu\mu} = \frac{1}{2}\Theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu}.$$

When non-vanishing, what is the sign of $\sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\omega_{\mu\nu}\omega^{\mu\nu}$?

4. Demonstrate the Raychaudhury equation

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\lambda}\Theta = -R_{\mu\nu}\ell^\mu \ell^\nu + \kappa\Theta - \frac{1}{2}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu}.} \tag{2.6}$$

Hint: Compute directly the derivative and use the commutation rule for covariant derivatives, as well as the various formulas established above.

2.2 *Schwarzschild in Painlevé–Gullstrand coordinates*

The Painlevé–Gullstrand coordinates T, r, ϑ, φ are obtained from the original Schwarzschild coordinates by setting

$$\mathrm{d}t = \mathrm{d}T + \frac{\mathrm{d}r}{\sqrt{2m/r} - \sqrt{r/2m}}.$$

The metric becomes *non-diagonal* and reads:

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}T^2 + \left(\mathrm{d}r + \sqrt{\frac{2m}{r}}\mathrm{d}T\right)^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right).}$$

A sample of Christoffel symbols is

$$\Gamma^T{}_{TT} = \frac{(2m/r)^{3/2}}{2r}, \quad \Gamma^r{}_{TT} = -\frac{m(2m-r)}{r^3}, \quad \Gamma^\vartheta{}_{TT} = \Gamma^\varphi{}_{TT} = 0.$$

2.2.1 *Radial geodesics*

1. Write down the *two first-order* equations (as done in the course, in the framework of Lagrangian formalism) that describe radial geodesic motion with affine parameter. Call $k = +1, 0, -1$ the parameter discriminating massive particles, light and tachyons; call $\mathcal{E}$ the "energy". The describing parameter is proper time τ for massive objects, and a parameter λ for light. The dot will be the derivative with respect to that parameter.

2. (a) What initial conditions (*i.e.* at spatial infinity $r \to \infty$) do guarantee $\dot{T} = 1$ for an incoming massive particle?

   (b) What is thus the interpretation of the coordinate T? Does anything happen across the horizon? Why?

3. Consider now the congruence of incoming radial light rays.

   (a) Show that these obey

   $$\begin{cases} \dot{r} = -\mathcal{E}, \\ T - T_0 = -\int_{r_0}^{r} \frac{\mathrm{d}r}{1+\sqrt{2m/r}}. \end{cases}$$

   (b) Write the tangent vector $\mathbf{l} = \ell^\mu \, \partial_\mu$ (leave r as an implicit function of λ). Determine its dual form $\mathbf{l} = \ell_\mu \, \mathrm{d}x^\mu$ and show it is hypersurface-orthogonal.

   (c) Find the vector field $\mathbf{n}$ obeying Eqs. (2.2), find its dual form and determine the corresponding $s_{\mu\nu}$ following Eq. (2.3).

   (d) Determine the expansion Θ (Eq. (2.5)) of the light congruence tangent to $\mathbf{l}$. Are the light rays at hand converging or diverging? What happens at the horizon? What happens at the Schwarzschild singularity?

   (e) Using the Raychaudhury equation (2.6), determine $\sigma_{\mu\nu}\sigma^{\mu\nu}$. What can we conclude about $\sigma_{\mu\nu}$?

2.2.2 *Light rays on the horizon*

Consider now the congruence tangent to the vector $\mathbf{k} = \partial_T$.

1. Show that this congruence is hypersurface-orthogonal. Determine[1] $S(T, r)$ such that the hypersurfaces $S(T, r) = s_0$ be orthogonal to $\mathbf{k}$.

2. Explain why this congruence is geodesic *on the horizon*. Show that it is non-affinely parameterized.

3. Using this congruence compute the surface gravity on Schwarzschild horizon.

4. Without any computation, determine $\nabla_{(\mu} k_{\nu)}$ and $\nabla_\mu k^\mu$.

5. Can we use in this case the formalism developed in Sec. 2.1.2 for computing the expansion Θ and its variation along the congruence?

[1] Do not perform explicitly the r-integral.

3 Test 2016-2017

- *The proposed exercises do not require long technical developments. We set here $c = 1$.*
- *The covariant and Lie derivative of a rank-2 tensor read:*

$$\nabla_\rho K_{\mu\nu} = \partial_\rho K_{\mu\nu} - \Gamma^\sigma{}_{\rho\mu} K_{\sigma\nu} - \Gamma^\sigma{}_{\rho\nu} K_{\mu\sigma},$$
$$\mathcal{L}_{\mathbf{v}} K_{\mu\nu} = v^\rho \partial_\rho K_{\mu\nu} + K_{\sigma\nu}\partial_\mu v^\sigma + K_{\mu\sigma}\partial_\nu v^\sigma.$$

- *The Christoffel symbols for the three-dimensional Euclidean metric in spherical coordinates $\{r, \vartheta, \varphi\}$,*

$$d\ell^2 = dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),$$

are

$$\Gamma^r{}_{\vartheta\vartheta} = -r, \quad \Gamma^r{}_{\varphi\varphi} = -r\sin^2\vartheta,$$
$$\Gamma^\vartheta{}_{r\vartheta} = \tfrac{1}{r}, \quad \Gamma^\vartheta{}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta,$$
$$\Gamma^\varphi{}_{r\varphi} = \tfrac{1}{r}, \quad \Gamma^\varphi{}_{\vartheta\varphi} = \cot\vartheta.$$

- *The third-degree equation*

$$x^3 + px + q = 0$$

with negative *discriminent $\Delta = 4p^3 + 27q^2$, admits* 3 real *solutions:*

$$y = \sqrt{-\frac{4p}{3}} \times \begin{cases} \cos\theta, \\ \cos\left(\theta + \frac{2\pi}{3}\right), \\ \cos\left(\theta + \frac{4\pi}{3}\right), \end{cases}$$

where

$$\cos 3\theta = -q\sqrt{-\frac{27}{4p^3}}.$$

3.1 Towards a binary system

3.1.1 Extrinsic curvature

Consider a spacetime manifold $\mathcal{M}$ foliated as $\cup_{t\in\mathbb{R}}\Sigma_t$. If $\left\{x^i,\ i = 1, 2, 3\right\}$ are coordinates on Σ_t equipped with the metric

$$\mathrm{d}\ell^2 = \gamma_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \tag{3.1}$$

the four-dimensional spacetime metric of $\mathcal{M}$ reads generally:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + B^i\mathrm{d}t\right)\left(\mathrm{d}x^j + B^j\mathrm{d}t\right), \tag{3.2}$$

where

$$\mathbf{n} = \frac{1}{N}\left(\partial_t - B^i\partial_i\right)$$

is the normalized timelike vector, orthogonal to the leaves Σ_t. One also introduces $B_i = \gamma_{ij}B^j$. Finally, one defines

$$\Delta_\mu^\nu = \delta_\mu^\nu + n_\mu\, n^\nu, \tag{3.3}$$

and the extrinsic curvature tensor of Σ_t:

$$\boxed{K_{\mu\nu} = -\frac{1}{2}\Delta_\mu^\rho\,\Delta_\nu^\sigma\,\mathcal{L}_\mathbf{n}g_{\rho\sigma}.} \tag{3.4}$$

1. Show that Δ_μ^ν displayed in (3.3) is the orthogonal projector on Σ_t, and compute its components in terms of N and B^i.

2. Show that the spatial components of the extrinsic curvature (3.4) are given by

$$\boxed{K_{ij} = -\frac{1}{2N}\left(\partial_t\gamma_{ij} - \mathcal{L}_\mathbf{B}\gamma_{ij}\right),}$$

   where $\mathcal{L}_\mathbf{B}\gamma$ is understood as the three-dimensional Lie derivative of the three-dimensional metric (3.1), along a three-dimensional vector $\mathbf{B} = B^i\partial_i$.

3. Consider the Schwarzschild metric in ordinary coordinates,

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right). \tag{3.5}$$

   This defines a foliation for which you are invited to

   (a) determine N, B^i and γ_{ij},

   (b) compute K_{ij}.

4. The Gullstrand–Painlevé coordinates $\{T, r, \vartheta, \varphi\}$ are obtained from the original Schwarzschild coordinates by setting

$$\mathrm{d}t = \mathrm{d}T + \frac{\mathrm{d}r}{\sqrt{2m/r} - \sqrt{r/2m}}.$$

   The metric now reads:

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \left(\mathrm{d}r + \sqrt{\frac{2m}{r}}\mathrm{d}T\right)^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right).$$

   The latter defines a slightly different foliation for which the same questions are asked:

   (a) determine N, B^i and γ_{ij},

   (b) compute K_{ij}.

3.1.2 *The ADM mass*

Assume a spacetime metric expressed as a foliation (3.2) with radial coordinate r such that

$$\gamma_{ij} = f_{ij} + \mathcal{O}\left(1/r\right), \quad \partial_r \gamma_{ij} = \partial_r f_{ij} + \mathcal{O}\left(1/r^2\right),$$
$$K_{ij} = \mathcal{O}\left(1/r^2\right), \quad \partial_r K_{ij} = \mathcal{O}\left(1/r^3\right),$$

where f_{ij} is the Euclidean flat three-dimensional metric.[1] Under these assumptions Arnowitt, Deser and Misner defined a conserved charge called ADM mass, which coincides with other definitions, when available:

$$\boxed{M_{\text{ADM}} = \frac{1}{16\pi G} \oint_{S^2_\infty} \mathrm{d}S_i \, \mathcal{E}^i,}$$

where

- the integral is performed over the two-sphere at spatial infinity $S^2_\infty = \lim_{r\to\infty} \partial\Sigma_t$;
- the measure on S^2_∞ is $\mathrm{d}S_i = 1/2\sqrt{f}\epsilon_{ijk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k$;
- $\mathcal{E}^i$ is computed as a three-dimensional object on flat space,[2] $\bar{\nabla}_i$ being the Levi–Civita covariant derivative associated with f_{ij}:

$$\boxed{\mathcal{E}^i = \bar{\nabla}_j \gamma^{ij} - \bar{\nabla}^i \gamma^k{}_k.}$$

In this expression[3] it is advised to keep only the leading orders in inverse powers of r since ultimately $r \to \infty$.

1. Compute M_{ADM} for Schwarzschild solution in ordinary coordinates (3.5).
2. Repeat te computation in Gullstrand–Painlevé coordinates (3.6) and comment the result.
3. For a binary system, the initial condition (metric on Σ_0) obeying the constraint subset of Einstein's equations reads:

$$\mathrm{d}\ell^2 = \Psi^4 f_{ij}\,\mathrm{d}x^i \mathrm{d}x^j,$$

   where f_{ij} is again the Euclidean flat three-dimensional metric, and

$$\Psi = 1 + \frac{\alpha_1}{\|\vec{r} - \vec{c}_1\|} + \frac{\alpha_2}{\|\vec{r} - \vec{c}_2\|}$$

   with $\vec{c}_{1,2}$ the location of the black holes on Σ_0. It is convient to work in spherical coordinates.

   (a) Compute the ADM mass of this system.

   (b) Chose $\vec{c}_1 = 0$ so that the coordinate system is centered on the first black hole. Hence $\vec{c}_2 = \vec{r}_{12}$. Assume that one is exploring the vicinity of the first black hole, *i.e.* $r \ll r_{12}$, and define

$$\rho = \frac{\alpha_1^2}{r}.$$

[1] Caution: in general, the coordinates x^i are not Cartesian. Hence the Christoffel symbols of $f_{ij} \neq \delta_{ij}$ do not vanish.

[2] Indices are lowered and raised with f_{ij} and its inverse f^{ij}: here $\gamma^{ij} = f^{ik} f^{j\ell} \gamma_{k\ell}$ and $\gamma^k{}_k = f^{ij}\gamma_{ij}$.

[3] Note that this coincides with the expression displayed in the lecture notes, when $f_{ij} = \delta_{ij}$.

i. Show that

$$\mathrm{d}\ell^2 \approx \left(1+\frac{4\alpha_1}{\rho}\left(1+\frac{\alpha_2}{r_{12}}\right)\right)\left[\mathrm{d}\rho^2+\rho^2\left(\mathrm{d}\vartheta^2+\sin^2\vartheta\,\mathrm{d}\varphi^2\right)\right].$$

ii. Read off the value of the mass of the first black hole. What is the binding energy of the system?

3.2 *The photon sphere of Kerr black hole*

The purpose of this exercise is to show that in Kerr geometry there exists two photon circular orbits in the equatorial plane ($\vartheta = \pi/2$), one direct and one retrograde with respect to the rotation of the black hole. If one relaxes the equatorial condition, further non-planar orbits exist, confined on a sphere called the photon sphere.[4]

The Kerr metric reads:

$$\begin{aligned}\mathrm{d}s^2 &= -\frac{\Delta}{\varrho^2}\left[\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\varphi\right]^2 + \frac{\varrho^2}{\Delta}\mathrm{d}r^2\\ &\quad + \varrho^2\mathrm{d}\vartheta^2 + \frac{\sin^2\vartheta}{\varrho^2}\left[a\,\mathrm{d}t - \left(r^2+a^2\right)\mathrm{d}\varphi\right]^2,\end{aligned}$$

where

$$\Delta = r^2 + a^2 - 2mr, \quad \varrho^2 = r^2 + a^2\cos^2\vartheta.$$

For convenience, the inverse metric is also displayed:

$$g^{tt} = -1 - \frac{4mr\left(r^2+a^2\right)}{(r^2+a^2-2mr)(2r^2+a^2+a^2\cos(2\vartheta))}, \quad g^{\varphi\varphi} = \frac{\left(2r^2+a^2+a^2\cos(2\vartheta)-4mr\right)}{\sin^2\vartheta(r^2+a^2-2mr)(2r^2+a^2+a^2\cos(2\vartheta))},$$

$$g^{t\varphi} = -\frac{4mra}{(r^2+a^2-2mr)(2r^2+a^2+a^2\cos(2\vartheta))}, \quad g^{\vartheta\vartheta} = \frac{1}{r^2+a^2\cos^2\vartheta}, \quad g^{rr} = \frac{r^2+a^2-2mr}{r^2+a^2\cos^2\vartheta}.$$

One assumes

$$\boxed{0 \leqslant a < m.}$$

The orbits of any point-like test particle are labeled by the conserved quantities $\mathscr{E}$ and $\mathscr{L}$, as defined and used in the lecture notes. One will assume without proof that $\vartheta = \pi/2$ solves the ϑ-Euler–Lagrange equation of motion.[5]

[4] The interested reader can find more on this subject in *Spherical photon orbits around a Kerr black hole*, E. Teo, General Relativity and Gravitation (2003) **35** 1909.

[5] Caution: *do not* delve into second-order Euler–Lagrange equations.

1. Show that circular orbits obey

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[g^{rr}\left(g^{\varphi\varphi}\mathscr{L}^2 - 2g^{\varphi t}\mathscr{L}\mathscr{E} + g^{tt}\mathscr{E}^2\right)\right] = 0, \tag{3.6}$$

$$g^{\varphi\varphi}\mathscr{L}^2 - 2g^{\varphi t}\mathscr{L}\mathscr{E} + g^{tt}\mathscr{E}^2 = 0 \tag{3.7}$$

with

$$g^{\varphi\varphi}\mathscr{L}^2 - 2g^{\varphi t}\mathscr{L}\mathscr{E} + g^{tt}\mathscr{E}^2 = \frac{-2m(\mathscr{L}-a\mathscr{E})^2 + \left(\mathscr{L}^2 - a^2\mathscr{E}^2\right)r - \mathscr{E}^2 r^3}{r\left(a^2+r^2-2mr\right)}.$$

2. The properties of these orbits are obtained in 3 steps:

(a) Using Eq. (3.6) compute their radius r_c as a function of m, a and

$$x = \frac{\mathscr{L}}{\mathscr{E}} + a. \tag{3.8}$$

(b) Determine x from Eq. (3.7).

(c) Putting everything together, find the 2 admissible radiuses $r_c^{\pm}$ associated with the direct and retrograde photons respectively. Discuss the $a \to 0$ limit.

3. Show that for equatorial photons on circular orbits, the orbital period in coordinate time is given by

$$T = 2\pi \left| \frac{\mathscr{L}}{\mathscr{E}} \right| .$$

4. What is the orbital period of a photon as measured by a static observer sitting at the photon orbital radius around a supermassive Schwarzschild black hole. The data are as follows:

- $M = 4 \times 10^6\, M_\odot$ with $M_\odot \approx 1.99 \times 10^{30}$ kg the solar mass,
- $G \approx 6.67 \times 10^{-11}\, \frac{\mathrm{m}^3}{\mathrm{kg\, s}^2}$,
- $c \approx 2.99 \times 10^8\, \frac{\mathrm{m}}{\mathrm{s}}$ (to be appropriately restored).

4 Test 2017-2018

- The proposed exercises do not require long technical developments. The longest (like 14, 15 or 19) should not exceed half a page each.
- The Riemann tensor of a metric and torsionless connection obeys the Bianchi identities. In components, these are

$$R^{\mu}{}_{[\nu\rho\sigma]} = 0, \quad \nabla_{[\lambda} R^{\mu}{}_{\nu|\rho\sigma]} = 0.$$

- The components of the Weyl tensor for a metric and torsionless (Levi–Civita) connection in n dimensions read:

$$\begin{aligned} C_{\mu\nu\rho\sigma} \quad = \quad & R_{\mu\nu\rho\sigma} - \frac{1}{n-2}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho}\right) \\ & + \frac{1}{(n-1)(n-2)} R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right). \end{aligned}$$

- The Cartan formalism in a natural frame ($\theta^{\mu} = \mathrm{d}x^{\mu}$) leads to the following equations for a Levi–Civita connection:

$$\begin{cases} \mathrm{d}g_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu} \text{ with } \omega_{\mu\nu} = g_{\mu\rho}\omega^{\rho}{}_{\nu}, \\ \omega^{\mu}{}_{\nu} \wedge \mathrm{d}x^{\nu} = 0. \end{cases}$$

  Caution: since in the natural frame $\mathrm{d}g_{\mu\nu} \neq 0$, $\omega_{\mu\nu}$ *is not* antisymmetric.

- Consider a one-parameter (σ) family of geodesics $x^{\mu} = x^{\mu}(\tau, \sigma)$ with τ an affine parameter along the curves. The velocity components are $u^{\mu} = \frac{\partial x^{\mu}}{\partial \tau}$, whereas the components of the deviation vector are $s^{\mu} = \frac{\partial x^{\mu}}{\partial \sigma}$. The deviation acceleration reads:

$$a^{\mu} \equiv \frac{\mathrm{D}^2 s^{\mu}}{\partial \tau^2} = R^{\mu}{}_{\nu\rho\sigma} u^{\nu} u^{\rho} s^{\sigma}.$$

4.1 The Schouten tensor and propagation of gravity

The Einstein equations in four spacetime dimensions are 10 second-order partial-differential equations for the spacetime metric. Alternatively, they are algebraic equations for 10 out of the 20 components of the Riemann tensor, namely those of the Ricci tensor (we assume Levi–Civita connection). The remaining 10 components of the Weyl tensor are not determined algebraically from the sources, but rather obey a differential equation following Einstein's equations. The purpose of this exercise is to set and interpret this equation. We work in arbitrary spacetime dimension n.

The Schouten tensor is defined algebraically from the Ricci tensor and the curvature scalar:

$$\boxed{P_{\mu\nu} = \frac{1}{n-2}\left(R_{\mu\nu} - \frac{1}{2(n-1)}Rg_{\mu\nu}\right).}$$

Show the following expressions:

1. $\nabla_\mu R^\mu{}_{\nu\rho\sigma} = \nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho}$ and $\nabla_\mu R^\mu{}_\sigma = \frac{1}{2}\nabla_\sigma R$;
2. $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu\rho}P_{\nu\sigma} - g_{\nu\sigma}P_{\mu\rho} + g_{\nu\rho}P_{\mu\sigma} + g_{\mu\sigma}P_{\nu\rho}$;
3. $\nabla_\mu P^\mu{}_\sigma = \frac{1}{2(n-1)}\nabla_\sigma R$ and $\nabla_\mu C^\mu{}_{\nu\rho\sigma} = (n-3)\left(\nabla_\rho P_{\nu\sigma} - \nabla_\sigma P_{\nu\rho}\right)$.

Using Einstein's equations:

4. prove

$$\boxed{\nabla_\mu C^\mu{}_{\nu\rho\sigma} = \frac{8\pi G}{c^4}J_{\nu\rho\sigma},} \quad (4.1)$$

where

$$J_{\nu\rho\sigma} = \frac{n-3}{n-2}\left[\nabla_\rho T_{\nu\sigma} - \nabla_\sigma T_{\nu\rho} - \frac{1}{n-1}\left(\nabla_\rho T^\lambda{}_\lambda\, g_{\nu\sigma} - \nabla_\sigma T^\lambda{}_\lambda\, g_{\nu\rho}\right)\right];$$

5. interpret Eq. (4.1).

4.2 *The pp-waves*

We will here focus on four-dimensional spacetimes equipped with a line element $\mathrm{d}s^2$ and a Levi–Civita connection.

4.2.1 *Metrics with a covariantly constant null vector*

Let us assume that $\mathrm{d}s^2$ admits a nowhere vanishing and regular, parallel *i.e.* covariantly constant, null vector $\boldsymbol{k} = k^\mu\partial_\mu$:

$$\nabla_\mu k^\nu = 0, \quad k_\mu k^\mu = 0$$

with $k_\mu = g_{\mu\nu}k^\nu$ the components of the associated form $\mathbf{k} = k_\mu \mathrm{d}z^\mu$, in a coordinate system $z^\mu, \mu = 0, \ldots, 3$. Show the following:

6. $\boldsymbol{k}$ is a Killing vector – *do not confuse* $\boldsymbol{k}$ *with* $\mathbf{k}$;
7. $\mathcal{L}_{\boldsymbol{k}}\mathbf{k} = 0$ *i.e.* the form $\mathbf{k}$ is invariant along the vector $\boldsymbol{k}$;
8. the form $\mathbf{k}$ is closed.

Hence, one can find a function $u(z)$ such that locally

$$\mathbf{k} = \mathrm{d}u.$$

9. Show that

$$\mathrm{i}_{\boldsymbol{k}}\mathrm{d}u = 0.$$

The function $u(z)$ is therefore constant along the integral lines of the vector field $\boldsymbol{k}$. Together with the function r along these lines defined as

$$\boldsymbol{k} = \partial_r, \tag{4.2}$$

u can be adopted in a new coordinate system $\{u, r, x^1 = x, x^2 = y\}$. *The latter will be used until the end.*

10. Justify why, in this coordinate system, the metric at hand reads:

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + K(u,x,y)\mathrm{d}u^2 + 2A_i(u,x,y)\mathrm{d}x^i\mathrm{d}u + g_{ij}(u,x,y)\mathrm{d}x^i\mathrm{d}x^j. \tag{4.3}$$

The coordinate r leading to this form is not unique.

11. Show that under

$$r \to r + \Lambda(u,x,y),$$

the metric (4.3) is form-invariant with

$$\begin{cases} K \to K + 2\partial_u\Lambda(u,x,y), \\ A_i \to A_i + \partial_i\Lambda(u,x,y). \end{cases}$$

12. Provide a heuristic argument, inspired by two-dimensional electromagnetism, to explain why A_i can be set to zero without loss of generality.

The resulting metric captures many of the remarkable solutions to Einstein's equations, such as Robinson–Trautman, Kundt or pp-waves.

4.2.2 *The gravitational pp-waves*

The pp-wave metric is obtained assuming that the two-dimensional spatial section at constant r and u is Euclidean space:

$$g_{ij}(u,x,y)\mathrm{d}x^i\mathrm{d}x^j = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2.$$

This leads to

$$\boxed{\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + K(u,x,y)\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2,} \tag{4.4}$$

representing a *plane-fronted* gravitational wave propagating along the *parallel* rays tangent to the vector field $\boldsymbol{k}$.

13. Write the inverse-metric components $g^{\mu\nu}$ for (4.4) and the relationship among the components v^μ and v_μ of a vector $\boldsymbol{v}$ and its associated form $\mathbf{v} = \mathbf{g}(\boldsymbol{v})$. These will be useful throughout the exercise.

14. In the *non-orthonormal* natural frame $\{\theta^u = \mathrm{d}u, \theta^r = \mathrm{d}r, \theta^i = \mathrm{d}x^i, i = 1,2\}$, show that the spin-connection elements

$$\begin{cases} \omega^u{}_u = \omega^u{}_r = \omega^u{}_i = \omega^i{}_j = \omega^i{}_r = \omega^r{}_r = 0, \\ \omega^r{}_u = \frac{1}{2}\mathrm{d}K = \frac{1}{2}\left(\partial_u K\mathrm{d}u + \partial_i K\mathrm{d}x^i\right), \\ \omega^r{}_i = \frac{1}{2}\partial_i K\mathrm{d}u, \\ \omega^i{}_u = -\frac{1}{2}\delta^{ij}\partial_j K\mathrm{d}u \end{cases}$$

obey the metricity and torsionless conditions.

15. The non-vanishing elements of the curvature two-form are $\mathcal{R}^r{}_i$ and $\mathcal{R}^i{}_u$. Compute them and show that

$$R_{uiju} = \frac{1}{2}\partial^2_{ij}K,$$

where $\partial^2_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}$.

16. Show that the only non-vanishing components of the Ricci and of the Weyl tensors are respectively

$$R_{uu} = -\frac{1}{2}\left(\partial^2_{xx}K + \partial^2_{yy}K\right) \quad \text{and} \quad C_{uiju} = \frac{1}{2}\left(\partial^2_{ij}K - \frac{1}{2}\delta_{ij}\left[\partial^2_{xx}K + \partial^2_{yy}K\right]\right).$$

17. Display Einstein's vacuum equations for the pp-waves. What is the remarkable property they satisfy?

One usually defines the following functions:

$$\begin{cases} A_0(u,x,y) = \frac{1}{2}\left(\partial^2_{xx}K + \partial^2_{yy}K\right), \\ A_+(u,x,y) = \frac{1}{2}\left(\partial^2_{xx}K - \partial^2_{yy}K\right), \\ A_\times(u,x,y) = \partial^2_{xy}K. \end{cases}$$

These allow to express and interpret the geodesic deviation inside a one-parameter geodesic congruence $x^\mu(\tau,\sigma)$ describing point-like test masses free-falling inside the pp-wave gravitational field.

18. Compute the geodesic-deviation acceleration vector components a^u, a^r, a^i.

19. Choose a gauge where $\dot{u} = 1$ and $u' = 0$,[1] and show that

$$\begin{cases} a^x + a^y = \frac{1}{2}(x'+y')A_0 + \frac{1}{2}(x'-y')A_+ + \frac{1}{2}(x'+y')A_\times, \\ a^x - a^y = \frac{1}{2}(x'-y')A_0 + \frac{1}{2}(x'+y')A_+ - \frac{1}{2}(x'-y')A_\times. \end{cases} \tag{4.5}$$

[1] The dot stands for the τ-derivative and the prime for σ-derivative.

20. Interpret Eqs. (4.5), making the distinction between vacuum and non-vacuum pp-wave solutions.

4.2.3 *The plane waves in Brinkmann coordinates*

Quadratic functions $K(u,x,y)$,

$$\boxed{K(u,x,y) = \frac{1}{2}K_{ij}(u)x^i x^j,}$$

define a subclass of pp-waves dubbed *plane waves*. The coordinate system in use is known as Brinkmann's.

21. What are now vacuum Einstein's equations?

22. Consider the following configuration:

$$K_{xy} = \kappa\delta(u), \quad K_{xx} = K_{yy} = 0 \tag{4.6}$$

with κ an arbitrary constant. Does (4.6) solve Einstein's equations? How would you interpret it?

A gravitational plane wave has extra symmetries, besides the original one generated by the null Killing vector field $\boldsymbol{k}$.

23. Show that the following vector fields:

$$\boldsymbol{\xi} = f^i(u)\partial_i - \delta_{ij}x^i \frac{\mathrm{d}f^j}{\mathrm{d}u}\partial_r, \tag{4.7}$$

are Killing for any solution $f^i(u)$ of the harmonic-oscillator equation

$$\frac{\mathrm{d}^2 f^i}{\mathrm{d}u^2} = \frac{1}{2}\delta^{ik}K_{kj}(u)f^j.$$

It can be shown that this equation possesses 4 linearly independent solutions and that the 5 Killing fields $\boldsymbol{k}$ and $\boldsymbol{\xi}$, Eqs. (4.2) and (4.7), obey the Heisenberg algebra.

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- The proposed exercises do not require long technical developments. The longest (23) does not exceed one page.
- Bold typefaces as $\boldsymbol{u}$ designate vectors, forms or more general tensors. The context makes the meaning clear. Anyway, the indices appearing in the components can be lowered or raised at wish using the metric tensor.
- The connection will always be metric and torsionless *i.e.* Levi–Civita.
- The fully antisymmetric tensor has components $\eta_{0123} = \sqrt{g}$ with g the absolute value of the metric determinant, whereas $\eta^{0123} = -1/\sqrt{g}$.
- Geodesics will always be affinely parameterized.
- Useful integrals (valid for $x > 1$):

$$\int \mathrm{d}x \frac{\sqrt{1/x}}{1-1/x} = 2\sqrt{x} - \ln\frac{\sqrt{x}+1}{\sqrt{x}-1}, \quad \int \mathrm{d}x \frac{\sqrt{x}}{1-1/x} = 2\sqrt{x}\left(1+\frac{x}{3}\right) + \ln\frac{\sqrt{x}+1}{\sqrt{x}-1}.$$

- Cartan's formalism in *orthonormal* frame leads to the following equations for a Levi–Civita connection:

$$\begin{cases} \omega_{ab} + \omega_{ba} = 0 \text{ with } \omega_{ab} = \eta_{ac}\omega^c{}_b, \\ \mathrm{d}\boldsymbol{\theta}^a + \omega^a{}_b \wedge \boldsymbol{\theta}^b = 0, \end{cases}$$

  where $\omega^a{}_b$ are the elements of the spin-connection one-form. The curvature two-form is given by the second Cartan structure equation:

$$\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b.$$

- Further formulas – *Christmas homework.* The Kerr metric reads:

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{\Delta}{\varrho^2}\left[c\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\varphi\right]^2 + \frac{\varrho^2}{\Delta}\mathrm{d}r^2 \\ &\quad + \varrho^2\mathrm{d}\vartheta^2 + \frac{\sin^2\vartheta}{\varrho^2}\left[ac\,\mathrm{d}t - \left(r^2+a^2\right)\mathrm{d}\varphi\right]^2, \end{aligned}$$

  where

$$\Delta = r^2 + a^2 - 2mr, \quad \varrho^2 = r^2 + a^2\cos^2\vartheta.$$

More explicitly, one finds:

$$g_{00} = -1 + \frac{2mr}{\rho^2}, \quad g_{\varphi\varphi} = \left(r^2 + a^2 + \frac{2mra^2}{\rho^2}\sin^2\vartheta\right)\sin^2\vartheta, \quad g_{0\varphi} = -\frac{2mra}{\rho^2}\sin^2\vartheta.$$

As usual $x^0 = ct$ and $m = {}^{MG}/_{c^2}$. For convenience, the inverse metric is also displayed:

$$g^{00} = -1 - \frac{4mr\left(r^2 + a^2\right)}{\left(r^2 + a^2 - 2mr\right)\left(2r^2 + a^2 + a^2\cos(2\vartheta)\right)},$$
$$g^{\varphi\varphi} = \frac{\left(2r^2 + a^2 + a^2\cos(2\vartheta) - 4mr\right)}{\sin^2\vartheta\left(r^2 + a^2 - 2mr\right)\left(2r^2 + a^2 + a^2\cos(2\vartheta)\right)},$$
$$g^{0\varphi} = -\frac{4mra}{\left(r^2 + a^2 - 2mr\right)\left(2r^2 + a^2 + a^2\cos(2\vartheta)\right)}.$$

5.1 *Local observers, free fall and Fermi–Walker transport*

5.1.1 *Setting the stage*

In general relativity we must define carefully the local frames with respect to which measurements are made. These frames are materialized in a time-like congruence $\mathscr{C}_{\boldsymbol{u}}$ tangent to a vector field $\boldsymbol{u}$. The later may or may not be geodesic, and it is associated with the velocity field of a family of observers. These usually carry another set of three space-like vectors $\boldsymbol{e}_i$, $i = 1, 2, 3$ forming with $\boldsymbol{e}_0 = {}^{\boldsymbol{u}}/_c$ a tetrad *i.e.* a local frame, which for convenience is often chosen orthonormal:

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \eta_{ab}, \quad a, b = 0, 1, 2, 3.$$

The associated coframe is as usual $\boldsymbol{\theta}^a$ with $\boldsymbol{\theta}^a(\boldsymbol{e}_b) = \mathrm{i}_{\boldsymbol{e}_b}\boldsymbol{\theta}^a = \delta^a_b$, and allows to express the background metric as

$$\mathrm{d}s^2 = \eta_{ab}\boldsymbol{\theta}^a\boldsymbol{\theta}^b.$$

In practice, the space-like triads are realized with gyroscopes.

A vector field $\boldsymbol{u}$ has acceleration $\boldsymbol{a}$ and vorticity two-form $\boldsymbol{\psi}$ defined as:

$$\boldsymbol{a} = \nabla_{\boldsymbol{u}}\boldsymbol{u}, \quad \boldsymbol{\psi} = \frac{1}{2}\left(\mathrm{d}\boldsymbol{u} + \frac{1}{c^2}\boldsymbol{u}\wedge\boldsymbol{a}\right). \tag{5.1}$$

One also defines the vorticity one-form as the Hodge–Poincaré dual of $\boldsymbol{u}\wedge\boldsymbol{\psi}$:

$$\boldsymbol{\Omega} = *\left(\boldsymbol{u}\wedge\boldsymbol{\psi}\right).$$

1. Show that the vector field $\boldsymbol{u}$ is hypersurface-orthogonal iff $\boldsymbol{\Omega} = 0$.

2. Show that the components of $\boldsymbol{\Omega}$ read: $\Omega^\sigma = \frac{1}{2}u_\mu\psi_{\nu\rho}\eta^{\mu\nu\rho\sigma}$.

3. Prove

$$\boxed{\nabla_{\boldsymbol{e}_b}\boldsymbol{\theta}^a = -\boldsymbol{\theta}^c\mathrm{i}_{\boldsymbol{e}_b}\omega^a{}_c.}$$

<u>Hint:</u> *use Leibniz rule.*

5.1.2 *Free-falling observers and application to Schwarzschild*

We assume the congruence $\mathscr{C}_{\boldsymbol{u}}$ be geodesic.

4. Show that if two vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallelly transported along $\mathscr{C}_{\boldsymbol{u}}$, then their scalar product $\boldsymbol{v}\cdot\boldsymbol{w}$ is constant along $\mathscr{C}_{\boldsymbol{u}}$. Conclude that a free-falling observer can carry parallelly without obstruction an orthonormal local frame $\boldsymbol{e}_a$ with $\boldsymbol{e}_0 = {}^{\boldsymbol{u}}/_{c}$.

   In Schwarzschild geometry,

$$\mathrm{d}s^2 = -\left(1-\frac{r_\mathrm{g}}{r}\right)c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{r_\mathrm{g}}{r}} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right),$$

the time-like geodesics obey (the dot is the derivative with respect to proper time τ):

$$c^2\dot{t} = \frac{\mathcal{E}}{1-\frac{r_\mathrm{g}}{r}}, \quad \dot{\varphi} = \frac{\mathcal{L}}{r^2\sin^2\vartheta}, \quad \dot{r}^2 - c^2\frac{r_\mathrm{g}}{r} + \frac{\mathcal{L}^2}{r^2} - \frac{\mathcal{L}^2 r_\mathrm{g}}{r^3} = \frac{\mathcal{E}^2 - c^4}{c^2}. \tag{5.2}$$

We consider observers moving *inwards* on a radial congruence tangent to the vector field

$$\boldsymbol{u} = \frac{1}{1-\frac{r_\mathrm{g}}{r}}\partial_t - \sqrt{\frac{r_\mathrm{g}}{r}}c\partial_r.$$

5. Show (using Eqs. (5.2)) that the congruence $\mathscr{C}_{\boldsymbol{u}}$ at hand is geodesic, and determine $\mathcal{E}$ and $\mathcal{L}$. What is the kinematical state of the considered observers at radial infinity?

6. For the geodesics $\mathscr{C}_{\boldsymbol{u}}$, determine the functions $r(\tau)$ and $t(r)$, assuming that the observer is at $r \to +\infty$ when $t \to -\infty$ and $\tau \to -\infty$.

7. When do these observers reach the horizon, in coordinate and in proper time?

   Together with this vector field $\boldsymbol{e}_t = {}^{\boldsymbol{u}}/_{c}$, we define three extra space-like[1] mutually orthogonal and norm-1 vector fields:

$$\boldsymbol{e}_r = -\frac{1}{\sqrt{\frac{r}{r_\mathrm{g}}} - \sqrt{\frac{r_\mathrm{g}}{r}}}\frac{1}{c}\partial_t + \partial_r, \quad \boldsymbol{e}_\vartheta = \frac{1}{r}\partial_\vartheta, \quad \boldsymbol{e}_\varphi = \frac{1}{r\sin\vartheta}\partial_\varphi.$$

[1] Space-like and time-like refer in this discussion to the asymptotic region $r > r_\mathrm{g}$.

8. Show that the coframe dual to $\boldsymbol{e}_a$ is

$$\boldsymbol{\theta}^t = c\mathrm{d}t + \frac{1}{\sqrt{\frac{r}{r_\mathrm{g}}} - \sqrt{\frac{r_\mathrm{g}}{r}}}\mathrm{d}r, \quad \boldsymbol{\theta}^r = \sqrt{\frac{r_\mathrm{g}}{r}}c\mathrm{d}t + \frac{1}{1-\frac{r_\mathrm{g}}{r}}\mathrm{d}r, \quad \boldsymbol{\theta}^\vartheta = r\mathrm{d}\vartheta, \quad \boldsymbol{\theta}^\varphi = r\sin\vartheta\mathrm{d}\varphi.$$

Alternatively

$$c\mathrm{d}t = \frac{1}{1-\frac{r_\mathrm{g}}{r}}\boldsymbol{\theta}^t - \frac{1}{\sqrt{\frac{r}{r_\mathrm{g}}} - \sqrt{\frac{r_\mathrm{g}}{r}}}\boldsymbol{\theta}^r, \quad \mathrm{d}r = -\sqrt{\frac{r_\mathrm{g}}{r}}\boldsymbol{\theta}^t + \boldsymbol{\theta}^r, \quad \mathrm{d}\vartheta = \frac{1}{r}\boldsymbol{\theta}^\vartheta, \quad \mathrm{d}\varphi = \frac{1}{r\sin\vartheta}\boldsymbol{\theta}^\varphi.$$

The independent spin-connection elements are

$$\begin{cases} \omega^r{}_t = \frac{1}{2}\sqrt{\frac{r_\mathrm{g}}{r^3}}\boldsymbol{\theta}^r, & \omega^\vartheta{}_t = -\sqrt{\frac{r_\mathrm{g}}{r^3}}\boldsymbol{\theta}^\vartheta, \quad \omega^\vartheta{}_r = \frac{1}{r}\boldsymbol{\theta}^\vartheta, \\ \omega^\varphi{}_t = -\sqrt{\frac{r_\mathrm{g}}{r^3}}\boldsymbol{\theta}^\varphi, & \omega^\varphi{}_r = \frac{1}{r}\boldsymbol{\theta}^\varphi, \quad \omega^\varphi{}_\vartheta = \frac{\cot\vartheta}{r}\boldsymbol{\theta}^\varphi. \end{cases}$$

9. Prove that $\boldsymbol{\theta}^a$ are parallelly transported along $\mathscr{C}_{\boldsymbol{u}}$. Be careful: *the proof fits in two lines, using appropriately the already established results.*

In summary, we have at our disposal a congruence of free-falling observers carrying parallelly orthonormal tetrads – since $\boldsymbol{\theta}^a$ are parallelly transported so are their duals $\boldsymbol{e}_a$ (we knew it already for $\boldsymbol{e}_0 = c\boldsymbol{u}$ because $\mathscr{C}_{\boldsymbol{u}}$ was proven to be geodesic).

10. Show that $\boldsymbol{u}$ is hypersurface orthogonal, and determine the equation of the corresponding hypersurface in the form $T(x) = T_0$ (T_0 is an arbitrary constant).

From this point on, it would be natural to define Fermi normal coordinates (often abusively called Riemann normal coordinates) around *a given* geodesic of the congruence. These include the proper time along the geodesic, and three spatial coordinates defined inside the above hypersurface, whose lines are tangent to the $\boldsymbol{e}_i$s at the geodesic. We will not delve into this quite technical subject.[2]

[2] Interested students are welcome to check the following publications:

- F. K. Manasse and C. W. Misner, *Fermi normal coordinates and some basic concepts in differential geometry*, J. Math. Phys. **4** (1963) 735;
- D. Bini, A. Geralico and R. T. Jantzen, *Fermi coordinates in Schwarzschild spacetime: closed form expressions*, Gen. Rel. Grav. **43** (2011) 1837, arXiv:1408.4947 [gr-qc].

5.1.3 *Energy measurements*

In the above setup, a radial photon comes from infinity with momentum $\boldsymbol{p} = p_0 c\mathrm{d}t + p_r \mathrm{d}r$. This photon is captured by the free-falling observer at r, who measures its energy E_{ff_r}.

11. We define $E = -cp_0$. Why is E conserved along the photon trajectory, and what is its meaning? Determine p_r. Be careful: *the photon is falling inwards.*

12. Compute E_{ff_r}. Compare with the energy E_{r_r} that would have measured an observer *at rest* at the same location r. Explain the difference.

5.1.4 *Accelerated observers*

Often observers are accelerated, as are ordinary people at rest or astronauts flying in rockets. These can still carry gyroscopes, and the concept of (no longer parallelly) transported tetrads should be made mathematically sound.

Given an accelerated congruence $\mathscr{C}_{\boldsymbol{u}}$ tangent to a normalized vector field $\boldsymbol{u}$ (*i.e.* $\|\boldsymbol{u}\|^2 = -c^2$) with acceleration $\boldsymbol{a}$, one defines the *Fermi derivative along* $\boldsymbol{u}$ of any vector field $\boldsymbol{v}$ as follows:

$$\boxed{D^{\mathrm{F}}_{\boldsymbol{u}}\boldsymbol{v} = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \frac{1}{c^2}\left(\boldsymbol{a}(\boldsymbol{v}\cdot\boldsymbol{u}) - \boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{a})\right)}$$

or, in components

$$D^{\mathrm{F}}_{\boldsymbol{u}}v^\mu = u^\nu\nabla_\nu v^\mu + \frac{1}{c^2}\left(a^\mu v^\nu u_\nu - u^\mu v^\nu a_\nu\right).$$

A vector field is transported *à la Fermi–Walker* along a congruence $\mathscr{C}_{\boldsymbol{u}}$ if its Fermi derivative vanishes along the vector field $\boldsymbol{u}$.

14. Show that any vector $\boldsymbol{u}$ such that $\|\boldsymbol{u}\|^2 = -c^2$ is always self-Fermi–Walker-transported.

15. Show that if $\boldsymbol{v}$ is orthogonal to $\boldsymbol{u}$ along $\mathscr{C}_{\boldsymbol{u}}$, then $D^{\mathrm{F}}_{\boldsymbol{u}}\boldsymbol{v}$ is also orthogonal to $\boldsymbol{u}$.

16. Show that if two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are Fermi–Walker-transported along a vector $\boldsymbol{u}$, then their scalar product $\boldsymbol{v}\cdot\boldsymbol{w}$ is constant along every orbit of the congruence $\mathscr{C}_{\boldsymbol{u}}$.

In summary, any observer of a family can carry an orthonormal local frame, consisting of the velocity field $\boldsymbol{e}_0 = \boldsymbol{u}/c$ and three space-like Fermi–Walker-transported vectors $\boldsymbol{e}_i$, $i = 1, 2, 3$.

We further impose that Fermi derivative (i) obeys Leibniz rule for tensor or contracted tensor product, and (ii) coincides for scalar functions f with the ordinary directional derivative: $D^{\mathrm{F}}_{\boldsymbol{u}} f = \nabla_{\boldsymbol{u}} f = \boldsymbol{u}(f)$.

17. Show that for a one-form $\boldsymbol{w} = w_\mu \mathrm{d}x^\mu$

$$D^{\mathrm{F}}_{\boldsymbol{u}} w_\mu = u^\nu \nabla_\nu w_\mu + \frac{1}{c^2}\left(a_\mu w_\nu u^\nu - u_\mu w_\nu a^\nu\right). \tag{5.3}$$

Tip: *start with $w_\mu v^\mu$ for arbitrary v^μ.*

18. Rewrite (5.3) in terms of the vorticity of $\boldsymbol{u}$ defined in (5.1).

Using Leibniz rule, the Fermi derivative can be generalized to any tensor, and for any tensor Fermi–Walker transport along a congruence $\mathscr{C}_{\boldsymbol{u}}$ demands vanishing Fermi derivative along the vector field $\boldsymbol{u}$.

5.1.5 *Back to Schwarzschild from hovering observers*

A hovering observer[3] is at rest at fixed "altitude" r and celestial inclination ϑ, φ. He needs for that a spacecraft with powerful engines in order to resist against falling. Assuming he is skilled and succeeds, his canonically normalized velocity vector will be

$$\boldsymbol{u} = \frac{1}{\sqrt{1-\frac{r_{\mathrm{g}}}{r}}}\partial_t.$$

[3] None of the quantities $\boldsymbol{u}$, $\boldsymbol{e}_a$, $\boldsymbol{\theta}^a$, $\boldsymbol{\omega}$ introduced here should be confused with those already met for free-falling observers.

We assume having a whole set of such observers filling the sky, and defining a genuine continuous congruence $\mathscr{C}_{\boldsymbol{u}}$.

Given $\boldsymbol{e}_t = \boldsymbol{u}/c$, it is natural to define a new orthonormal coframe as

$$\boldsymbol{\theta}^t = \sqrt{1-\frac{r_{\mathrm{g}}}{r}}c\mathrm{d}t, \quad \boldsymbol{\theta}^r = \frac{1}{\sqrt{1-\frac{r_{\mathrm{g}}}{r}}}\mathrm{d}r, \quad \boldsymbol{\theta}^\vartheta = r\mathrm{d}\vartheta, \quad \boldsymbol{\theta}^\varphi = r\sin\vartheta\mathrm{d}\varphi.$$

19. Determine the spatial frame vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\vartheta$ and $\boldsymbol{e}_\varphi$ (*read them off without much computation*).

20. Using Cartan's formalism, determine the independent spin-connection elements:

$$\omega^r{}_t, \quad \omega^\vartheta{}_r, \quad \omega^\varphi{}_r, \quad \omega^\varphi{}_\vartheta.$$

Be systematic: *compute* $d\boldsymbol{\theta}^a$, $a = t, r, \vartheta, \varphi$, *express them in terms of* $\boldsymbol{\theta}^a$, *read off the spin connection. Each element turns out to be single-term. The other two independent elements* $\omega^{\vartheta}{}_t$ *and* $\omega^{\varphi}{}_t$ *vanish.*

21. Compute the acceleration form $\boldsymbol{a} = a_b \boldsymbol{\theta}^b$. Determine its norm $a = \sqrt{a^b a_b}$. How would you interpret it? What happens at the horizon? Renormalize the blue-shift divergence of a by appropriate multiplication. What is the renormalized acceleration at the horizon, $a_{\mathrm{r}}(r_{\mathrm{g}})$?

22. Prove that the coframe tetrad is Fermi–Walker-transported along $\mathscr{C}_{\boldsymbol{u}}$. Again: *do not embark in long technicalities.*

23. Determine the independent elements of the curvature two-form:

$$\mathcal{R}^r{}_t, \quad \mathcal{R}^{\vartheta}{}_t, \quad \mathcal{R}^{\vartheta}{}_r, \quad \mathcal{R}^{\varphi}{}_t, \quad \mathcal{R}^{\varphi}{}_r, \quad \mathcal{R}^{\varphi}{}_{\vartheta}$$

(*each contains at the end a single and simple term*).

5.2 *Killing & Killing–Yano tensors and conservation laws*

5.2.1 *General properties*

Killing vectors $\boldsymbol{\xi} = \xi^\mu \partial_\mu$ provide the mean for constructing conserved quantities for geodesic motion, which are linear in the velocity of the point-like free-falling object: $\xi^\mu u_\mu$. This is Nœther's theorem.

We work here in natural frames.

- Killing tensors are *symmetric*, rank-n tensor fields $\boldsymbol{K}$ with components $K^{\nu_1 \dots \nu_n} = K^{(\nu_1 \dots \nu_n)}$ obeying

$$\nabla^{(\mu} K^{\nu_1 \dots \nu_n)} = 0.$$

- Killing–Yano tensors are *antisymmetric*, rank-n tensor fields $\boldsymbol{y}$ (*i.e.* n-forms) with components $y_{\nu_1 \dots \nu_n} = y_{[\nu_1 \dots \nu_n]}$ obeying

$$\nabla_{(\mu} y_{\nu_1) \dots \nu_n} = 0.$$

24. Show that the scalar $K^{\nu_1 \dots \nu_n} u_{\nu_1} \dots u_{\nu_n}$ is conserved along geodesics.

25. Give a trivial example for a rank-2 Killing tensor, always present. What is the corresponding conserved quantity along geodesics?

26. Show that the $n-1$-form with components $\mathrm{i}_{\boldsymbol{u}} \boldsymbol{y}$ is parallelly transported along geodesics.

27. Suppose $\boldsymbol{y}$ be a rank-2 Killing–Yano tensor with components $y_{\mu\nu}$. Show that $K_{\mu\nu} = y_\mu{}^\rho y_{\rho\nu}$ are the components of a rank-2 Killing tensor. Tip: *write explicitly* $\nabla_{(\mu} K_{\nu\rho)} = 0$.

5.2.2 *Kerr geometry* – Christmas homework

A less trivial rank-2 Killing tensor is available in Kerr's rotating black-hole solution. This was discovered in 1968 by Brandon Carter, demonstrating thereby that geodesic motion (and other field equations) was integrable in that background.[4] For that purpose, one introduces two vector fields:

[4] B. Carter, *Hamilton–Jacobi and Schrödinger separable solutions of Einstein's equations*, Commun. Math. Phys. **10** (1968) 280.

$$\begin{aligned} \boldsymbol{\ell} &= \frac{r^2+a^2}{c\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\varphi, \\ \boldsymbol{k} &= \frac{r^2+a^2}{2c\rho^2}\partial_t - \frac{\Delta}{2\rho^2}\partial_r + \frac{a}{2\rho^2}\partial_\varphi. \end{aligned}$$

28. Show that $\boldsymbol{\ell}$ and $\boldsymbol{k}$ are light-like vectors with $\boldsymbol{\ell}\cdot\boldsymbol{k} = -1$.

29. Show that
$$K^{\mu\nu} = \rho^2\left(\ell^\mu k^\nu + \ell^\nu k^\mu\right) + r^2 g^{\mu\nu}$$
are the components of a Killing tensor.

30. What happens in the limit of vanishing a?

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