ICFP Masters program
École Normale Supérieure

M2 *(Fall/Winter 2017)*

Final Exam: Quantum Field Theory

This exam is closed book, closed notes. It consists of six pages. Useful formulae are collected in appendix A and the Feynman rules of QED are summarized in appendix B. It is essential that you exhibit your calculations with all required intermediate steps. You have 3 hours. Good luck!

1 The g-factor predicted by the Dirac equation

From non-relativistic quantum mechanics, we know that a particle of spin $\vec{S}$ couples to a magnetic field $\vec{B}$ as

$$H = \frac{\vec{p}\cdot\vec{p}}{2m} + \frac{e}{2m} g \vec{B}\cdot\vec{S}\,. \qquad (1)$$

The coupling g determines the magnetic dipole moment of the particle. The Dirac equation predicts a value for g, the so-called g-factor of the electron, which you will derive in this problem.

1. Starting from the Dirac equation for a spinor ψ of charge $-e$, derive

$$(\not{D}^2 + m^2)\psi = 0\,. \qquad (2)$$

2. Derive

$$\not{D}^2 = D_\mu D^\mu + \frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu}\,, \quad \text{where} \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,. \qquad (3)$$

The term proportional to $\sigma^{\mu\nu}$ shows that spinors couple differently to the electromagnetic field than scalars do. You will next compute the value that the Dirac equation predicts for the g-factor of the electron.

3. Express $F_{\mu\nu}\sigma^{\mu\nu}$ in terms of the three vectors $\vec{\sigma}$, $\vec{E}$, and $\vec{B}$. Use the Weyl representation of the γ matrices.

4. As a warm-up for the final step in this problem, derive the standard expression for non-relativistic kinetic energy from the mass-shell relation $p^2 = m^2$.

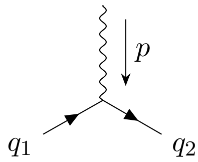
5. Now introduce the variables $p_A^\mu = p^\mu - eA^\mu$, and identify $H = p_A^0$. By invoking (3) and passing to momentum space, obtain the value of g for an electron as implied by this analysis. Make sure to identify the operator $\vec{S}$ correctly in terms of the Pauli matrices $\vec{\sigma}$!

2 The g factor from the electron vertex

Note: for this problem, we will not need renormalized perturbation theory. We will thus omit the subscript $_R$ from both m_R and e_R.

Consider the electron vertex at tree level

$$i\mathcal{M}_0^\mu =$$



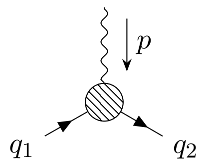
$$= -ie\bar{u}(q_2)\gamma^\mu u(q_1)\,. \tag{4}$$

Note that the photon leg is amputated, and that the electron legs are on-shell.

1. Show that
$$\bar{u}(q_2)\gamma^\mu u(q_1) = \bar{u}(q_2)\left(\frac{q_2^\mu + q_1^\mu}{2m} + i\sigma^{\mu\nu}\frac{(q_2)_\nu - (q_1)_\nu}{2m}\right)u(q_1)\,, \tag{5}$$
where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. This is called the Gordon identity.

Next consider the loop corrected electron vertex

$$i\mathcal{M}^\mu =$$



$$= -ie\,\bar{u}(q_2)\Gamma^\mu u(q_1)\,, \tag{6}$$

where $i\mathcal{M}^\mu$ is to equal the sum over all Feynman diagrams with the indicated external legs, with the free photon propagator amputated.

2. Without assuming anything about the polarization tensors $u(q_1)$ and $\bar{u}(q_2)$, and without imposing momentum conservation, what is the most general form that Γ^μ can take, based solely on Lorentz invariance?

3. Now assume that the electron is on-shell, and that momentum conservation is imposed at the vertex. By imposing the Ward identity, show that the most general form of $\mathcal{M}^\mu$ is
$$i\mathcal{M}^\mu = (-ie)\bar{u}(q_2)\left[F_1\left(p^2\right)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m}p_\nu F_2\left(p^2\right)\right]u(q_1)\,. \tag{7}$$
Carefully justify each step!

4. F_1 and F_2 are the form factors of the electron vertex. Determine their values at tree level.

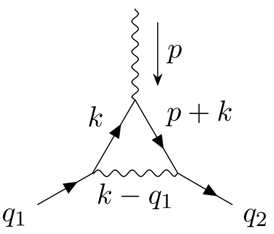
The g-factor is corrected beyond tree level as

$$g = 2 + 2F_2(0)\,. \tag{8}$$

3 Loop correction to the g factor

You will now compute the loop correction

$$i\mathcal{M}_2^\mu = \tag{9}$$



to the g factor.

1. Read off the expression for the Feynman diagram (9) in Feynman gauge (i.e. at $\xi = 1$), with the same prescription as in (4) for the external legs: the electrons are on-shell, $p = q_2 - q_1$, and the external photon propagator is amputated.

2. Use Feynman parametrization to put the denominator of your expression in the form $(\tilde{k} - \Delta + i\epsilon)^3$, with the only momentum dependence of Δ being on p. Give explicitly the expression for $\tilde{k}$ and for Δ.

3. Your expression should now be in the form

$$i\mathcal{M}_2^\mu = -2e^3 \int_0^1 dx\, dy\, dz\, \delta(x + y + z - 1) \int \frac{d^4k}{(2\pi)^4} \frac{N^\mu}{(\tilde{k} - \Delta + i\epsilon)^3}\,. \tag{10}$$

   Show that the contribution N^μ evaluates to

$$N^\mu = 2\bar{u}(q_2)\left[-\not{k}\gamma^\mu\not{p} - \not{k}\gamma^\mu\not{k} + 2m(2k^\mu + p^\mu) - m^2\gamma^\mu\right] u(q_1)\,. \tag{11}$$

   This computation involves roughly one page of algebra. Don't get bogged down here! If you are having trouble reproducing (11), you can return to this at the end.

4. Upon substituting $\tilde{k} \mapsto k$ and further algebra, this expression can be replaced by

$$\begin{aligned} -\frac{1}{2}N^\mu \;=\; & \left[-\frac{1}{2}k^2 + (1-x)(1-y)p^2 + (1 - 4z + z^2)m^2\right] \bar{u}(q_2)\gamma^\mu u(q_1) \\ & + imz(1-z)\, p_\nu \bar{u}(q_2)\sigma^{\mu\nu}u(q_1) + m(z-2)(x-y)p^\mu\, \bar{u}(q_2)u(q_1)\,. \end{aligned} \tag{12}$$

   You are NOT asked to derive this result.

   To arrive at (12), one must use that under the d^4k integral, the substitution

$$k^\mu k^\nu \mapsto \frac{1}{4}g^{\mu\nu}k^2 \tag{13}$$

   is possible. Justify this substitution.

5. Show that $\mathcal{M}_2^\mu$ satisfies the Ward identity, i.e. $p_\mu \mathcal{M}_2^\mu = 0$.

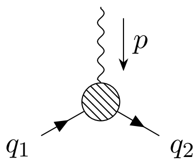
6. Comparing to (7), extract the expression for $F_2(p^2)$. Determine the superficial degree of divergence of the d^4k integral. Then evaluate the d^4k integral explicitly. Does the prediction from studying the superficial degree of divergence hold true?

7. Using (8), determine the order α correction to g (recall that $\alpha = \frac{e^2}{4\pi}$).

4 Vertex renormalization

You will now compute the renormalization of the electron vertex in renormalized perturbation theory,

$$-ie_R\bar{u}(q_2)\Gamma^\mu(p)u(q_1) = \quad (14)$$



Take the RHS of (14) to equal the sum of all Feynman diagrams in renormalized perturbation theory with the given external legs such that

- the free photon propagator is amputated,
- it is not possible to cut the diagram into two disconnected pieces, one of which is a contribution to the electron 2-point function, upon cutting a single line.

1. Draw all diagrams that contribute to Γ^μ at order α.
2. $\Gamma^\mu(p)$ takes the form

$$\Gamma^\mu(p) = F_1(p^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m_R}p_\nu F_2(p^2)\,. \tag{15}$$

Find the expression for $F_1(p^2)$ in Feynman gauge (i.e. at $\xi = 1$) as an integral over the internal momentum and Feynman parameters, making maximal use of your results from problem 3.

3. Including the tree level results, $F_1(p^2)$ takes the form

$$F_1(p^2) = 1 + \delta_1 + f(p^2) + \mathcal{O}(e_R^4)\,. \tag{16}$$

To evaluate this result in dimensional regularization, we need to revisit the computation of $i\mathcal{M}_2^\mu$, and perform the manipulations involving the γ matrices in d dimensions. The result (which you do not need to derive) is

$$f(p^2) = -2ie_R^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int dx\,dy\,dz\,\delta(x+y+z-1) \tag{17}$$

$$\frac{\frac{(d-2)^2}{d}k^2 - \left[(d-2)xy + 2z\right]p^2 + \left[2 + 2z^2 - d(1-z)^2\right]m_R^2}{(k^2 - \Delta + i\epsilon)^3}\,, \tag{18}$$

with

$$\Delta = (1-z)^2m_R^2 - xyp^2 + zm_\gamma^2\,. \tag{19}$$

The infrared divergence of the diagram has been regulated by introducing a photon mass m_γ. Evaluate $f(p^2)$ via dimensional regularization. You can simplify your calculation by performing an analysis of superficial degrees of divergence to identify the UV divergent contributions to the integral. You are only required to perform those integrals over Feynman parameters which are straightforward.

4. Renormalize the electron vertex by imposing

$$\Gamma^\mu(0) = \gamma^\mu\,. \tag{20}$$

What is the value of δ_1 in this renormalization scheme to order α? You can express your result in terms of an integral over one Feynman parameter.

5 The vertex correction in terms of Green's functions

1. Describe succinctly the Feynman diagrams that contribute to the time-ordered product

$$\int d^4x\, d^4y\, d^4z\, e^{iq_1x} e^{-iq_2x} e^{ipz} \langle\Omega|T\{\psi(x)\bar{\psi}(y)A^\mu(z)\}|\Omega\rangle\,. \tag{21}$$

2. Explain, again at the level of Feynman diagrams, how the 3-point function (21) differs from the 3-point function

$$\int d^4x\, d^4y\, d^4z\, e^{iq_1x} e^{-iq_2x} e^{ipz} \langle\Omega|T\{\psi(x)\bar{\psi}(y)J^\mu(z)\}|\Omega\rangle\,, \tag{22}$$

   where $J^\mu(z) = \bar{\psi}(z)\gamma^\mu\psi(z)$ is the electromagnetic current.

3. Relate (22) to the electromagnetic vertex $\Gamma^\mu(p)$ which is defined as the sum of all Feynman diagrams with one ingoing electron leg of momentum q_1, one outgoing electron leg of momentum q_2, one photon leg of momentum p, such that

   - the free photon propagator is amputated,
   - it is not possible to cut the diagram into two disconnected pieces, one of which is a contribution to the electron 2-point function, upon cutting a single line.

A Useful formulae

The fine structure constant

$$\alpha = \frac{e^2}{4\pi}\,. \tag{23}$$

Dirac equation for a charge $Q = -1$ fermion

$$(i\not{D} - m)\psi = (i\not{\partial} - e\not{A} - m)\psi = 0\,. \tag{24}$$

Electron and positron polarization tensor

$$(\not{p} - m)u(p) = 0\,, \quad (\not{p} + m)v(p) = 0\,. \tag{25}$$

Weyl representation of γ matrices

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \tag{26}$$

where

$$\sigma^\mu = (1, \vec{\sigma})\,, \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})\,, \tag{27}$$

and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices, which satisfy

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k\,. \tag{28}$$

Feynman parameters

$$\frac{1}{AB} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{[xA+yB]^2} \tag{29}$$

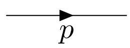
$$\frac{1}{ABC} = \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{[xA+yB+zC]^3} \tag{30}$$

Dimensional regularization

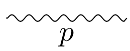
$$\int \frac{d^dk}{(2\pi)^d}\frac{k^{2a}}{(k^2-\Delta)^b} = i(-1)^{a-b}\frac{1}{(4\pi)^{d/2}}\frac{1}{\Delta^{b-a-\frac{d}{2}}}\frac{\Gamma(a+\frac{d}{2})\Gamma(b-a-\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})} \tag{31}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{32}$$

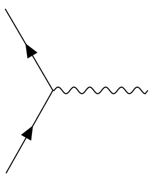
B Feynman rules for renormalized perturbation theory of QED



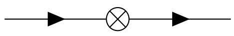
$$= \frac{i(\not{p}+m_R)}{p^2-m_R^2+i\epsilon}$$



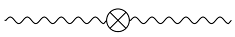
$$= \frac{-i}{p^2+i\epsilon}\left[g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$



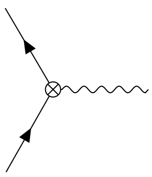
$$= -ie_R\gamma^\mu$$



$$= i(\not{p}\delta_2-(\delta_m+\delta_2)m_R)$$



$$= -i\delta_3 p^2 g^{\mu\nu} \qquad \text{(in Feynman gauge)}$$



$$= -ie_R\delta_1\gamma^\mu$$