ICFP M2 - Statistical physics 2 – Exam

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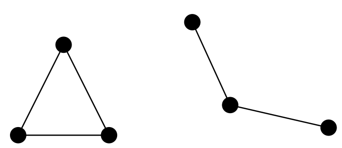
The exam is made of two parts. The first one is a series of short independent exercices to check your knowledge and understanding of the contents of some of the lectures, the second one is a longer problem with partially independent subparts.

No document, calculator nor phone is allowed.

You can write your answers in English or French.

1 Questions on the lectures

1. Consider a random variable X with the probability density $f_X(x) = \begin{cases} C \frac{1}{x^{\frac{5}{4}}} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$.
   (a) Compute the value of the constant C.
   (b) Does X admits a variance? an average value?
   (c) Recall, without a long derivation, the scaling with n of the maximum $M_n = \max(X_1, \dots, X_n)$ of n independent copies of X, when $n \to \infty$.
   (d) Same question for the sum $S_n = X_1 + \cdots + X_n$.
2. What is a self-averaging quantity? Give an example of such a quantity.
3. The Binary Symmetric Channel (BSC) takes as an input a variable $X \in \{0, 1\}$, and outputs $Y \in \{0, 1\}$ with a probability p of flipping the output with respect to the input.
   (a) Give the Shannon entropy (in bits) $S(Y|X)$ of the output conditional on the input.
   (b) Suppose that $X = 0$ with a probability denoted q; describe the marginal law of Y, and give the Shannon entropy $S(Y)$.
   (c) The capacity $C(p)$ of the BSC channel is defined as the maximal mutual information between the output and the input, i.e. $C(p) = \sup_q I(X; Y) = \sup_q [S(Y) - S(Y|X)]$. Compute $C(p)$.
4. Consider the graph of the figure :



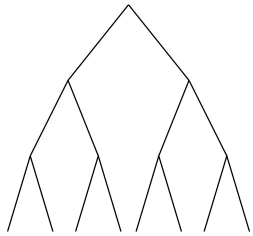
   (a) How many connected components does it contain?
   (b) What is the probability that it becomes connected if one adds one edge, chosen uniformly at random among the absent ones?
5. Let us denote X a random variable that takes the value 1 with probability p, and the value -1 with probability $1 - p$. Consider the sum $S_N = X_1 + \cdots + X_N$ of N independent copies of X.
   (a) What are the possible values of $s = \frac{1}{N} S_N$?
   (b) Give an exact expression at finite N of the probability $\mathbb{P}[S_N = Ns]$.
   (c) Compute the rate of large deviation $\omega(s)$, defined as

$$\omega(s) = \lim_{N \to \infty} \frac{1}{N} \ln \mathbb{P}[S_N = Ns] \ .$$

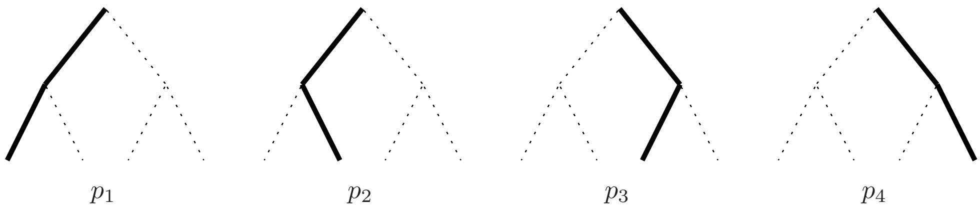
   (d) Draw the shape of $\omega(s)$, indicating in particular the location of its maximum, and its behavior in $s \to \pm 1$. You may consider first the case $p = \frac{1}{2}$, then generalize your answers to an arbitrary value of p.

2 A model for a polymer in a disordered environment

An important class of disordered physical systems is constituted by elastic interfaces in random environments. In this problem we shall study a simple model for a polymer in presence of disorder. The model is defined on a tree in which each vertex has $k \geq 2$ descendents for N generations, the root being the generation 0, the leaves at the N-th generation having no descendent. For instance the figure on the right represents such a tree for $k = 2$ and $N = 3$:



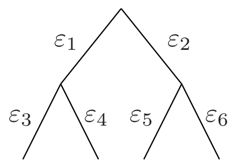
A configuration of the polymer is represented in this model as a self-avoiding path p from the root to one of the leaves, whose length (number of edges it contains) is thus N. On the figure below the four configurations in the case $k = 2$, $N = 2$ are represented as bold lines :



Each edge e of the tree is assigned an energy ε_e, and a configuration p of the polymer has an energy equal to the sum of the energies of the N edges it crosses, to be denoted

$$E(p) = \sum_{e \in p} \varepsilon_e \ .$$

For instance in the special case $k = 2$, $N = 2$ we label the six edges according to the figure on the right, in such a way that the four configurations above have the energies :



$$\begin{aligned} E(p_1) &= \varepsilon_1 + \varepsilon_3 \ , \qquad E(p_2) = \varepsilon_1 + \varepsilon_4 \ , \\ E(p_3) &= \varepsilon_2 + \varepsilon_5 \ , \qquad E(p_4) = \varepsilon_2 + \varepsilon_6 \ . \end{aligned}$$

We assume that the system is at equilibrium with a thermal bath of inverse temperature $\beta = \frac{1}{T}$ and we denote the partition function for a system of size (length of the polymer) N

$$Z_N = \sum_p e^{-\beta E(p)} \ ,$$

where p runs over all possible configurations of the polymer ; the parameter k is kept understood to lighten the notations.

The energies ε_e are random, independent from one edge to another, and identically distributed, Z_N is thus a random variable. We shall denote $\mathbb{E}[\bullet]$ the average over this quenched disorder, and $\rho(\varepsilon)$ the probability density of the energy on one edge.

The thermodynamic limit corresponds to $N \to \infty$ with k held fixed.

2.1 Basic properties

1. How many configurations are there for generic values of k and N ?
2. Describe qualitatively the physical behavior of the model in the limit of infinite and zero temperature.
3. In the case $k = N = 2$ represented on the figures above, compute

$$\mathbb{E}[E(p_1)^2] \ , \qquad \mathbb{E}[E(p_1)E(p_2)] \ , \qquad \text{and} \quad \mathbb{E}[E(p_1)E(p_3)] \ , \tag{1}$$

   assuming $\mathbb{E}[\varepsilon_e] = 0$ and $\mathbb{E}[\varepsilon_e^2] = \sigma^2$.

4. In the general case (k and N arbitrary), discuss briefly the existence and form of correlations between the energies $E(p)$ for the various configurations of the polymer.

2.2 The annealed computation

1. Compute, for a generic value of k and N, the average partition function $\mathbb{E}[Z_N]$, and deduce the value of the annealed free-energy $f_{\rm a}(\beta) = \lim_{N\to\infty} -\frac{1}{N\beta}\ln\mathbb{E}[Z_N]$.

   For simplicity we assume in the rest of this part of the problem that the energies ε_e on the edges are i.i.d. Gaussian random variables with $\mathbb{E}[\varepsilon_e]=0$ and $\mathbb{E}[\varepsilon_e^2]=\sigma^2$.

2. Simplify your expression of $f_{\rm a}(\beta)$.
3. Compute the associated entropy $s_{\rm a}(\beta) = \beta^2 \frac{{\rm d}f_{\rm a}}{{\rm d}\beta}$.
4. Draw the shape of $f_{\rm a}$ as a function of the temperature $T=1/\beta$, specifying the location of its maximum $T_{\rm c}$ and the value of $f_{\rm a}$ at the maximum.
5. Recall (and justify briefly) the inequality between the quenched free-energy $f_{\rm q}(\beta) = \lim_{N\to\infty} -\frac{1}{N\beta}\mathbb{E}[\ln Z_N]$ and $f_{\rm a}(\beta)$.
6. Can $f_{\rm q}$ be equal to $f_{\rm a}$ for $T<T_{\rm c}$? By analogy with the behavior of the Random Energy Model studied during the TDs and lectures, make a conjecture on the value of $f_{\rm q}$ at all temperatures.

2.3 The quenched computation via a wave equation

We come back now to an arbitrary distribution $\rho(\varepsilon)$ for the energies on the edges.

1. Write down an induction relation between Z_{N+1} and i.i.d. copies of the random variables Z_N (that you might denote $Z_N^{(1)}$, $Z_N^{(2)}$, $\dots$) and ε (that you can write $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\dots$)
2. We define $G_N(x) = \mathbb{E}[\exp(-e^{-\beta x}Z_N)]$ (keeping the dependency on β and k understood for the sake of simplicity). Show, as a consequence of your answer to the previous question, the functional induction relation between G_{N+1} and G_N,
$$G_{N+1}(x) = \left(\mathbb{E}[G_N(x+\varepsilon)]\right)^k \ , \qquad (2)$$
   where in the right-hand-side the average is over ε with its distribution $\rho(\varepsilon)$.
3. Show that for any positive real $z>0$,
$$\ln z = \int_{-\infty}^{\infty} {\rm d}t \left(e^{-e^{-t}} - e^{-ze^{-t}} \right) \ , \qquad \text{by} \qquad (3)$$
   - justifying the convergence of the integral.
   - checking that the equality is true for a well-chosen value of z.
   - checking that the derivatives with respect to z of the left and right hand sides coincide for all z.
4. Write explicitly $Z_{N=1}$, and deduce the value that should be assigned to $Z_{N=0}$ to initiate the induction. What is the corresponding value of $G_0(x)$?
5. Conclude that
$$\mathbb{E}[\ln Z_N] = \beta \int_{-\infty}^{\infty} {\rm d}x \left(G_0(x) - G_N(x) \right) \ . \qquad (4)$$

2.4 The asymptotic solution of the wave equation

In order to complete the computation of the quenched free-energy density from equation (4) it remains to understand the behavior of the function $G_N(x)$ in the large N limit. This behavior is determined by the initial condition $G_0(x)$, and by the induction equation (2). It is useful to think of N as a (discrete) time variable and x as a (continuous) space variable, in such a way that (2) can be seen as a wave equation encoding the time evolution of a space-dependent profile.

1. Draw the shape of $G_0(x)$ as a function of x, paying special attention to its behavior in $x\to-\infty$ and $x\to+\infty$, as well as its monotonicity properties.
2. Argue that $G_N(x) = \mathbb{E}[\exp(-e^{-\beta x}Z_N)]$ has qualitatively the same shape.
3. Suppose that in the right hand side of (2) $G_N(x)=\alpha_N$ for all x, i.e. that at time N the profile of the wave is homogeneous in space, with $\alpha_N\in[0,1]$. Show that $G_{N+1}(x)=\alpha_{N+1}$ for all x, and study the mapping $\alpha_N \to \alpha_{N+1} = r(\alpha_N)$, in particular its fixed points and their stabilities.
4. Suppose that in the right hand side of (2) $G_N(x) = H(x-x_0)$, with H the Heaviside function, i.e. that at time N the profile of the wave is an abrupt step. Draw qualitatively the shape of $G_{N+1}(x)$ (you can think for instance that ε has a Gaussian distribution).

5. The previous questions lead naturally to the study of solutions of (2) that takes the form of travelling waves, $G_N(x) = g(x - Nv)$, where $g(x)$ is a scaling function, independent of N, that describes the shape of the travelling front connecting the stable and unstable fixed points of the mapping $r(\alpha)$ studied above, and v is the velocity of the front. Write down the equation that g must satisfy in order for $G_N(x) = g(x - Nv)$ to be a solution of (2).
6. Argue qualitatively that the velocity v should be positive.
7. The equation on g admits a priori one solution for all positive v (up to an irrelevant shift of the origin of the x axis), that we shall denote g_v. Writing $g_v(x) = 1 - h_v(x)$, and assuming that $h_v(x) \to 0$ as $x \to +\infty$, show that in this limit h_v obeys the following linearized equation

$$h_v(x) = k \int_{-\infty}^{\infty} \mathrm{d}\varepsilon \, \rho(\varepsilon) \, h_v(x + v + \varepsilon) + O(h_v^2) \ . \tag{5}$$

8. Suppose that $h_v(x)$ has an exponential tail behavior, $h_v(x) \sim c \, e^{-\lambda x}$ when $x \to +\infty$, with $\lambda > 0$ and c an arbitrary constant, and find a relation $v(\lambda)$ between the tail behavior and the velocity of the front. Simplify your answer assuming that ε is Gaussian with variance σ^2, and draw the shape of $v(\lambda)$ in this case.
9. We assume now that the initial condition $G_0(x)$ evolves under the wave equation (2) to reach at large N the traveling wave form $g_{\widehat{v}(\beta)}(x - N\widehat{v}(\beta))$, with a velocity $\widehat{v}(\beta)$ selected by the initial condition. Using the identity (4), relate $f_{\mathrm{q}}(\beta)$ and $\widehat{v}(\beta)$.
10. Show that the natural identification of the tail behavior of the travelling wave with the one of the initial condition, namely $\widehat{v}(\beta) = v(\beta)$, leads to $f_{\mathrm{q}}(\beta) = f_{\mathrm{a}}(\beta)$.
11. A more detailed analysis shows that the velocity of the front is fixed by the initial condition according to $\widehat{v}(\beta) = \min_{\lambda \in]0,\beta]} v(\lambda)$. Interpret qualitatively this relation from the physical properties of the wave equation established previously. Deduce then the value of f_{q} at all temperatures, and compare with your answer to question 2.2.6.

2.5 The replica computation

We will see now how this result for the quenched free-energy can be recovered with a replica computation. For simplicity we consider that the energies on the edges ε_e are Gaussian of zero mean and variance σ^2.

1. Recall the replica trick that allows to compute $\mathbb{E}[\ln Z_N]$ from a well-chosen limit of $\mathbb{E}[Z_N^n]$.
2. Consider p_a and p_b, two configurations of the polymer of size N, and let us define the overlap $q(p_a, p_b)$ as the number of common edges crossed by the two configurations, divided by N. Express $\mathbb{E}[E(p_a)E(p_b)]$ in terms of $q(p_a, p_b)$.
3. Show that for an integer value of n one can write

$$\mathbb{E}[Z_N^n] = \int \prod_{1 \le a < b \le n} \mathrm{d}q_{a,b} \, e^{Ns(Q)} \exp\left[N \frac{1}{2} \beta^2 \sigma^2 \sum_{a,b=1}^{n} q_{a,b} \right] \ , \tag{6}$$

where Q denotes the $n \times n$ matrix containing as matrix elements the overlaps $q_{a,b}$; you will give a formal expression for the entropic term $e^{Ns(Q)}$, and specify the value of the diagonal terms $q_{a,a}$.
4. In order to complete the replica computation one needs to make an ansatz on the form of Q that brings the dominant contribution in the thermodynamic limit. We will take the "one step of Replica Symmetry Breaking" one (1RSB), in which the n replicas are divided into n/m groups of m replicas and with $q_{a,b} = 1$ if the replicas a and b are in the same group, 0 otherwise. Evaluate, at the leading exponential order in N, the energetic and entropic terms in equation (6).
5. Evaluate $f_{\mathrm{1RSB}}(\beta; m)$ by taking the thermodynamic limit within the 1RSB ansatz ; show that $f_{\mathrm{1RSB}}(\beta; m) = f_{\mathrm{a}}(\beta m)$. Conclude that $f_{\mathrm{q}}(\beta) = \sup_{m \in [0,1]} f_{\mathrm{1RSB}}(\beta; m)$.

To learn more about this problem you can consult the paper B. Derrida, H. Spohn, *Polymers on disordered trees, spin glasses, and traveling waves*, J. Stat. Phys **51**, 817 (1988).