

ICFP M2 - STATISTICAL PHYSICS 2 – TD n° 1

Extreme values distributions

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We consider a sequence X_1, X_2, \dots of independent identically distributed (i.i.d.) random variables, with a common distribution equal to the one of X . One is often interested in the behavior as $n \rightarrow \infty$ of :

- the maximum $M_n = \max(X_1, \dots, X_n)$, and
- the sum $S_n = X_1 + \dots + X_n$

of n such random variables. In both cases there only exists a few “universality classes” for the possible limit behaviors, independently on most of the “microscopic” details of the law of X . The lecture has covered the case of the sum (with stable random variables generalizing the central limit theorem), we consider in this problem the behavior of the maximum M_n .

We will denote R_X the right-edge of the support of X , defined as $R_X = \inf\{x : F_X(x) = 1\}$, which can be equal to $+\infty$.

One says that a random variable Z is degenerate if it takes a constant value c almost surely, then $F_Z(z) = 0$ for $z < c$ and $F_Z(z) = 1$ for $z \geq c$.

1 Extreme value distributions

1. Explain why $F_{M_n}(x) = (F_X(x))^n$.
2. Draw the shape of $F_{M_n}(x)$ as n grows, distinguishing the cases :
 - (a) $R_X = +\infty$,
 - (b) $R_X < \infty$ and F_X is continuous in R_X ,
 - (c) $R_X < \infty$ and F_X is discontinuous in R_X .
3. Show that $M_n \xrightarrow{d} R_X$ as $n \rightarrow \infty$.
4. In order to have a more precise description of the behavior of M_n one needs to shift and rescale it, we thus define $\widehat{M}_n = \frac{M_n - a_n}{b_n}$, where a_n and $b_n > 0$ are two sequences chosen appropriately so that \widehat{M}_n converges in distribution to a non degenerate random variable (if possible) when $n \rightarrow \infty$.
 - (a) Express the distribution function $F_{\widehat{M}_n}$ of the rescaled maximum in terms of F_X .
 - (b) Show that if a_n and b_n are chosen in such a way that $F_X(a_n + \widehat{x} b_n) = 1 - \frac{\gamma(\widehat{x})}{n} + o(\frac{1}{n})$, where \widehat{x} and $\gamma(\widehat{x})$ are finite when $n \rightarrow \infty$, then \widehat{M}_n has indeed a non-trivial limit, and express its distribution function.
5. Consider first that X has an exponential distribution of parameter 1, i.e. $F_X(x) = 1 - e^{-x}$ for $x \geq 0$, $F_X(x) = 0$ for $x \leq 0$. Find the values of a_n and b_n that ensures the convergence of \widehat{M}_n towards a random variable whose distribution function is

$$G_G(x) = e^{-e^{-x}} . \quad (1)$$

Draw the shape of this distribution function, and of the associated density.

6. Same questions when X takes values between 0 and 1, with a distribution function on this interval given by $F_X(x) = 1 - (1 - x)^\alpha$, with $\alpha > 0$ a parameter ; the limit distribution function for \widehat{M}_n is now

$$G_W(x) = \begin{cases} e^{-(-x)^\alpha} & \text{for } x \leq 0 \\ 1 & \text{for } x \geq 0 \end{cases} . \quad (2)$$

7. Same questions when X is a Pareto random variable, taking values on $[1, \infty[$, with a distribution function $F_X(x) = 1 - x^{-\alpha}$, with again $\alpha > 0$, the limit distribution being

$$G_F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-x^{-\alpha}} & \text{for } x > 0 \end{cases} . \quad (3)$$

8. Consider finally the case where X admits the density $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, i.e. X is a standard Gaussian random variable. Check that with the choice

$$a_n = \sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi}{2\sqrt{2 \ln n}} , \quad b_n = \frac{1}{\sqrt{2 \ln n}} , \quad (4)$$

the normalized maximum \widehat{M}_n converges to the distribution given in (1). *Indication* : the distribution function of Gaussian random variables admits the asymptotic expansion

$$F_X(x) = 1 - \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + O\left(\frac{e^{-\frac{x^2}{2}}}{x^3}\right) \quad \text{as } x \rightarrow \infty . \quad (5)$$

It turns out that these three distributions, named respectively after Gumbel, Weibull and Fréchet, are the only possible non degenerate limits for \widehat{M}_n , modulo a shift and a rescaling, whatever the original distribution of X . Note that for some X there is no choice of a_n and b_n that allows for a non degenerate limit (this is the case of the example of question 2c, but also of Poisson random variables).

The type of the distribution function for the limit of \widehat{M}_n depends on the behavior of F_X at the edge of its support R_X :

- If R_X is finite and F_X reaches 1 with a power-law behavior (possibly multiplied by a slowly varying function) then X is in the domain of attraction of the Weibull law.
- The Fréchet law is instead observed if R_X is infinite and F_X reaches 1 with an inverse power-law behavior (possibly multiplied by a slowly varying function).
- If F_X reaches 1 faster than any power-law at R_X (be this edge finite or not) then the Gumbel distribution will apply.

2 Proof of the three types theorem

We sketch here the proof of the statement made above : the only possible limit distributions for \widehat{M}_n are the Gumbel, Weibull and Fréchet laws.

1. We will admit the following result. If $(X_n)_{n \geq 1}$ is a sequence of random variables, with two rescalings allowing for a non-trivial limit,

$$\frac{X_n - a_n}{b_n} \xrightarrow{d} Y, \quad \frac{X_n - c_n}{d_n} \xrightarrow{d} Z, \quad (6)$$

with Y and Z non-degenerate, then

$$\frac{b_n}{d_n} \rightarrow B \in]0, \infty[, \quad \frac{a_n - c_n}{d_n} \rightarrow A \text{ with } A \text{ finite and } Z \stackrel{d}{=} A + BY. \quad (7)$$

Translate this statement in terms of distribution functions.

2. Suppose that a_n and b_n are chosen such that \widehat{M}_n converges towards a non degenerate random variable Y with a distribution function G . Show that for any positive integer m there exists $A(m)$ and $B(m) > 0$ such that

$$G^m(x) = G(A(m) + B(m)x) \quad \text{i.e.} \quad \max(Y_1, \dots, Y_m) \stackrel{d}{=} \frac{Y - A(m)}{B(m)} \quad (8)$$

where Y_1, \dots, Y_m are independent random variables with the same distributions as Y ; Y is thus said to be stable under the max operation. Generalize this result to

$$G^s(x) = G(A(s) + B(s)x) \quad (9)$$

for all reals $s > 0$.

3. Show that the functions $A(s)$ and $B(s)$ are solutions of

$$\begin{cases} B(st) = B(s)B(t), \\ A(st) = A(s) + B(s)A(t) = A(t) + B(t)A(s), \end{cases} \quad (10)$$

for all $s, t > 0$.

4. In the following we assume for simplicity that A and B are differentiable. Show that $B(s) = s^\theta$, where θ is an arbitrary real parameter.
5. Show that if $\theta = 0$ the distribution function G is of the Gumbel form (modulo an affine change of variables).
6. Assuming now $\theta > 0$, prove that G is of the Weibull type with $\alpha = 1/\theta$. Similarly the Fréchet distribution is obtained when $\theta < 0$.