Statistical Field Theory and Applications: An Introduction for (and by) Amateurs.

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The $O(N)$ vector model with $N \to \infty$ in $D = 3$

We study a model of N-component spins $\vec{\Phi}$ governed by the Hamiltonian

$$H[\vec{\Phi}] = \frac{1}{2} \int d\vec{x} \, \left\{ \sum_{\alpha=1}^{N} \left(\frac{\partial \Phi_\alpha}{\partial \vec{x}} \right)^2 + r_0 \sum_{\alpha=1}^{N} (\Phi_\alpha)^2 + \frac{u}{12N} \left(\sum_{\alpha=1}^{N} (\Phi_\alpha)^2 \right)^2 \right\} .$$

The dimension of the embedding space is fixed as $D = 3$. In this exercise we shall compute the critical exponents for large N, and more precisely, first for infinite N and then the corrections to order $1/N$.

(i) Write the propagator and interaction vertex in Fourier space. Which diagrams contribute to the correlation function

$$G(k) = \langle \tilde{\Phi}_\alpha(-\vec{k}) \tilde{\Phi}_\alpha(\vec{k}) \rangle \ ?$$

Show that in the limit $N \to \infty$, the only surviving diagrams are of the "cactus" type (see Figure 1), where the solid line represents the bare (free) propagator.

(ii) Deduce the implicit equation

$$\frac{1}{G_\Phi^\infty(k)} = k^2 + r_0 + \frac{u}{6} \int_{q<\Lambda} \frac{d^3 q}{(2\pi)^3} \, G_\Phi^\infty(q)$$

satisfied by the dressed propagator $G_\Phi^\infty(k)$, where Λ denotes an ultraviolet cut-off. (*Hint*: Formally sum up subclasses of diagrams in a geometric series.)

(iii) Interpret the identity (). Use it to compute first the critical temperature and next the exponents η^∞ and ν^∞ in the limit $\Lambda \to \infty$. (*Hint*: The critical temperature is such that the renormalised mass vanishes.)

(iv) We wish to recover this result by the saddle point method. By introducing a new scalar field $\sigma(\vec{x})$, show that one may rewrite the partition function of the above model in the form

$$Z = \int D\vec{\Phi}(\vec{x}) \, D\sigma(\vec{x}) \, \exp\left(-H[\vec{\Phi}, \sigma] \right) ,$$

where

$$H[\vec{\Phi}, \sigma] = \frac{1}{2} \int d\vec{x} \, \left\{ \sum_{\alpha=1}^{N} \left(\frac{\partial \Phi_\alpha}{\partial \vec{x}} \right)^2 + \left(r_0 + i\sqrt{\frac{u}{3N}} \, \sigma \right) \sum_{\alpha=1}^{N} (\Phi_\alpha)^2 + \sigma^2 \right\} .$$

(v) Integrate over the fields $\vec{\Phi}$. Which effective action for the field σ does one arrive at?

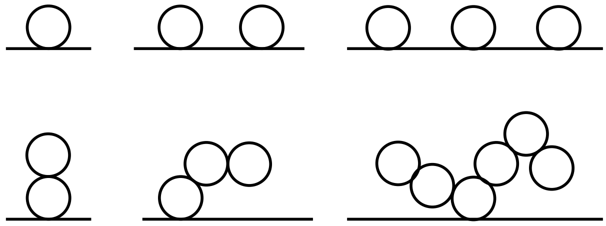

Figure 1: Some examples of “cactus” diagrams.

(vi) In the limit $N \to \infty$ one can obtain Z by computing the saddle point of the preceeding action. One supposes that this saddle point is uniform (that is, independent of x). Show that we hence recover the implicit equation ().

(vii) Verify that the classical solution obtained is indeed a local minimum of the action.

(viii) We now use the action found in (v) as the starting point for computing the corrections of order $1/N$ to the critical exponents. Write down the bare propagators of the fields σ and $\vec{\Phi}$, as well as the interaction vertex.

In the following questions we focus on obtaining the propagator of σ in the $N \to \infty$ limit. This step is necessary in order to go to the next order in the computation of the Φ-propagator.

(ix) Show that in the limit $N \to \infty$ the dressed propagator $G_\sigma^\infty(k)$ of σ satisfies the implicit equation illustrated in Figure **??**. (Here the solid line represents the propagator of Φ and the dashed line that of σ. The presence of a point on a propagator means that it is dressed.) Could this equation have been anticipated from the answer to question (iv)?

(x) Compute the integral

$$I = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(\mathbf{1} - \mathbf{q})^2} \cdot \frac{1}{\mathbf{q}^2} = \frac{1}{8},$$

where $\mathbf{1}$ represents a unit vector. (*Hint*: use polar coordinates.)

(xi) Deduce that at the critical temperature we have

$$G_\sigma^\infty(k) \simeq \frac{48}{u}\, k \qquad (k \to 0)\,.$$

The final stage of the exercise is now to obtain the propagator of Φ to order $1/N$.

(xii) Show that the self-energy $\Sigma_\Phi^N(k)$ of the dressed propagator $G_\Phi^N(k)$ of Φ is given by Figure 2, where the first term represents the finite contribution for $N \to \infty$ that has been studied in question (iv), while the second term is the sought-for contribution at order $1/N$.

(xiii) Infer that

$$\Sigma_\Phi^N(k) - \Sigma_\Phi^N(0) \simeq \frac{8}{3\pi^2 N}\, k^2 \ln k \qquad (k \to 0)\,.$$

(xiv) Deduce the value of the exponent η to order $1/N$.

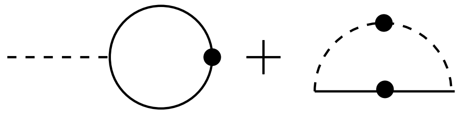

Figure 2: Development of the self-energy.

Correction :

(i) We work in three spatial dimensions. Let us set

$$\tilde{\phi}_\alpha(\mathbf{k}) = \int \mathrm{d}^3x \phi_\alpha(\mathbf{x})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}},$$

and transform Fourier transform the Hamiltonian. Doing the integral d^3x produces a momentum conserving delta function:

$$\begin{aligned} H[\vec{\tilde{\phi}}] &= \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{\alpha=1}^{N}\tilde{\phi}_\alpha(\mathbf{k})(k^2+r_0)\tilde{\phi}_\alpha(-\mathbf{k}) \\ &+ \frac{u}{24N}\sum_{\alpha,\beta=1}^{N}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\frac{\mathrm{d}^3k_3}{(2\pi)^3}\tilde{\phi}_\alpha(\mathbf{k}_1)\tilde{\phi}_\alpha(\mathbf{k}_2)\tilde{\phi}_\beta(\mathbf{k}_3)\tilde{\phi}_\beta(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \end{aligned} \tag{1}$$

From this expression we immediately read off the bare propagator

$$\left\langle \tilde{\phi}_\alpha(\mathbf{k})\tilde{\phi}_\beta(\mathbf{k}')\right\rangle = \frac{1}{k^2+r_0}\,\delta(\mathbf{k}+\mathbf{k}')\delta_{\alpha,\beta}$$

and the interaction vertex

$$-\frac{u}{24N}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}.$$

By convention, the four legs carry vector indices α_i and wave vectors $\mathbf{k}_i$ with $i = 1,2,3,4$. By convention, the latter are counted positively (resp. negatively) at a vertex when their orientation is incoming (resp. outgoing). The summations $\sum_{\alpha,\beta=1}^{N}$ in (1) teach us that $\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ equals 1 if the indices are equal in pairs (when $\alpha\neq\beta$) and also 1 if all four indices are equals (when $\alpha=\beta$ in the sums). Otherwise $\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is zero. This can be written explicitly (albeit in a cumbersome manner):

$$\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4} + \delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} + \delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3} - 2\delta_{\alpha_1,\alpha_2}\delta_{\alpha_1,\alpha_3}\delta_{\alpha_1,\alpha_4}.$$

Let us know examine a diagram that contributes to $G(\mathbf{k})$; see Figure 3. To leading order in N we can suppose that $\alpha\neq\beta$. We then need to contract

$$\langle\phi_\mathrm{l}\cdot\phi_\alpha\phi_\alpha\phi_\beta\phi_\beta\cdot\phi_\mathrm{r}\rangle,$$

where ϕ_g and ϕ_d are the external legs on the left and the right of the diagram. One should first choose whether to contract the external legs with the two α or with the two β indices (yielding a factor of 2) and next which one of the indices connects to ϕ_l (giving another factor of 2). The sum over β produces a factor of N. The diagram therefore equals

$$-\frac{u}{24N}\times 4N\times\int \mathrm{d}^3q\frac{1}{q^2+r_0}\left(\frac{1}{k^2+r_0}\right)^2 + \mathcal{O}(1/N)$$

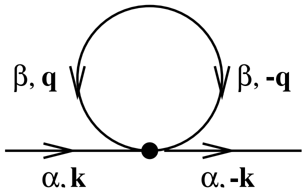

Figure 3: A diagram contributing to $G(\mathbf{k})$.

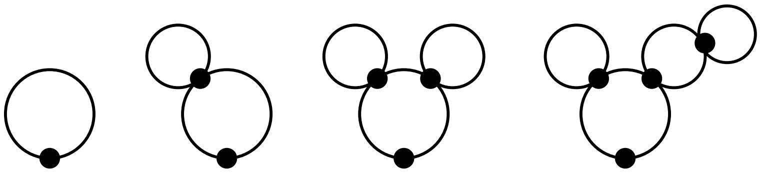

Figure 4: Diagrams contributing to the selfenergy in the limit $N \to \infty$.

which is finite in the $N \to \infty$ limit. More generally, we see that to stay of order 1, one is obliged to compensate all the factors of $1/N$ coming from the vertices by the same number of factors of N comming from the summation over the loops. The contributing diagrams therefore have an equal number of vertices and *independent* loops. Put otherwise, each diagram which is not a tadpole of a tadpole vanishes in the limit, and all finite diagrams have the "cactus" from shown in Figure 1. *Example*: Consider the 2nd diagram in Figure 1, but modified so that the two loops touch one another in a point. There are then *three* independent loops for the wave vectors, but the conservation of the $\mathrm{O}(N)$ indices at the vertices leaves only two independent loops for the vector indices. This modified diagram is thus of order $1/N$.

(ii) Let us now isolate the one-particle irreducible (1PI) contributions to $G(\mathbf{k})$. The corresponding diagrams are those contributing to the selfenergy $\Sigma(\mathbf{k})$ in the limit $N \to \infty$; they are shown in Figure 4. Denoting by $G_0 = 1/(k^2 + r_0)$ the bare propagator, we have the identity

$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \cdots = G_0 \left(\frac{1}{1 - \Sigma G_0} \right),$$

or

$$\frac{1}{G} = \frac{1}{G_0} - \Sigma.$$

Notice that since the entire "current" flows between the external legs, we have necessarily $\Sigma(\mathbf{k}) = \Sigma(\mathbf{0})$.

To produce the diagrams contributing to Σ, it suffices to take those of G, bend down the two external legs and connect them in a point. Since this operation adds one more vertex, the contribution must be multiplied by $-\frac{u}{24N} \times 4N$ (where the factor $4N$ occurs for the same combinatorial reason as explained above). Therefore,

$$\Sigma(\mathbf{0}) = -\frac{u}{6} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G(\mathbf{q}),$$

and we have the desired identity:

$$\frac{1}{G(\mathbf{k})} = k^2 + r_0 + \frac{u}{6} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G(\mathbf{q}).$$

(iii) One can use this identity to study the renormalisation of the mass (squared) away from its bare value r. To this end, write the dressed propagator $G(\mathbf{k}) = \frac{1}{k^2+r}$ with

$$r = r_0 + \frac{u}{6}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{q^2+r}. \tag{2}$$

Now to go spherical coordinates and impose an ultraviolet cut-off Λ so that

$$r_0 - r = -\frac{u}{6}\int_0^\Lambda \frac{\mathrm{d}q}{(2\pi)^3}\frac{4\pi q^2}{q^2+r}. \tag{3}$$

The critical temperature is related with the value of the bare mass for which the renormalised mass vanishes: $r(r_0) = 0$. We have then

$$r_{\mathrm{c}} = -\frac{u}{6}\int_0^\Lambda \mathrm{d}q \frac{1}{2\pi^2} = -\frac{u\Lambda}{12\pi^2}$$

which depends explicitly on the UV cut-off. It is not a problem that r_{c} diverges for $\Lambda \to \infty$. In fact, we need the finite cut-off in order to control precisely what it means to be in the vicinity of the critical point. Let us subtract and add r_{c}:

$$r = r_0 - r_{\mathrm{c}} + \frac{u}{12\pi^2}\int_0^\Lambda \mathrm{d}q\left(\frac{q^2}{q^2+r} - 1\right). \tag{4}$$

A change of variables, $\tilde{q} = q/\sqrt{r}$, then produces

$$\mathrm{d}q\left(\frac{q^2}{q^2+r} - 1\right) = -\sqrt{r}\frac{\mathrm{d}\tilde{q}}{\tilde{q}^2+1},$$

and we can now take the $\Lambda \to \infty$ limit in the integral:

$$\int_0^\Lambda \mathrm{d}q\left(\frac{q^2}{q^2+r} - 1\right) = -\sqrt{r}\,[\arctan\tilde{q}]_0^{\Lambda/\sqrt{r}} \to -\sqrt{r}\frac{\pi}{2}.$$

Obviously this computation is only correct to order $\sqrt{r}$, so to remain consistent we can set to zero the square of the small quantity, i.e., set $r = 0$ dans (4):

$$0 \simeq (r_0 - r_{\mathrm{c}}) - \frac{u}{24\pi}\sqrt{r} + \mathcal{O}(r).$$

Finally, we define the reduced temperature as $\tau = r_0 - r_{\mathrm{c}}$. Since r plays the role of the mass squared, we can set $r \sim 1/\xi^2$. Thus,

$$\tau \sim \frac{u}{24\pi}\sqrt{r} \sim \frac{u}{24\pi}\frac{1}{\xi},$$

whence $\xi \sim \tau^{-1}$, and we have thus identified the critical exponent

$$\nu^\infty = 1.$$

We recall that the mean-field result, valid for $D \to \infty$, is $\nu_{\mathrm{MF}} = \frac{1}{2}$. So we have already found something non-trivial.
At r_{c} the renormalised mass vanishes, so we have $G_\infty(\mathbf{q}) = 1/q^2$, to be compared with the general form of the propagator $G(\mathbf{q}) \sim 1/q^{2-\eta}$. We have thus

$$\eta^\infty = 0.$$

This looks a bit trivial, and in particular on may well wonder how it would be possible to obtain a non-zero value for η; the answer to this question will appear at the end of the exercise.

(iv) We can get rid of the quartic term at the price of introducing a new scalar field $\sigma(\mathbf{x})$:

$$\exp\left[-\int \mathrm{d}^3x \frac{u}{24N}\left(\vec{\phi}^2\right)^2\right] = c\int \mathcal{D}\sigma \exp\left[-\int \mathrm{d}^3x \left\{\frac{\sigma^2}{2} - i\sqrt{\frac{u}{12N}}\sigma\vec{\phi}^2\right\}\right],$$

as can be easily seen by rewriting

$$\frac{\sigma^2}{2} - i\sqrt{\frac{u}{12N}}\sigma\vec{\phi}^2 = \frac{1}{2}\left(\sigma - i\sqrt{\frac{u}{12N}}\vec{\phi}^2\right)^2 + \frac{u}{24N}\left(\vec{\phi}^2\right)^2.$$

(The value of the constant c is not important.) The partition function then becomes

$$Z = c\int \mathcal{D}\sigma \exp\left[-\int \mathrm{d}^3x \frac{\sigma^2}{2}\right]\int \mathcal{D}\vec{\phi} \exp\left[-\frac{1}{2}\int \mathrm{d}^3x \left\{\left(\nabla\vec{\phi}\right)^2 + \left(r_0 + i\sqrt{\frac{u}{3N}}\sigma\right)\vec{\phi}^2\right\}\right],$$

(v) We can now perform the integral over $\mathcal{D}\vec{\phi}$ with result

$$\left(\det\left[\delta(\mathbf{x}-\mathbf{y})\left(-\nabla_{\mathbf{x}}^2 + r_0 + i\sqrt{\frac{u}{3N}}\sigma(\mathbf{x})\right)\right]\right)^{-N/2}.$$

Change the scale: $\tilde{\sigma}(\mathbf{x}) = N^{-1/2}\sigma(\mathbf{x})$, to find

$$Z = \int \mathcal{D}\tilde{\sigma}(\mathbf{x})\mathrm{e}^{-\frac{N}{2}S[\tilde{\sigma}]}$$

where the action reads

$$S[\tilde{\sigma}] = \int \mathrm{d}^3x\tilde{\sigma}(\mathbf{x})^2 + \log\det\left[\delta(\mathbf{x}-\mathbf{y})\left(-\nabla_{\mathbf{x}}^2 + r_0 + i\sqrt{\frac{u}{3}}\sigma(\mathbf{x})\right)\right]. \tag{5}$$

The fact that S is multiplied by a global factor of N shows that the saddle point method is *exact* in the limit $N \to \infty$.

(vi) Let us look for a saddle point of the form

$$\tilde{\sigma}(x) = \tilde{\sigma}_{\mathrm{c}} + \epsilon(x),$$

where $\epsilon \ll \tilde{\sigma}$ represents the small fluctuations around the classical solution. The classical action for a system of finite volume V is then

$$S[\tilde{\sigma}_{\mathrm{c}}] = V\cdot\tilde{\sigma}_{\mathrm{c}}^2 + \log\det\left[\delta(\mathbf{x}-\mathbf{y})\left(-\nabla_{\mathbf{x}}^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\mathrm{c}}\right)\right].$$

The last term can be rewritten as

$$\mathrm{Tr}\log\left[\delta(\mathbf{k}-\tilde{\mathbf{k}})\left(\mathbf{k}^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\mathrm{c}}\right)\right] = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\log\left(\mathbf{k}^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\mathrm{c}}\right)\langle\mathbf{k}|\mathbf{k}\rangle$$

where we have, in Fourier space,

$$\langle\mathbf{k}|\mathbf{k}\rangle = \delta(\mathbf{0}) = \int \mathrm{d}\mathbf{x}\mathrm{e}^{i\mathbf{x}\cdot\mathbf{0}} = V.$$

The result for the classical is therefore

$$S[\tilde{\sigma}_{\rm c}] = V\left[\tilde{\sigma}_{\rm c}^2 + \int \frac{{\rm d}^3k}{(2\pi)^3}\log\left(\mathbf{k}^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\rm c}\right)\right].$$

The corresponding equation of motion reads $\partial S/\partial\tilde{\sigma} = 0$, or

$$2\tilde{\sigma} + \int \frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{k^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}}\cdot i\sqrt{\frac{u}{3}} = 0.$$

To recover the form previously encountered, we must set $r = r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}$. Multiplying by $\frac{i}{2}\sqrt{\frac{u}{3}}$ we then obtain

$$i\sqrt{\frac{u}{3}}\tilde{\sigma} + \frac{1}{2}\left(i\sqrt{\frac{u}{3}}\right)^2\int\frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{k^2+r} = 0,$$

or

$$(r - r_0) - \frac{u}{6}\int\frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{k^2+r} = 0,$$

which coincides with relation (2).

(vii) This is a rather technical question, which however shows a few neat tricks. Let us omit the tildes on $\tilde{\sigma}$ for notational convenience. We compute the Hessian ("matrix" of second derivatives) of the action (5):

$$\frac{\delta^2 S[\sigma]}{\delta\sigma(x)\delta\sigma(y)} = 2\delta(x-y) + \frac{\delta^2}{\delta\sigma(x)\delta\sigma(y)}\log\det M(x,y),$$

where we have defined the "matrix" (with continuous indices)

$$M(x,y) = \delta(x-y)\left(-\nabla_x^2 + r_0 + i\sqrt{\frac{u}{3}}\sigma(x)\right).$$

To compute the second term, we first remark that

$$\begin{aligned}
\frac{\delta\log\det M}{\delta M(z_1,z_2)} &= \frac{1}{\det M}\cdot\frac{\delta\det M}{\delta M(z_1,z_2)} \\
&= \frac{1}{\det M}\times[\text{co-facteur de } M](z_1,z_2) \\
&= M^{-1}(z_1,z_2),
\end{aligned}$$

and that

$$\begin{aligned}
\frac{\delta M(z_1,z_2)}{\delta\sigma(x)} &= i\sqrt{\frac{u}{3}}\delta(z_1-z_2)\frac{\delta\sigma(z_1)}{\delta\sigma(x)} \\
&= i\sqrt{\frac{u}{3}}\delta(z_1-z_2)\delta(z_1-x) \\
&= i\sqrt{\frac{u}{3}}\delta(z_1-x)\delta(z_2-x).
\end{aligned}$$

By generalising the chain rule of differentiation to continuous indices, we thus have

$$\begin{aligned}
\frac{\delta\log\det M}{\delta\sigma(x)} &= \int {\rm d}z_1{\rm d}z_2\frac{\delta\log\det M}{\delta M(z_1,z_2)}\cdot\frac{\delta M(z_1,z_2)}{\delta\sigma(x)} &(6)\\
&= \int {\rm d}z_1{\rm d}z_2 M^{-1}(z_1,z_2) i\sqrt{\frac{u}{3}}\delta(z_1-x)\delta(z_2-x) &(7)\\
&= i\sqrt{\frac{u}{3}}M^{-1}(x,x).
\end{aligned}$$

To derive one more term, we need the lemma

$$\frac{\delta M^{-1}(x_1,x_2)}{\delta M(z_1,z_2)} = -M^{-1}(x_1,z_1)M^{-1}(z_2,x_2).) \tag{8}$$

This can be proven by deriving the trivial identity

$$\int \mathrm{d}x_3 M^{-1}(x_1,x_3)M(x_3,x_4) = \delta(x_1,x_4)$$

with respect to $M(z_1,z_2)$. We find

$$0 = \int \mathrm{d}x_3 \left(\frac{\partial M^{-1}(x_1,x_3)}{\partial M(z_1,z_2)} M(x_3,x_4) + M^{-1}(x_1,x_3)\delta(x_3,z_1)\delta(x_4,z_2) \right).$$

One next multiplies by $\int \mathrm{d}x_4 M^{-1}(x_4,x_2)$:

$$\begin{aligned} &\int \mathrm{d}x_3 \mathrm{d}x_4 \frac{\partial M^{-1}(x_1,x_3)}{\partial M(z_1,z_2)} M(x_3,x_4)M^{-1}(x_4,x_2) = \\ &-\int \mathrm{d}x_3 \mathrm{d}x_4 M^{-1}(x_1,x_3)M^{-1}(x_4,x_2)\delta(x_3,z_1)\delta(x_4,z_2). \end{aligned}$$

The left-hand side becomes

$$\int \mathrm{d}x_3 \frac{\partial M^{-1}(x_1,x_3)}{\partial M(z_1,z_2)} \delta(x_3,x_2) = \frac{\partial M^{-1}(x_1,x_2)}{\partial M(z_1,z_2)},$$

whereas the right-hand side can be written $-M^{-1}(x_1,z_1)M^{-1}(z_2,x_2)$. This concludes the proof of the lemma (8).
Using this, we have finally

$$\begin{aligned} \frac{\delta^2 \log \det M}{\delta\sigma(x)\delta\sigma(y)} &= i\sqrt{\frac{u}{3}} \int \mathrm{d}z_1 \mathrm{d}z_2 \frac{\delta M^{-1}(x,x)}{\delta M(z_1,z_2)} \cdot \frac{\delta M(z_1,z_2)}{\delta\sigma(y)} \\ &= i\sqrt{\frac{u}{3}} \int \mathrm{d}z_1 \mathrm{d}z_2 \left[-M^{-1}(x,z_1)M^{-1}(x,z_2)\right] \cdot \left[i\sqrt{\frac{u}{3}} \delta(z_1-y)\delta(z_2-y) \right] \\ &= \frac{u}{3}\left[M^{-1}(x,y)\right]^2. \end{aligned}$$

At the saddle point, this equals

$$\begin{aligned} &\frac{u}{3}\left(\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{e}^{ik(x-y)}}{k^2 + r_0 + i\sqrt{\frac{u}{3}}\sigma_\mathrm{c}} \right)^2 = \\ \frac{u}{3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} &\frac{\mathrm{e}^{ik(x-y)}}{\left(q^2 + r_0 + i\sqrt{\frac{u}{3}}\sigma_\mathrm{c}\right)\left((\mathbf{k}-\mathbf{q})^2 + r_0 + i\sqrt{\frac{u}{3}}\right)}, \end{aligned}$$

where we have named the integration variables $\mathbf{q}$ and $\mathbf{k}-\mathbf{q}$, respectively.
The conclude, the Hessian can be diagonalised by Fourier transform, and its eigenvalues are

$$\lambda_k = 2 + \frac{u}{3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\left(q^2 + r_0 + i\sqrt{\frac{u}{3}}\sigma_\mathrm{c}\right)\left((\mathbf{k}-\mathbf{q})^2 + r_0 + i\sqrt{\frac{u}{3}}\right)}.$$

One can easily convince oneself that λ_k is well-defined for all $r_0 > 0$, and that the real part is always positive. It follows that the fluctuations around the saddle point are bounded, as required.

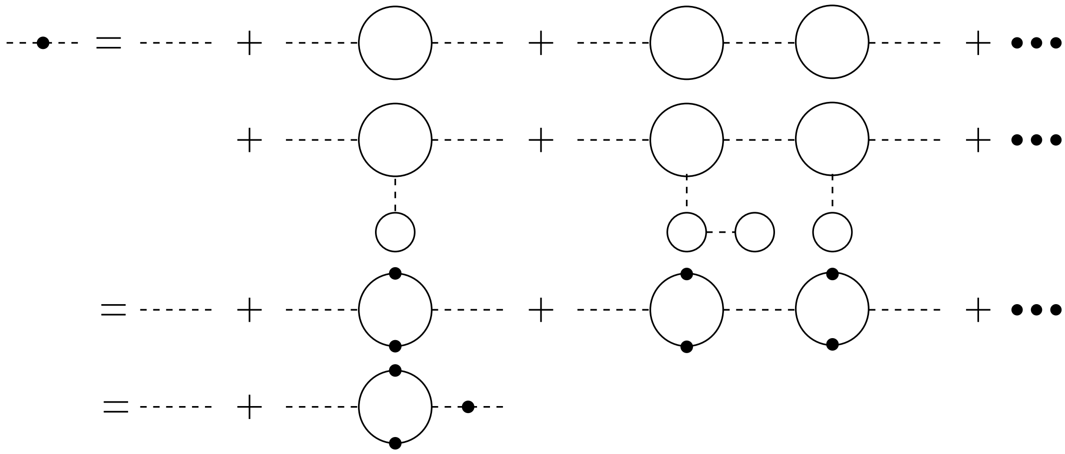

Figure 5: Derivation of the implicit relation satisfied by the propagators.

(viii) We use the action found in (v) to read off the diagrammatic rules. The bare propagator of the scalar field σ is simply

$$\langle\tilde{\sigma}(\mathbf{k})\tilde{\sigma}(-\mathbf{k}\rangle = 1,$$

and that of the vector field ϕ is

$$\langle\tilde{\phi}_\alpha(\mathbf{k})\tilde{\phi}_\beta(-\mathbf{k})\rangle = \frac{\delta_{\alpha,\beta}}{k^2 + r_0}$$

where we note the usual conservation of $\mathrm{O}(N)$ indices.
Finally, there is a trivalent interaction vertex $\sigma\phi_\alpha\phi_\beta$ with value

$$-i\sqrt{\frac{u}{12N}}\delta_{\alpha,\beta}\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)(2\pi)^3.$$

We stress that the decomposition of a tetravalent interaction into a pair of trivalent interactions, by means of an auxiliary field (here σ), is a very common and useful trick in field theory.

(ix) The equation satisfied by the propagators in the limit $N \to \infty$ is shown graphically in Figure 5. Each trivalent vertex contributes a factor of $1/\sqrt{N}$, and each independent loop gives a factor of N. In the limit $N \to \infty$ we thus need exactly twice more vertices than loops to get a non-zero result, so only tadpoles of tadpoles survive (first line of Fig. 5). The insertion of ϕ-loops into other ϕ lines amounts to renormalising the propagator of ϕ (second line). And finally, we dress the propagator of σ in order to count all the dressed tadpoles in one fell swoop (third line).
In algebraic terms, Fig. 5 reads

$$G_\sigma^\infty(\mathbf{k}) = 1 + 1 \cdot \int \frac{\mathrm{d}^3q}{(2\pi)^3} G_\phi^\infty(-\mathbf{q}) G_\phi^\infty(-\mathbf{k}+\mathbf{q}) G_\sigma^\infty(\mathbf{k}) \cdot \left(-i\sqrt{\frac{u}{12N}}\right)^2 \cdot 2N,$$

where the factor of $2N$ comes from the two ways of contracting the ϕ in $\langle\sigma\phi\phi|\phi\phi\sigma\rangle$, and from the sum over the vector index in the loop. We can then isolate $G_\sigma^\infty(\mathbf{k})$:

$$G_\sigma^\infty(\mathbf{k}) = \frac{1}{1 + \frac{u}{6}\int \frac{\mathrm{d}^3q}{(2\pi)^3} G_\phi^\infty(\mathbf{k}-\mathbf{q}) G_\phi^\infty(\mathbf{q})}. \tag{9}$$

At the critical point, $G_\phi^\infty(\mathbf{q}) = 1/\mathbf{q}^2$ and one can compute the integral. Let us first set

$$\begin{aligned}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} G_\phi^\infty(\mathbf{k}-\mathbf{q}) G_\phi^\infty(\mathbf{q}) &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{q})^2} \cdot \frac{1}{\mathbf{q}^2} \\ &= \frac{1}{|\mathbf{k}|} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(\mathbf{1}-\mathbf{q})^2} \cdot \frac{1}{\mathbf{q}^2} \equiv \frac{1}{|\mathbf{k}|} \cdot I,\end{aligned}$$

where $\mathbf{1}$ is an arbitrary unit vector (we integrate over the orientations of $\mathbf{q}$), and the factor $\frac{1}{|\mathbf{k}|}$ can be found by dimensional analysis.

(x) We now compute the integral I. Let θ be the angle between $\mathbf{1}$ and $\mathbf{q}$. Going to polar coordinates,

$$I = \int_0^\infty \frac{\mathrm{d}q}{(2\pi)^3} \frac{q^2}{q^2} \int_0^\pi \mathrm{d}\theta 2\pi \sin\theta \frac{1}{1+q^2-2q\cos\theta},$$

and with the change of variables $x = -\cos\theta$:

$$\begin{aligned} I &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}q \int_{-1}^1 \mathrm{d}x \frac{1}{1+q^2-2qx} \\ &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}q \frac{1}{2q} \left\{ \log(1+q^2+2q) - \log(1+q^2-2q) \right\} \\ &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}q \frac{1}{q} \log\left| \frac{1+q}{1-q} \right|. \end{aligned}$$

One easily sees that $\int_1^\infty = \int_0^1$ by making the change of variables $q = 1/u$. Therefore,

$$I = \frac{1}{2\pi^2} \int_0^1 \mathrm{d}q \frac{1}{q} \log\left(\frac{1+q}{1-q}\right),$$

and we can expand

$$\log\left(\frac{1+q}{1-q}\right) = \sum_{n=0}^\infty \left[\left(1 + \frac{q^n}{n}\right) - \left(1 + \frac{(-q)^n}{n}\right) \right] = 2 \sum_{p=0}^\infty \frac{1}{2p+1} q^{2p+1}.$$

Integrating term by term leads to

$$I = \frac{1}{2\pi^2} \cdot 2 \sum_{p=0}^\infty \frac{1}{(2p+1)^2},$$

and since $\sum_{n=1,3,5,\ldots} = \sum_{n=0}^\infty - \sum_{n=0,2,4,\ldots}$,

$$\sum_{p=0}^\infty \frac{1}{(2p+1)^2} = \sum_{n=0}^\infty \left(\frac{1}{n^2} - \frac{1}{(2n)^2} \right) = \left(1 - \frac{1}{4}\right) \sum_{n=0}^\infty \frac{1}{n^2} = \frac{3}{4} \cdot \frac{\pi^2}{6}.$$

Finally,

$$I = \frac{1}{2\pi^2} \cdot 2 \cdot \frac{\pi^2}{8} = \frac{1}{8}. \tag{10}$$

(xi) Combining these pieces, it follows that (9) becomes, in the limit $k \to 0$,

$$G_\sigma^\infty(\mathbf{k}) \simeq \frac{48}{u} |\mathbf{k}|. \tag{11}$$

(xii) Let us return to ():

$$\frac{1}{G_\phi^N(\mathbf{k})} = k^2 + r_0 - \Sigma_\phi^N(\mathbf{k}). \tag{12}$$

The first term in Fig. 2 is the contribution to the self-energy which remains finite when $N \to \infty$. This contribution has already been accounted for—more precisely in Fig. 1—but it reappears here in the formalism using trivalent vertices. Indeed, if we provide external legs of the ϕ type, the first term yields precisely the diagrams in Fig. 4). The second term, however, is of order $1/N$, since there is no ϕ-loop to provide the compensating factor of N. On can verify that no other diagrams contribute at this order.

(xiii) We therefore set out to compute the new diagram. There is a combinatorial factor due to the contraction $\langle \phi_\mathrm{l} \cdot \phi^1\phi^1\sigma | \sigma\phi^2\phi^2 \cdot \phi_\mathrm{r}\rangle$:

- A factor $\frac{1}{2}$ coming from the expansion of the exponential (there are two vertices).
- A factor 2 due to the choice of contracting ϕ_l with either a ϕ^1 or a ϕ^2.
- A factor 2^2, since we must decide which one of the ϕ^1 (or ϕ^2) we wish to contract with an external leg.

Donc, au total un facteur de 4.
The diagram, expressed as a function of the wave number $\mathbf{k}$ flowing between external legs, then leads to the integral

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} G_\sigma^\infty(\mathbf{q}) G_\phi^\infty(\mathbf{k}-\mathbf{q}) \cdot 4 \left(i\sqrt{\frac{u}{12N}} \right)^2,$$

which is seen to have an explicit factor of $1/N$. It is therefore consistent to evaluate the propagators for $N \to \infty$. (This is a usual phenomenon in perturbation theory, seen also in ϕ^4 theory for example: some quantities are lagging one order behind others, in order to define a consistent perturbative framework.)
We are not interested in this integral *per se*, but only in its dependence on $\mathbf{k}$, so it is permissible to subtract the same equation with $\mathbf{k} = \mathbf{0}$:

$$\Delta\Sigma_\phi^N(\mathbf{k}) = 4 \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G_\sigma^\infty(\mathbf{q}) \left[G_\phi^\infty(\mathbf{k}-\mathbf{q}) - G_\phi^\infty(-\mathbf{q}) \right] \left(i\sqrt{\frac{u}{12N}} \right)^2,$$

where $[\cdots]$ equals $\frac{1}{(\mathbf{k}-\mathbf{q})^2} - \frac{1}{\mathbf{q}^2}$ at the critical point.
The behaviour of $\Delta\Sigma$ for small $|\mathbf{k}|$ (i.e., at large distances in real space) is determined by the small values of $\mathbf{q}$ in the integral over $\mathbf{q}$. When using (11), we thus suppose that $G_\sigma^\infty(\mathbf{q}) \simeq \frac{48}{u}|\mathbf{q}|$ when $q < \Lambda$, where Λ denotes an appropriate UV cut-off. In the limit $\mathbf{k} \to 0$, we therefore have

$$\begin{aligned}
\Delta\Sigma_\phi^N(\mathbf{k}) &\simeq 4 \int_{q<\Lambda} \frac{\mathrm{d}^3 q}{(2\pi)^3} |\mathbf{q}| \left[(\mathbf{k}-\mathbf{q})^{-2} - \mathbf{q}^{-2} \right] \frac{48}{u} \cdot \frac{-u}{12N} \\
&= -\frac{2}{\pi^3 N} \int_{q<\Lambda} \frac{\mathrm{d}^3 q}{(2\pi)^3} |\mathbf{q}| \left[(\mathbf{k}-\mathbf{q})^{-2} - \mathbf{q}^{-2} \right]
\end{aligned}$$

We still need to compute the integral. Unlike the previous case, we do not need an explicit evaluation, but only the leading-order asymptotics for small $\mathbf{k}$. We go to polar coordinates:

$$I(k) = 2\pi \int_0^\Lambda \mathrm{d}q q^3 \int_0^\pi \mathrm{d}\theta \sin\theta \left[\frac{1}{k^2 + q^2 - 2kq\cos\theta} - \frac{1}{q^2} \right].$$

We first perform the integral $\int \mathrm{d}\theta$ by substituting $x = -\cos\theta$:

$$\int_{-1}^1 \mathrm{d}x \left[\frac{1}{k^2 + q^2 + 2kqx} - \frac{1}{q^2} \right] = \frac{1}{kq} \log\left| \frac{k+q}{k-q} \right| - \frac{2}{q^2},$$

and next set $y = q/k$ to arrive at

$$I(k) = 2\pi k^2 \int_0^{\Lambda/k} \mathrm{d}y\, y^2 \left[\log\left|\frac{1+y}{1-y}\right| - \frac{2}{y}\right].$$

For $k \ll 1$, we have $y \gg 1$, and the first non-zero order in $[\cdots]$ yields $\simeq \frac{2}{3y^3}$. Notice that there is no infrared divergence, as can be seen *before* doing the series expansion. Thus,

$$\begin{aligned} I(k) &\simeq 2\pi k^2 \cdot \frac{2}{3}\log\left|\frac{\Lambda}{k}\right| \\ &\simeq -\frac{4\pi}{3}k^2\log k + \cdots . \end{aligned}$$

There is a small subtlety here: At the critical point, we are interested in large distances, hence small k (IR limit). However, the change of variables means that we are interested in large Λ/k, so we should study the UV limit of the one-loop integral. The final result is therefore:

$$\Delta\Sigma_\phi^N(k) \simeq \frac{8}{3\pi^2 N}k^2\log k.$$

(xiv) At the critical point

$$G(k) \sim \frac{1}{k^{2-\eta}} \simeq \frac{1}{k^2(1-\eta\log k + \cdots)},$$

and from (12) we see that

$$\Sigma(k) = \eta k^2 \log k + \cdots .$$

The above result therefore implies that

$$\eta = \frac{8}{3\pi^2 N} + \mathcal{O}\left(\frac{1}{N^2}\right).$$

The subject is treated in chapter 26 of the book by Zinn-Justin, but omitting all the technical details of the computation (see also the book by Parisi). It is mentioned there that one can push the computation to higher order:

$$\begin{aligned} \eta &= \frac{\eta_1}{N} + \frac{\eta_2}{N^2} + \frac{\eta_3}{N^3} + \ldots \\ \eta_1 &= \frac{8}{3\pi^2} \simeq 0.27019 \\ \eta_2 &= -\frac{8}{3}\eta_1^2 \simeq -0.19467 \\ \eta_3 &= \eta_1^3\left[-\frac{797}{84} - \frac{61}{24}\pi^2 + \frac{27}{8}\psi''\left(\frac{1}{2}\right) + \frac{9}{2}\pi^2\ln 2\right] \simeq -1.19502. \end{aligned}$$

Here $\psi(x) = \frac{\mathrm{d}\log\Gamma(x)}{\mathrm{d}x}$ denotes the digamma function.
To study the 3D Ising model one could naively set $N = 1$ in these results. A high-temperature series expansion gives $\eta = 0.055 \pm 0.014$. This seems to compare rather favourably with the change from the first-order result $\eta_1 = 0.27$ to the second-order one $\eta_1 + \eta_2 = 0.076$. Naive hopes are quickly quenched when observing that at third order $\eta_1 + \eta_2 + \eta_3 = -1.12$. The resolution of this situation is that the perturbative series is only asymptotic and should in fact be handled with resommation techniques (more details in the second semester optional course given by Kay Wiese).