Statistical Field Theory and Applications:
An Introduction for (and by) Amateurs.

by Denis BERNARD & Jesper JACOBSEN

The XY model

The XY model is a statistical spin model with spin variables $\vec{S}_i$, on each site i of the lattice Λ, which are two component unit vectors, $\vec{S}_i^2 = 1$. The energy of a configuration $[\vec{S}]$ is defined as $E[\vec{S}] = -\sum_{[ij]} \vec{S}_i \cdot \vec{S}_j$ where the sum runs over neighboor points on Λ. Parametrising the unit spin vectors $\vec{S}_i$ by an angle Θ_i defined modulo 2π, we write the configuration energy as

$$E[\vec{S}] = -\sum_{[ij]} \cos(\Theta_i - \Theta_j).$$

The partition function is $Z = \int [\prod_i \frac{d\Theta_i}{2\pi}] \exp\left(\beta \sum_{[i,j]} \cos(\Theta_i - \Theta_j)\right)$ with $\beta = 1/k_B T$ the inverse temperature.

Here is the solution of the problem on the XY model given in Section 9.1.

• *IA- The XY model on a lattice: High temperature expansion*

The aim of this section is to study the high temperature ($\beta \ll 1$) behavior of the XY model. It is based on rewriting the Boltzmann sums in terms of dual flow variables.

IA-1 Explain why we can expand $e^{\beta \cos\Theta}$ in series as $e^{\beta\cos\Theta} = I(\beta)\left(1 + \sum_{n\neq 0} t_n(\beta) e^{in\Theta}\right)$, where $I(\beta)$ and $t_n(\beta)$ are some real β-dependent coefficients. We set $t_0(\beta) = 1$.

IA-2 By inserting this series in the defining expression of the partition function and by introducing integer variables $u_{[ij]}$ on each edge $[ij]$ of the lattice Λ, show that the partition function can be written as $Z = I(\beta)^{N_\mathrm{e}} \cdot \hat{Z}$ with N_e the number of edges and

$$\hat{Z} = \sum_{[u],\ [\partial u = 0]} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the partition sum is over all configurations $[u]$ of integer edge variables $u_{[ij]}$ such that, for any vertex $i \in \Lambda$, the sum of these variables arriving at i vanishes, i.e. $\sum_j u_{[ij]} = 0$.
Remark: The variables u are attached to the edge of the lattice and may be thought of as 'flow variables'. The condition that their sum vanishes at any given vertex is a divergence free condition. The divergence at a vertex i of a configuration $[u]$ is defined as $(\partial u)_i := \sum_j u_{[ij]}$.

IA-3 Let i_1 and i_2 be two points of Λ and $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2}\rangle$ be the two-point spin correlation function. Explain why $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2}\rangle = \mathrm{Re}\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})}\rangle$.
Show that,

$$\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})}\rangle = \frac{1}{\hat{Z}} \cdot \sum_{\substack{[u] \\ [\partial u = \delta_{\cdot;i_1} - \delta_{\cdot;i_2}]}} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the sum is over all integer flow configurations such that their divergence is equal to $+1$ at point i_1, to -1 at point i_2, and vanishes at any other vertex.

IA-4 Show that $t_n(\beta) = t_{-n}(\beta) \simeq \frac{\beta^n}{2^n n!}$ as $\beta \to 0$.
Argue, using this asymptotic expression for the $t_n(\beta)$'s, that the leading contribution to the spin correlation functions at high temperature comes from flow configurations with $u = 0$ or $u = \pm 1$ on each edge of the lattice.

IA-5 Deduce that, at high temperature, the correlation function $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ decreases exponentially with the distance between the two points i_1 and i_2.
Show that the correlation length behaves as $\xi \simeq a/\log(2/\beta)$ at high temperature.

Correction :

IA-1: $e^{\beta \cos \Theta}$ is a periodic function of Θ (with period 2π) so that it can be represented as a Fourier series. By reality, we have $t_n(\beta) = t_{-n}(\beta)$.
IA-2 We insert the representation $e^{\beta \cos \Theta} = I(\beta) \sum_n t_n(\beta) e^{in\Theta}$ in the partition function to write

$$\begin{aligned}\hat{Z} &= \int [\prod_i \frac{d\Theta_i}{2\pi}] \prod_{[ij]} \Big(\sum_{u_{[ij]}} t_{u_{[ij]}}(\beta)\, e^{iu_{[ij]}\Theta} \Big), \\ &= \int [\prod_i \frac{d\Theta_i}{2\pi}] \sum_{[u]} \Big(\prod_{[ij]} t_{u_{[ij]}}(\beta)\, e^{iu_{[ij]}\Theta} \Big).\end{aligned}$$

Integration of the Θ_i's yields the constraint $(\partial u)_i := \sum_j u_{[ij]} = 0$.
IA-3 By reality $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle = \mathrm{Re}\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle$. By definition

$$\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle = \frac{1}{\hat{Z}} \times \int [\prod_i \frac{d\Theta_i}{2\pi}] \sum_{[u]} \Big(\prod_{[ij]} t_{u_{[ij]}}(\beta)\, e^{iu_{[ij]}\Theta} \Big) \cdot e^{-i(\Theta_{i_1} - \Theta_{i_2})}.$$

Integration over the Θ_i's gives $\sum_j u_{[ij]} = 0$ for all $i \neq i_1, i_2$, but $\sum_j u_{[i_1 j]} = 1$ and $\sum_j u_{[i_2 j]} = -1$.
IA-4 By definition $t_n(\beta) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} e^{\beta \cos \theta}$. For small β we can expand $e^{\beta \cos \theta}$ in Taylor series. The first term of this series which contributes non-trivially to the integral is $\frac{1}{n!}(\beta \cos \theta)^n$. Its integral yields $t_n(\beta) \simeq \beta^n / 2^n n!$ as $\beta \to 0$.
Since $t_u \simeq \text{const.}\, (\beta/2)^{|u|}$ the leading contribution to the correlation function is for $u = 0$ or $|u| = 1$ (and the weight of configuration is independent of sign of u).
IA-5 Selecting the configuration with $u = 0$ or $u = \pm 1$ on each lattice edge compatible with the divergence constraints $\sum_j u_{[ij]} = 0$ for all $i \neq i_1, i_2$, but $\sum_j u_{[i_1 j]} = 1$ and $\sum_j u_{[i_2 j]} = -1$ selects a path from i_1 to i_2. The weight of such path γ is proportional to $(\beta/2)^{d_{12}(\gamma)}$ with $d_{12}(\gamma)$ its length (measured as the number of steps of the path). The leading contribution comes from the shortest path, and the correlation function decreases exponentially with the distance d_{12} between the two points as $(\beta/2)^{d_{12}}$. The correlation length is thus $a/\log(2/\beta)$, asymptotically at high temperature. The high temperature phase is disordered.

- *IB- Low temperature expansion*

The aim of this section is to study the low temperature ($\beta \gg 1$) behavior of the XY model. It consists in expanding the interaction energy $\cos(\Theta_i - \Theta_j)$ to lowest order in the angle variables so that we write the configuration energy as (up to an irrelevant additive constant)

$$E[\vec{S}] = \text{const.} + \frac{1}{2} \sum_{[i,j]} (\Theta_i - \Theta_j)^2 + \cdots .$$

This approximation neglects the 2π-periodicity of the angle variables.

IB-1 Argue that the higher order terms in this expansion, say the terms proportional to $\sum_{[i,j]} (\Theta_i - \Theta_j)^4$, are expected to be irrelevant and can be neglected.

IB-2 Write the expression of the partition function Z of the model within this approximation. Explain why, in this approximation, the theory may be viewed as a Gaussian theory.

IB-3 Let $G_\beta(x)$ be the two-point function of this Gaussian theory. Show that $G_\beta(x) = \beta^{-1}\, G(x)$ with

$$G(x) = \int_{-\pi/a}^{+\pi/a} \frac{d^2p}{(2\pi/a)^2} \frac{e^{ip\cdot x}}{4-2(\cos ap_1 + \cos ap_2)},$$

with p_1, p_2 the two components of the momentum p and a the lattice mesh.

IB-4 Let i_1 and i_2 be two points on Λ and x_1 and x_2 be their respective Euclidean positions. Let $C_\alpha(x_1,x_2) = \langle e^{i\alpha(\Theta_{i_1}-\Theta_{i_2})}\rangle$ with α integer. Show that

$$C_\alpha(x_1,x_2) = e^{-\frac{\alpha^2}{\beta}\left(G(0)-G(x_1-x_2)\right)}.$$

IB-5 Explain why $G(x)$ is actually IR divergent[1] and what is the origin of this divergence, but that $G(0)-G(x)$ is finite for all x. Show that

$$G(0)-G(x) = \frac{1}{2\pi}\log(|x|/a) + \text{const.} + O(1/|x|).$$

IB-6 Deduce that the correlation functions C_α decrease algebraically at large distance according to

$$C_\alpha(x_1,x_2) \simeq \text{const.}\,(a/|x_1-x_2|)^{\alpha^2/2\pi\beta}.$$

Compare with the high temperature expansion.

Correction :

IB-1 This expansion is a gradient expansion, the leading term is $(\nabla\Theta)^2$. The other terms $(\nabla\Theta)^p$, with higher powers of the gradient, are irrelevant (when estimated using the leading Gaussian contribution $\int(\nabla\Theta)^2$).
IB-2 In this approximation, the partition reads (up to an irrelevant multiplicative constant)

$$Z = \int [\prod_i \frac{d\Theta_i}{2\pi}]\, e^{-\frac{\beta}{2}\sum_{[i,j]}(\Theta_i-\Theta_j)^2}.$$

This is a Gaussian theory.
IB-3 The two point function is given by the inverse of the quadratic form defining the action. Hence it is $\beta^{-1}G(x)$ with G the Green function of the lattice Laplacian on $(a\mathbb{Z}^2)$. Thus $\hat{G}(p)$, the Fourier transform of $G(x)$ is solution of

$$\left(4-2(\cos ap_1+\cos ap_2)\right)\hat{G}(p) = 1.$$

This yields the formula for $G(x)$ given in the text[2].
IB-4 The formula for C_α follows from the fact that the theory is Gaussian (with a even translation invariant two point function).
IB-5 The function $G(x)$ is IR divergent because the (discrete) Laplacian has the constant function as zero mode (the constant function is in the kernel of the Laplacian) and the inverse Laplacian does not exist. For this inverse to exist one has to impose boundary conditions (say periodicity, or Dirichlet, etc.) which eliminate the constant zero mode. The IR divergence of $G(x)$ is of the form $\frac{1}{2\pi}\log L$ with L the linear size of system box (this can be seen by looking at the small momenta contribution to the integral:

[1] So that, when defining $G(x)$, we implicitly assumed the existence of an IR cut-off, say $|p| > 2\pi/L$ with L the linear size of the box on which the model is considered.

[2] There is actually an IR divergence is this formula – due to constant zero mode of the Laplacian – so that we implicitly assume an IR regularization.

$\frac{1}{(2\pi)^2}\int_{|p|>2\pi/L}\frac{d^2p}{(p)^2}.$). This divergence cancels in the difference $G(0)-G(x)$ (because this is independent of the constant zero mode). Alternatively, we can write

$$G(0)-G(x)=\int_{-\pi/a}^{+\pi/a}\frac{d^2p}{(2\pi/a)^2}\frac{1-\cos(p\cdot x)}{4-2(\cos ap_1+\cos ap_2)}.$$

which is explicitly convergent.

Since $G(0)-G(x)$ only depends on $|x|/a$, the long distance behavior is identical to the continuous limit ($a\to 0$). In this limit $G(0)-G(x)$ is (minus) a Green function of the 2D Euclidean Laplacian. Hence it is equal to $\frac{1}{2\pi}\log|x|$ up to an additive constant. Dimensional analysis then fixes the constant as in the text.

IB-6 Direct application of the above formula.

- *II- The role of vortices in the XY field theory*

The previous computations show that the model is disordered at high temperature but critical at low temperature with temperature dependent exponents. The aim of this section is to explain the role of topological configurations, called vortices, in this transition.

We shall now study the model in continuous space, the Euclidean plane $\mathbb{R}^2$, but with an explicit short distance cut-off a. We shall consider the XY system in a disc of radius L.

In the continuous formulation, the spin configurations are then maps Θ from $\mathbb{R}^2$ to $[0,2\pi]$ modulo 2π. The above Gaussian energy is mapped into the action

$$S_0[\Theta]=\frac{\kappa}{2}\int d^2x(\nabla\Theta)^2,$$

with a coefficient κ proportional to β.

II-1 Argue that the coefficient κ cannot be absorbed into a rescaling of the field variable Θ?

II-2 A vortex, centred at the origin, is a configuration such that $\Theta^{\pm}_{\rm v}(z)=\pm\mathrm{Arg}(z)$, with z the complex coordinate on $\mathbb{R}^2$, or in polar coordinates[3], $\Theta^{\pm}_{\rm v}(r,\phi)=\pm\phi$.
Show that $\Theta^{\pm}_{\rm v}$ is an extremum of S_0 in the sense that $\nabla^2\Theta^{\pm}_{\rm v}=0$ away from the origin.
Show that $\oint_{C_0}d\Theta^{\pm}_{\rm v}=\pm 2\pi$ for C_0 a small contour around the origin.

II-3 Let a_0 be a small short distance cut-off and let $\mathbb{D}(a_0)$ be the complex plane with small discs of radius a_0 around the vortex positions cut out. Prove that, evaluated on $\Theta^{\pm}_{\rm v}$, the action S_0 integrated over $\mathbb{D}(a_0)$ (with an IR cut-off L) is

$$S^{(1)}_{\rm vortex}=\frac{\kappa}{2}\int_{\mathbb{D}(a_0)}d^2x\,(\nabla\Theta^{\pm}_{\rm v})^2=\pi\kappa\log\big[L/a_0\big].$$

Give an interpretation of the divergence as $a_0\to 0$.

II-4 What is the entropy of single vortex configurations? Show that the contribution of single vortex configurations to the free energy is

$$e^{-F^{(1)}_{\rm vortex}}\simeq \text{const.}\,\big(\frac{L}{a_0}\big)^2\,e^{-\pi\kappa\log[L/a_0]}$$

Conclude that vortex configurations are irrelevant for $\pi\kappa>2$ but relevant for $\pi\kappa<2$.

[3] We recall the expression of the gradient in polar coordinates: $\nabla\Theta=(\partial_r\Theta,\frac{1}{r}\partial_\phi\Theta)$. The Laplacian is $\nabla^2F=\frac{1}{r}\partial_r(r\partial_r)F+\frac{1}{r^2}\partial^2_\phi F$.

Correction :

II-1 Since Θ is 2π-periodic we cannot rescale it to absorb the parameter κ in a redefinition of Θ (unless we redefine the periodicity).
II-2 In polar coordinate $\nabla^2\Theta = \frac{1}{r}\partial_r(r\partial_r)\Theta + \frac{1}{r^2}\partial_\phi^2\Theta$. Thus, $\nabla^2\Theta_v^\pm = 0$ away from the origin.
The gradient is $\nabla\Theta_v^\pm = \pm(0, \frac{1}{r})$. Hence, $\oint_{C_0} d\Theta_{\rm v}^\pm = \pm\int_0^{2\pi} d\phi = \pm 2\pi$.
II-3 Using $(\nabla\Theta_v^\pm)^2 = \frac{1}{r^2}$, we get (doing the integration using polar coordinate)

$$S^{(1)}_{\rm vortex} = \frac{\kappa}{2}\int_{a_0}^{L}\frac{2\pi\, rdr}{r^2} = \pi\kappa \log\big[L/a_0\big].$$

The UV divergence (with $a_0 \to 0$) is an echo of the fact that the naive continuous limit we are using is ill-defined near the core of the vortex at which the field Θ becomes singular.
II-4 The vortex center may be positioned at any position, with a typical size of diameter a_0. Hence the entropy of single vortex configuration is $\simeq \log(L/a_0)^2$. This yields the expression of $F^{(1)}_{\rm vortex}$ given in the text. And $e^{-F^{(1)}_{\rm vortex}}$ is significantly large for $\pi\kappa < 2$ but negligeable for $\pi\kappa > 2$.

- *III- The XY field theory and the sine-Gordon model*

The aim of this section is to analyse this phase transition using renormalization group arguments via a mapping to the so-called sine-Gordon field theory.

We shall consider a gas of vortices. The field configuration $\Theta_{\rm v}^{(M)}$ for a collection of M vortices of charges q_a centred at positions x_a is given by the sum of single vortex configuration:

$$\Theta_{\rm v}^{(M)} = \sum_{a=1}^{M} q_a\, {\rm Arg}(z - z_a).$$

We shall admit that the action of such configuration is

$$S^{(M)}_{\rm vortex} = -2\pi(\frac{\kappa}{2})\sum_{a\neq b} q_a q_b\, \log\big(\frac{|x_a - x_b|}{a_0}\big) + 2\pi(\frac{\kappa}{2})(\sum_b q_b)^2 \log\big(\frac{L}{a_0}\big) + \sum_a \beta\epsilon_c,$$

where ϵ_c is a 'core' energy (which is not taken into account by the previous continuous description).

- *IIIA- The XY field theory: Mapping to the sine-Gordon theory*

The aim of this section is to analyse this phase transition using renormalization group arguments via a mapping to the so-called sine-Gordon field theory.

We shall consider a gas of vortices. The field configuration $\Theta_{\rm v}^{(M)}$ for a collection of M vortices of charges q_a centred at positions x_a is given by the sum of single vortex configuration:

$$\Theta_{\rm v}^{(M)} = \sum_{a=1}^{M} q_a\, {\rm Arg}(z - z_a).$$

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$$S^{(M)}_{\rm vortex} = -2\pi(\frac{\kappa}{2})\sum_{a\neq b} q_a q_b\, \log\big(\frac{|x_a - x_b|}{a_0}\big) + 2\pi(\frac{\kappa}{2})(\sum_a q_b)^2 \log\big(\frac{L}{a_0}\big) + \sum_a \beta\epsilon_c,$$

where ϵ_c is a 'core' energy (which is not taken into account by the previous continuous description).

This mapping comes about when considering a gas of pairs of vortices of opposite charges $\pm$, so that the vortex system is neutral ($\sum_a q_a = 0$). We denote x_j^+ (resp. x_j^-) the positions of the vortices of charge $+$ (resp. $-$).

The vortex gas is defined by considering all possible vortex pair configurations (with arbitrary number of pairs) and fluctuations around those configurations. We set $\Theta = \Theta_{\rm v}^{(2n)} + \theta_{\rm sw}$ and associate to each such configuration a statistical weights e^{-S} with action given by

$$S = S_{\rm vortex}^{(2n)}[x_j^+, x_j^-] + S_0[\theta_{\rm sw}],$$

with $S_0[\theta_{\rm sw}]$ the Gaussian action $\frac{\kappa}{2}\int d^2x (\nabla\theta_{\rm sw})^2$. We still assume a short-distance cut-off a.

IIIA-1 Write the expression of the action $S_{\rm vortex}^{(2n)}[x_j^+, x_j^-]$ for a collection of n pairs of vortices at positions $x_j^\pm$, $j = 1, \cdots, n$.

IIIA-2 Argue that the partition function of the gas of vortex pairs is given by the product $Z = Z_{\rm sw} \times Z_{\rm vortex}$ with $Z_{\rm sw}$ the partition function for the Gaussian free field $\theta_{\rm sw}$ and

$$Z_{\rm vortex} = \sum_{n\geq 0} \frac{\mu^{2n}}{n!\cdot n!} \times \int (\prod_{j=1}^n d^2x_j^+ \prod_{j=1}^n d^2x_j^-)\, \frac{\prod_{i<j}(|x_i^+ - x_j^+|/a)^{2\pi\kappa}(|x_i^- - x_j^-|/a)^{2\pi\kappa}}{\prod_{i,j}(|x_i^+ - x_j^-|/a)^{2\pi\kappa}},$$

with $\mu = \big(\frac{a_0}{a}\big)^{\pi\kappa} e^{-\beta\epsilon_c}$.

IIIA-3 The aim of the following questions is to express $Z_{\rm vortex}$ as a path integral over an auxiliary bosonic field φ. Let $\tilde S_\kappa[\varphi] = \frac{1}{2\kappa}\int d^2x (\nabla\varphi)^2$ be a Gaussian action. Show that, computed with this Gaussian action,

$$\langle e^{i2\pi\varphi(x)} e^{-i2\pi\varphi(y)}\rangle_{\tilde S_\kappa} = \frac{1}{|x-y|^{2\pi\kappa}}.$$

Hint: The Green function associated to the action $\tilde S_\kappa[\varphi]$ is $G(x,y) = -\frac{\kappa}{2\pi}\log\big(|x-y|/a\big)$.

IIIA-4 What is the scaling dimension (computed with the Gaussian action $\tilde S_\kappa[\varphi]$) of the operators $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$?
Deduce that the perturbation $\cos(2\pi\varphi)$ is relevant for $\pi\kappa < 2$ and irrelevant for $\pi\kappa > 2$.
Is the the perturbation $(\nabla\varphi)^2$ relevant or irrelevant?

IIIA-5 Show that $Z_{\rm vortex}$ can be written as the partition function of Gaussian bosonic field with action $S_{sG}[\varphi]$,

$$Z_{\rm vortex} = \int [D\varphi]\, e^{-S_{sG}[\varphi]},$$

where the action S_{sG} is defined as

$$S_{sG}[\varphi] = \int d^2x \Big[\frac{1}{2\kappa}(\nabla\varphi)^2 - 2\mu\cos(2\pi\varphi)\Big].$$

This is called the sine-Gordon action.
Hint: Compute perturbatively the above partition function as a series in μ while paying attention to combinatorial factors.

Correction :

IIIA-1 Vortices with charge + are at positions x_j^+, those of charge − are at positions x_j^- with $j = 1, \cdots, n$. Hence, the total sum of the charges vanishes, and

$$S^{(2n)}_{\text{vortex}}[x_j^+, x_j^-] \;=\; -2\pi\kappa \sum_{i<j} \log\Big[\frac{|x_i^+ - x_j^+||x_i^- - x_j^-|}{a_0\, a_0}\Big] + 2\pi\kappa \sum_{i\le j} \log\Big[\frac{|x_i^+ - x_j^-|}{a_0}\Big] + 2n\beta\,\epsilon_c.$$

IIIA-2 Since the statistical weights of vortex configurations and of the spin waves θ_{sw} factorize, it is clear that the partition function factorizes as $Z = Z_{\text{rm}} \times Z_{\text{vortex}}$. Next we rewrite $S^{(2n)}_{\text{vortex}}$ in term of lattice cut-off a: this amounts to replace a_0 by a in the log's and to add the term $-2n\,\pi\kappa\,\log(\frac{a}{a_0})$. The vortex gas partition function is then

$$Z_{\text{vortex}} = \sum_{n\ge 0} \frac{1}{n!} \times \frac{1}{n!} \times \int \Big(\prod_{j=1}^{n} d^2x_j^+ \prod_{j=1}^{n} d^2x_j^-\Big)\, e^{-S^{(2n)}_{\text{vortex}}[x_j^+, x_j^-]},$$

where the factor $\frac{1}{n!} \times \frac{1}{n!}$ comes from the indistinguishability of the vortices of given charge. This proves the claim.
IIIA-3 Since $\langle \varphi(x)\varphi(0)\rangle = -\frac{\kappa}{2\pi} \log\big(|x-y|/a\big)$ (w.r.t the Gaussian theory with action $\tilde{S}_\kappa[\varphi] = \frac{1}{2\kappa}\int d^2x (\nabla\varphi)^2$), we get the result.
IIIA-4 W.r.t $\tilde{S}_\kappa[\varphi]$, the operator $\cos(2\pi\varphi)$ has scaling dimension $\pi\kappa$. It is relevant (in dimension $D = 2$) for $\pi\kappa < 2$ and irrelevant for $\pi\kappa > 2$.
The operator $(\nabla\varphi)^2$ has dimension 2 and it is marginal. Its RG behaviour depends on the details of the perturbation. If this is only perturbing operator, it is exactly marginal. If $(\nabla\varphi)^2$ is accompanied with other perturbing operators, then it may be relevant or irrelevant depending on the OPE structure.
IIIA-5 We expand the sine-Gordon partition function is Taylor series in μ. By charge conservation, only the even terms in the expansion are non-vanishing. Hence,

$$Z_{\text{sG}} = \sum_n \frac{(2\mu)^{2n}}{(2n)!} \,\langle \Big[\int d^2x\, \cos(2\pi\varphi)\Big]^{2n} \rangle_{\tilde{S}_\kappa},$$

where the expectations are computed using the Gaussian action $\tilde{S}_\kappa$. In these expectations, only the charge neutral combinations contribute. The terms of order $2n$ involve n operators $e^{+i2\pi\varphi}$, with charge +, and n operators $e^{-i2\pi\varphi}$, with charge −. Integrating over the positions of these charges and taking into account the combinatorial factor (corresponding to choose n amongst $2n$) we get

$$Z_{\text{sG}} = \sum_n \frac{(2\mu)^{2n}}{(2n)!} \binom{2n}{n} \frac{1}{2^{2n}} \int \Big(\prod_{j=1}^{n} d^2x_j^+ d^2x_j^-\Big) \langle \prod_j e^{+i2\pi\varphi(x_j^+)}\, e^{-i2\pi\varphi(x_j^-)} \rangle_{\tilde{S}_\kappa}.$$

This proves the result.

- *The XY field theory: The renormalization group analysis*

IIIB-1 We now study the renormalization group flow of the action S_{sG} for κ close to the critical value $\kappa_c = 2/\pi$. We let $\kappa^{-1} = \kappa_c^{-1} - \delta\kappa$ and write

$$S_{sG}[\varphi] = \tilde{S}_{\kappa_c}[\varphi] - \int d^2x \Big[\frac{1}{2}(\delta\kappa)(\nabla\varphi)^2 + 2\mu\cos(2\pi\varphi)\Big]$$

Show that, to lowest order, the renormalization group equations for the coupling constants $\delta\kappa$ and μ are of the following form:

$$\begin{aligned} (\dot{\delta\kappa}) = \ell\partial_\ell\,(\delta\kappa) \;&=\; b\,\mu^2 + \cdots \\ \dot{\mu} = \ell\partial_\ell\,\mu \;&=\; a\,(\delta\kappa)\,\mu + \cdots \end{aligned}$$

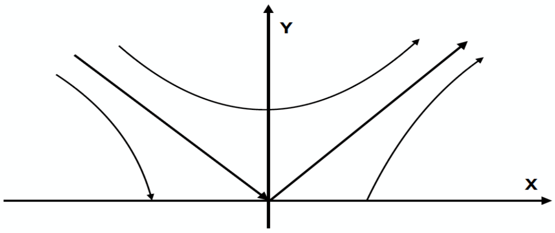

Figure 1: The XY RG flow.

with a and b some positive numerical constants.
Hint: It may be useful to first evaluate the OPE of the fields $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$.

IIIB-2 We redefine the coupling constants and set $X = a\,(\delta\kappa)$ and $Y = \sqrt{ab}\,\mu$ such that the RG equations now reads $\dot{X} = Y^2$ and $\dot{Y} = XY$.
Show that $Y^2 - X^2$ is an invariant of this RG flow.
Draw the RG flow lines in the upper half plane $Y > 0$ near the origin.

IIIB-3 We look at the flow with initial condition $X_I < 0$ and Y_I.
Show that if $Y_I^2 - X_I^2 < 0$ and $X_I < 0$, then the flow converges toward a point on the line $Y = 0$. Deduce that for such initial condition the long distance theory is critical. Compare with section I-B.

IIIB-4 Show that if $Y_I^2 - X_I^2 > 0$ and $X_I < 0$, the flow drives X and Y to large values.
Let $Y_0^2 = Y_I^2 - X_I^2$ with $Y_0 > 0$. Show that the solution of the RG equations are

$$\log\big(\frac{\ell}{a}\big) = \frac{1}{Y_0}\Big[\arctan\big(\frac{X(\ell)}{Y_0}\big) - \arctan\big(\frac{X_I}{Y_0}\big)\Big].$$

IIIB-5 The initial condition X_I and Y_I are smooth functions of the temperature T of the XY model. The critical temperature T_c is such that $X_I + Y_I = 0$. We take the initial condition to be near the critical line $X_I + Y_I = 0$ with $X_I < 0$. We let $X_I = -Y_I(1+\tau)$ in which $\tau \ll 1$ is interpreted at the distance from the critical temperature: $\tau \propto (T - T_c)$.
For $\tau > 0$, we define the correlation length as the length ξ at which $X(\ell)$ is of order 1.
Why is this a good definition?
Show that

$$\xi/a \simeq \text{const. } e^{\text{const.}/\sqrt{\tau}}.$$

Correction :

IIIB-1 W.r.t to $\tilde{S}_{\kappa_c}$ both operators $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$ are marginal. To lowest order the beta-functions are given by the OPE coefficients. These OPE are of the form:

$$\begin{aligned}(\nabla\varphi)^2 \times \cos(2\pi\varphi) &\Longrightarrow \cos(2\pi\varphi) + \text{irrelevant}\\ \cos(2\pi\varphi) \times \cos(2\pi\varphi) &\Longrightarrow (\nabla\varphi)^2 + \text{irrelevant}\end{aligned}$$

This implies the structure of the beta-function given in the text. The coefficients are positive because these OPE coefficients are positive (one also has to take into account the signs introduced when defining the perturbed action).
IIIB-2 $Y^2 - X^2$ is proved to be a constant of the RG flow by computing its derivative.

See the picture for a representation of the flow lines.
IIIB-3 If $Y_I^2 - X_I^2 < 0$ and $X_I < 0$, the picture shows that the flow converges to the axis $Y = 0$. Alternatively, $\dot{Y} = XY$ implies that Y decreases until the flow reaches $Y = 0$ with $X = -\sqrt{X_I^2 - Y_I^2}$. The large distance behaviour is critical because the theory at $Y = 0$ is a massless Gaussian theory which is critical. It corresponds to the low temperature phase of the XY model discussed in section IB.
IIIB-4 If $Y_I^2 - X_I^2 > 0$ and $X_I < 0$, the picture shows that the flow drives X and Y to large values (along the curve $Y^2 - X^2 = \text{const}$).
Using the fact that $Y^2 - X^2 = Y_0^2$ is an invariant of the RG flow, the latter can be written as $\dot{X} = Y_0^2 + X^2$, or alternatively $\frac{d\ell}{\ell} = \frac{dX}{Y_0^2 + X^2}$. The solution given in the text is checked by computing its ℓ derivative (using that $d \arctan x = dx/(x^2+1)$.
IIIB-5 For $X_I < 0$ and $X_I = -Y_I(1+\tau)$, we get $Y_0 = Y_I \sqrt{2\tau}$, and $Y_0 \to 0$ as $\tau \to 0$. At the length scale $\ell = \xi$ the correlation length, $X(\ell)$ is of order one, as is its initial value X_I, and thus $X(\ell)/Y_0 \to \infty$ and $X_I/Y_0 \to -\infty$ as $\tau \to 0$. Hence, from the explicit solution above we get

$$\log(\xi/a) = \pi/Y_0 = (\pi/Y_I) \times (1/\sqrt{2\tau}),$$

as claimed in the text.
The correlation diverges as $\xi \simeq e^{\text{const}/\sqrt{T-T_c}}$ near the transition. The transition is of infinite order. This is the Kosterlitz-Thouless (KT) transition.