ICFP Masters program
École Normale Supérieure
M1 *(Fall/Winter 2017/2018)*

Final Exam: Advanced Mathematics for Physicists

This exam is closed book, closed notes. It consists of four pages. You have 3 hours. Carefully justify each step in your reasoning. Good luck!

1 Finite groups and representations

1. **The group of quaternions**

   Let Q be the group generated by the elements $\bar{e}, i, j, k$ which satisfy the relations $\bar{e}^2 = e, i^2 = j^2 = k^2 = ijk = \bar{e}$,
   $$Q = \langle \bar{e}, i, j, k : \bar{e}^2 = e, i^2 = j^2 = k^2 = ijk = \bar{e} \rangle \,.$$
   Q is called the group of quaternions.

   (a) Show that $\rho : Q \to GL(2, \mathbb{C})$ defined by
   $$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \rho(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$
   $$\rho(\bar{e}) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(i^{-1}) = -\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(j^{-1}) = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(k^{-1}) = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$
   is an irreducible representation.

   (b) Show that $N = \langle e, \bar{e} \rangle$ is a normal subgroup of Q, and that $Q/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

   (c) Find 4 inequivalent one dimensional representations of Q (the previous subproblem can help).

   (d) Show that the conjugacy classes of Q are $\{e\}, \{\bar{e}\}, \{i, i^{-1}\}, \{j, j^{-1}\}, \{k, k^{-1}\}$.

   (e) Write down the complete character table of Q.

2. **The kernel of a character**

   For this problem, you can use the following proposition without proof:

   Proposition: *Let G be a finite group and (V, ρ) a finite dimensional representation. If $g \in G$, then there exists a basis of V such that the automorphism $\rho(g) : V \to V$ in this basis is given by a diagonal matrix. If $g^n = e$, i.e. g has order n, then the entries on the diagonal of this matrix are n^{th} roots of unity.*

   Now let ρ be an n dimensional representation of the finite group G, and let χ be the character of ρ.

(a) For $g \in G$, let $\rho(g) = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{C}$, where id indicates the identity automorphism. Show that $|\chi(g)| = \chi(1)$.

(b) Now assume that for $g \in G$, $|\chi(g)| = \chi(1)$. Show that $\rho(g) = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.

(c) Show that $\ker \rho = \{g \in G : \chi(g) = \chi(1)\}$.

The theorem you just proved motivates the following

Definition: *If χ is a character of G, then the kernel of χ, written* $\ker \chi$*, is defined by*

$$\ker \chi = \{g \in G : \chi(g) = \chi(1)\} \; .$$

3. **Normal subgroups and lifted characters**

In the following, G will always denote a finite group. Note that the last subproblem of this problem can be solved without solving the previous subproblems, by simply taking the claims stated there to be true. You may also use the propositions presented in problem 2.

(a) Assume that $N \lhd G$, i.e. that N is a normal subgroup of G, and let $\tilde{\chi}$ be a character of G/N. Define $\chi : G \to \mathbb{C}$ by

$$\chi(g) = \tilde{\chi}(Ng) \quad \forall g \in G\,.$$

Show that χ is a character of G, and that the representations corresponding to χ and to $\tilde{\chi}$ have the same dimension.

The proposition you just proved motivates the following

Definition: *If $N \lhd G$ and $\tilde{\chi}$ is a character of G/N, then the character χ of G which is given by*

$$\chi(g) = \tilde{\chi}(Ng) \quad \forall g \in G$$

is called the lift of $\tilde{\chi}$ to G.

(b) Assume that $N \lhd G$. By associating each character of G/N with its lift to G, show that we obtain a bijective correspondence between the set of characters of G/N and the set of characters χ of G which satisfy $N \leq \ker \chi$ (i.e. N is a subgroup of $\ker \chi$). What is more, show that irreducible characters of G/N correspond to irreducible characters of G which have N in their kernel.

(c) i. Let χ be an irreducible character of G. Show that $\ker \chi \lhd G$.

ii. Let $\chi_1, \ldots, \chi_s$ be irreducible characters of G. Show that $\bigcap_{i=1}^{s} \ker \chi_i \lhd G$.

iii. If $N \lhd G$, show that there exist irreducible characters $\chi_1, \ldots, \chi_s$ of G such that

$$N = \bigcap_{i=1}^{s} \ker \chi_i \lhd G\,.$$

(d) The dihedral group D_{2n} of order $2n$ is the group generated by two elements a, b which satisfy the relations $a^n = 1, b^2 = 1, b^{-1}ab = a^{-1}$,

$$D_{2n} = \langle a, b : a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle\,.$$

The character table of the group D_8 is given in table 1. With its help, determine all normal subgroups of D_8.

	1	a^2	a, a^3	b, a^2b	ab, a^3b
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 1: The character table of the group D_8.

2 Differential manifolds, Lie groups, and Lie algebras

1. **The symplectic group**

   Let $I_n \in GL(n, \mathbb{C})$ be the identity matrix. Let

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} .$$

   We define the complex symplectic group as the space of linear transformations that preserves Ω, i.e.

$$Sp(2n, \mathbb{C}) = \{A \in GL(2n, \mathbb{C}) : A^T \Omega A = \Omega\} .$$

   As a closed subspace of $GL(2n, \mathbb{C})$, the symplectic group is itself a Lie group.

   (a) Identify the Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ of the Lie group $Sp(2n, \mathbb{C})$.

   (b) Prove your claim.

   (c) Give a basis of the complex vector space underlying $\mathfrak{sp}(2n, \mathbb{C})$ in the case $n = 1$, and work out the Lie brackets of its elements.

   (d) Identify this Lie algebra with a Lie algebra we encountered in class. What is the corresponding Lie group? Is it isomorphic to $Sp(2, \mathbb{C})$?

2. **G, $\mathfrak{g}$, and $G \times \mathfrak{g}$**

   (a) Let (U, φ) be a coordinate system of the Lie group G, with $e \in U$. Show that this induces a coordinate system on $\mathfrak{g}$, the space of left-invariant vector fields on G.

   (b) Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be differential manifolds of dimension d_1 and d_2 respectively. Then $M_1 \times M_2$ becomes a differential manifold of dimension $d_1 + d_2$ with differential structure $\mathcal{F}$ defined as the maximal collection containing

$$\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta) : (U_\alpha, \varphi_\alpha) \in \mathcal{F}_1, (V_\beta, \psi_\beta) \in \mathcal{F}_2\} .$$

   Consider the product manifold $M \times N$ with the canonical projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$. Let $(m_0, n_0) \in M \times N$, and define injections $i_{n_0} : M \to M \times N$ and $i_{m_0} : N \to M \times N$ by setting

$$i_{n_0}(m) = (m, n_0) , \quad i_{m_0}(n) = (m_0, n) .$$

   Let $v \in T_{(m_0,n_0)} M \times N$, and let $v_1 = d\pi_1(v) \in T_{m_0} M$, and $v_2 = d\pi_2(v) \in T_{n_0} N$. Let $f : M \times N \to \mathbb{R}$ be C^∞. Prove that

$$v(f) = v_1(f \circ i_{n_0}) + v_2(f \circ i_{m_0}) .$$

(c) Consider the differential manifold $G \times \mathfrak{g}$. Prove that the vector field defined by

$$V(\sigma, X) = (X(\sigma), 0)\,, \quad (\sigma, X) \in G \times \mathfrak{g}$$

is smooth.

3. **Vector bundles**

Definition: *Let M be a differentiable manifold. A (real) smooth vector bundle of rank k over M is a differentiable manifold E together with a smooth surjective continuous map $\pi : E \to M$ satisfying*

- *For each $p \in M$, the set $E_p = \pi^{-1}(p) \subset E$ (called the fiber over p) is endowed with the structure of a k-dimensional real vector space.*
- *For each $p \in M$, there exists a neighborhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a local trivialization of E over U) such that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

(where π_1 denotes projection on the first factor); and such that for each $q \in U$, the restriction of Φ to E_q is a linear isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

(a) Show that the tangent bundle is a smooth vector bundle. What is its rank?

(b) Let $\pi : E \to M$ be a smooth vector bundle, and suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E such that $U \cap V \neq \emptyset$. Show that there exists a smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)\,,$$

where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.