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Statistical Field Theory and Applications : An Introduction for (and by) Amateurs

Exercice Book - 2020

(without corrections)

by

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Contents

1 Exercise corrections

1.1 Chapter 2: Brownian motions and random paths

• *Exercise 2.1: Random variables and generating functions.*

Let X be a real random variable. Let its characteristic function (also called generating function) be defined by

$$\Phi(z) := \mathbb{E}[\mathrm{e}^{\mathrm{i}zX}]\,.$$

We assume henceforth $z \in \mathbb{R}$.

(i) Show that $\Phi(z)$ is always well defined for $X \in \mathbb{R}$ and $z \in \mathbb{R}$.

(ii) Define also the function

$$W(z) := \log \Phi(z)$$

or conversely $\Phi(z) = \mathrm{e}^{W(z)}$.

Expand Φ and W in powers of z and identify the first few Taylor coefficients.

(iii) Suppose that X is an integer-valued discrete random variable having the Poisson distribution

$$\mathbb{P}[X = n] = \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}$$

for $n \in \mathbb{N}$, with parameter λ.

What are its mean, its covariance and its generating function?

(iv) Suppose now that X is a Gaussian variable with probability distribution density

$$\mathbb{P}\left[X \in [x, x + \mathrm{d}x]\right] = \frac{\mathrm{d}x}{\sqrt{2\pi\sigma}}\mathrm{e}^{-\frac{x^2}{2\sigma}}\,.$$

Verify that $\mathbb{P}$ is correctly normalised and compute its characteristic function.

• *Exercise 2.2: Random Gaussian vectors.*

Let $\vec{X}$ be an N-dimensional Gaussian random vector with real coordinates X^i, for $i = 1, \ldots, N$. By definition its probability distribution is

$$\mathbb{P}(X)\mathrm{d}^N X = \mathrm{d}^N X \left(\frac{\det G}{(2\pi)^N}\right)^{1/2} \exp\left(-\frac{1}{2}\langle X|G|X\rangle\right)\,,$$

with $\langle X|G|X\rangle := \sum_{ij} X^i G_{ij} X_j$, where the real symmetric form G_{ij} is supposed to be non-degenerate. Denote by $\hat{G}$ its inverse: $\sum_j G_{ij}\hat{G}^{jk} = \delta_i^k$.

(i) Verify that this distribution is normalised, that is:

$$\int \frac{\mathrm{d}^N X}{(2\pi)^{N/2}}\,\mathrm{e}^{-\frac{1}{2}\langle X|G|X\rangle} = (\det G)^{-1/2}\,.$$

(ii) For a vector U living in the dual space with respect to X, we define $\langle U|X\rangle = \sum_i U_i X^i$. Show that the corresponding generating function is

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle U|X\rangle}] = \mathrm{e}^{-\frac{1}{2}\langle U|\hat{G}|U\rangle}\,.$$

(iii) Show that the mean $\mathbb{E}[X^i] = 0$ and the covariance $\mathbb{E}[X^i X^j] = \hat{G}^{ij}$.

• *Exercise 2.3: The law of large number and the central limit theorem.*

The aim of this exercise is to prove (a simplified version of) the central limit theorem. Let ϵ_k, with $k = 1, \ldots, n$, be independent identically distributed (ii.d) variables. Each $\epsilon_k = \pm 1$ with equal probabilities.

In this case, the central limit theorem states that the sum $\hat{S}_n = \frac{1}{\sqrt{n}} \sum_k \epsilon_k$ converges (to be precise, in law) in the $n \to \infty$ limit to a Gaussian variable.

(i) Prove that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}z\hat{S}_n}] = \Big[\cos\big(\frac{z}{\sqrt{n}}\big)\Big]^n \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{-\frac{z^2}{2}}\,,$$

and conclude.
Hint: Recall the Taylor expansion $\cos(\frac{z}{\sqrt{n}}) = 1 - \frac{z^2}{2n} + o(\frac{1}{n})$ and use $\lim_{n\to\infty}[1 - \frac{y}{n} + o(\frac{1}{n})]^n = e^{-y}$ (which can be proved by taking the logarithm).

• *Exercise 2.4: Free random paths.*

The scaling limit of free random paths has been treated in section 2.2 of the main text. We recall that such paths are defined on a hypercubic lattice in D dimensions. Each step can be written $\pm a\mathbf{e}_j$, where $\mathbf{e}_j$ for $j = 1, \ldots, D$ is a basis of orthonormal vectors in $\mathbb{R}^D$, and is associated with a Boltzmann weight (fugacity) μ. The partition function for paths going from 0 to $\mathbf{x}$ is

$$Z(\mathbf{x}) = \sum_{\Gamma:0\to\mathbf{x}} \mu^{|\Gamma|}\,,$$

where $|\Gamma|$ denotes the length of the path Γ. It satisfies the difference equation

$$Z(x) = \delta_{x;0} + \mu \sum_{j=1}^{D} \big(Z(x + a\mathfrak{e}_j) + Z(x - a\mathfrak{e}_j)\big). \tag{1}$$

(i) Compute the Fourier transform of $Z(x)$ and prove that

$$Z(x) = \int_{\mathrm{BZ}} \frac{\mathrm{d}^D\mathbf{k}}{(2\pi/a)^D} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}}{1 - 2\mu\sum_j \cos(a\mathbf{k}\cdot\mathbf{e}_j)}\,, \tag{2}$$

where $\mathrm{BZ} = \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]^D$ is the Brillouin zone of the square lattice.

(ii) Let $\Delta^{\text{dis.}}$ be the discrete Laplacian and write $\Delta^{\text{dis.}} = \Theta - 2D\,\mathbb{I}$ with Θ the lattice adjacency matrix and $\mathbb{I}$ the identity matrix. We view Θ as acting on functions via $(\Theta \cdot f)(x) = \sum_{j=1}^{D} \big(f(x + a\mathfrak{e}_j) + f(x - a\mathfrak{e}_j)\big)$. Show that:

$$Z(x) = \langle x|\, \frac{1}{\mathbb{I} - \mu\,\Theta}\, |0\rangle,$$

with $|x\rangle$ the δ-function peaked at x, i.e. $\langle y|x\rangle = \delta_{y;x}$.
Give an expression of $W_N^{\text{free}}(\mathbf{x}$ as matrix elements of powers of the matrix Θ and give a geometrical interpretation of this formula.

(iii) Deduce from this formula that $Z(x)$ converges for $|\mu| < \mu_c$ with $\mu_c = 1/2D$.

(iv) Prove the formula for the Green function $G(x)$ given in the main text.

- *Exercise 2.5: Computation of a path integral Jacobian determinants.*

The aim of this exercise is to compute the determinant $\text{Det}[\partial_t - A(t)]$ of the linear map acting functions $f(t)$ as follows $f(t) \to (Jf)(t) = f'(t) - A(t)f(t)$ with $A(t)$ a given function. Instead of computing directly this determinant we factorize the derivation operator and we write $\text{Det}[\partial_t - A(t)] := \text{Det}[\partial_t] \times \text{Det}[1 - K]$. The operator K is defined by integration as follows:

$$K : f(t) \to (Kf)(t) = \int_0^t ds A(s) f(s),$$

for any function f defined on the finite interval $[0, T]$. The aim of this exercise is thus to compute the determinant $\text{Det}[1 - K]$ and to prove that

$$\text{Det}[1 - K] = e^{-\alpha \int_0^T ds\, A(s)},$$

with α a parameter depending on the regularization procedure ($\alpha = 0$ for Itô and $\alpha = \frac{1}{2}$ for Stratonovich conventions). This illustrates possible strategy to define and compute functional –infinite dimensional– determinants.

To define the determinant $\text{Det}[1 - K]$ we need to discretize it by representing the integral of any function by a Riemann sum. Let us divide the interval $[0, T]$ in N sub-interval $[n\delta, (n+1)\delta]$ with $n = 0, \cdots N-1$ and $\delta = T/N$. We will then take the limit $N \to 0$. To simplify notation we denote $f_n := f(n\delta)$. There are many possible discretizations but we shall only consider two of them (which correspond to the Itô and Stratonovich conventions):

$$\text{Ito} \quad : \quad \int_0^t f(t)dt := \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} f_k,$$

$$\text{Stratonovich} \quad : \quad \int_0^t f(t)dt := \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} \frac{1}{2}(f_k + f_{k+1}).$$

(i) Write the regularized action of the operator K on function f by writing the expression of $(Kf)_n$.

(ii) Show that the operator $1-K$ is lower triangular and determine the diagonal entries (which are convention dependent).

(iii) Deduce, by taking the large N limit, the formula for the determinant:

$$\begin{aligned}\text{Ito} \quad &: \quad \mathrm{Det}[1-K] = 1, \\ \text{Stratonovich} \quad &: \quad \mathrm{Det}[1-K] = e^{-\frac{1}{2}\int_0^T ds\, A(s)}.\end{aligned}$$

More general discretization are defined by sampling differently the Riemann sum as follows: $\int_0^t f(t)dt = \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} ((1-\alpha)f_k + \alpha f_{k+1})$. Following the same strategy as above, it is then clear that $\mathrm{Det}[1-K] = e^{-\alpha \int_0^T ds\, A(s)}$.

• *Exercise 2.6: Levy's construction of the Brownian motion.*

The path integral representation is actually closely related to an older (!) construction of the Brownian motion due to P. Levy. The aim of this exercise is to present the main point of Levy's approach which constructs the Brownian paths by recursive dichotomy.

We aim at constructing the Brownian curves on the time interval $[0,T]$ starting point x_0. The construction is recursive:

(a) First, pick the end point x_T with the Gaussian probability density $\frac{dx_T}{\sqrt{2\pi T}}\, e^{-(x_T-x_0)^2/2T}$ and draw (provisionally) a straight line from x_0 to x_T.

(b) Second, construct the intermediate middle point $x_{T/2}$ at time $T/2$ by picking it randomly from the Gaussian distribution centered around the middle of the segment joining x_0 to x_T, and with the appropriate covariance to be determined. Then, draw (provisionally) two straight lines from x_0 to $x_{T/2}$ and from $x_{T/2}$ to x_T.

(c) Next, iterate by picking the intermediate points at times $T/4$ and $3T/4$, respectively, from the Gaussian distribution centered around the middle point of the two segments drawn between x_0 and $x_{T/2}$ and between $x_{T/2}$ and x_T, respectively, and with the appropriate covariance. Then draw (provisionally) all four segments joining the successive points x_0, $x_{T/4}$, $x_{T/2}$, $x_{3T/4}$ and $x_{T/2}$.

(d) Iterate ad infinitum...

Show that this construction yields curves sampled with the Brownian measure.
Hint: This construction works thanks to the relation

$$\frac{(x_i-x)^2}{2(t/2)} + \frac{(x-x_f)^2}{2(t/2)} = \frac{(x_i-x_f)^2}{2t} + \frac{(x-\frac{x_i+x_f}{2})^2}{2(t/4)}$$

• *Exercise 2.7: The over-damped limit of the noisy Newtonian particle.*

Consider Newton's equation for a particle of mass m subject to a friction and random forcing (white noise in time). That is, consider the SDEs:

$$dX_t = \frac{P_t}{m}\, dt, \quad dP_t = -\gamma\, dX_t + dB_t,$$

with X_t the position and P_t the momentum. We are interested in the limit $m \to 0$ (or equivalently γ large). Let us set $m = \epsilon^2$ to match the Brownian scaling. Then show that:

(i) the process γX_t^ϵ converges to a Brownian motion B_t;

(ii) $Y_t^\epsilon := \epsilon\, \dot{X}_t^\epsilon$ converges to a finite random variable with Gaussian distribution.

That is: Introducing the mass, or ϵ, is a way to regularize the Brownian curves in the sense that X_t^ϵ admits a time derivative contrary to the Brownian motion. But quantities such as Y_t^ϵ, which are naively expected to vanish in the limit $\epsilon \to 0$, actually do not disappear because the smallness of ϵ is compensated by the irregularities in $\dot{X}_t^\epsilon$ as $\epsilon \to 0$. For instance $\mathbb{E}[\frac{1}{2} m \dot{X}_t^2]$ is finite in the limit $m \to 0$. Such phenomena—the existence of naively zero but nevertheless finite quantities due to the emergence of irregular structures in absence of regularizing—are common in statistical field theory, and are (sometimes) called 'anomaly'.

- *Exercise 2.8: SDEs with 'multiplicative' noise.*

Generalize the results described above for a more general SDE of the form

$$dX_t = a(X_t)dt + b(X_t)dB_t$$

with $a(x)$ and $b(x)$ smooth non constant functions. To deal with the small noise limit one may introduce a small parametr ϵ by rescaling $b(x)$ via $b(x) \to \epsilon\, b(x)$.

(i) Prove that the Fokker-Planck operator for SDEs reads

$$\mathcal{H} = \partial_x \big(\frac{1}{2} \partial_x\, b^2(x) - a(x) \big)$$

(ii) Verify that the invariant measure (if normalizable) is

$$\mathbb{P}_{\rm inv}(x)\, dx = b^{-2}(x)\, e^{-2s(x)}\, dx, \quad s(x) := -\int^x dy \frac{a(y)}{b^2(y)}.$$

What is the invariant measure if the later is not normalizable?
What is then the physical interpretation of this new measure?

(iii) Show that the action of the path integral representation of these SDEs is

$$S = \frac{1}{2} \int_0^T ds\, \frac{(\dot{x}_s - a(x_s))^2}{b^2(x_s)}.$$

in the small noise limit $\epsilon \ll 1$. Verify (by going back to the discret formulation) that this way of writing the action is still valid away from the small noise limit provided that one carefully defined the integrals.

• *Exercise 2.9: Multivariable SDEs*

Generalize all these results for multivariable SDEs of the form $dX^i = a^i(X)\, dt + b^i_j(X)\, dB^j_t$ where B^j are Brownian motions with covariance $\mathbb{E}[B^i_t B^j_s] = \delta^{ij} \min(t, s)$.

1.2 Chapter 3: Statistical lattice models

- *Exercise 3.1: Fermionic representation of the 2D Ising model*

The aim of this exercise is to complete the study of the 2D Ising model presented in the lecture notes. Recall the definition of the 2D Ising model given in the text.

(i) Prove—or argue—that the Ising model is described by the Hamiltonian

$$\mathcal{H} = -\gamma \sum_{x=1}^{N_x} \tau_x^3 - \beta \sum_{x=1}^{N_x} \tau_x^1 \tau_{x+1}^1 ,$$

where $\gamma = \mathrm{e}^{-2K_y}$ and $\beta = K_x$ are related to the anisotropic coupling constants (K_x, K_y) in the completely anisotropic limit, $K_x \ll 1$ and $K_y \gg 1$.

(ii) Recall the Jordan-Wigner transformations given in the main text which construct fermionic operators in terms of Pauli matrices via

$$a_x = e^{i\pi \sum_{y=1}^{x-1} \tau_y^- \tau_y^+}\, \tau_x^+, \quad a_x^\dagger = e^{-i\pi \sum_{y=1}^{x-1} \tau_y^- \tau_y^+}\, \tau_x^- .$$

Show that we may alternatively write

$$a_x = \big(\prod_{y=1}^{x-1} \tau_y^z\big)\, \tau_x^+, \quad a_x^\dagger = \big(\prod_{y=1}^{x-1} \tau_y^z\big)\, \tau_x^- .$$

Verify that they satisfy the canonical fermionic relation $a_x^\dagger a_y + a_y a_x^\dagger = \delta_{x,y}$.

(iii) Show that the Hamiltonian becomes

$$\begin{aligned}\mathcal{H} &= -\gamma \sum_{x=1}^{N_x} \tau_x^3 - \beta \sum_{x=1}^{N_x} \tau_x^1 \tau_{x+1}^1 \\ &= \gamma \sum_{x=1}^{N_x} \left(a_x^\dagger a_x - a_x a_x^\dagger\right) - \beta \sum_{x=1}^{N_x} \left(a_x^\dagger - a_x\right) \left(a_{x+1}^\dagger + a_{x+1}\right) .\end{aligned}$$

(iv) Complete the proof of the diagonalisation of the Ising hamiltonian and its spectrum. Proof that, after an appropriate Bogoliubov transformation on the fermion operators, the Ising hamiltonian can be written in the final form given in the main text, which we recall here,

$$\mathcal{H} = \sum_{k>0} h_k \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger\right),$$

with single particle spectrum $h_k = \left[(\gamma-\beta)^2 + 4\gamma\beta \sin^2(k/2)\right]^{1/2}$.

- *Exercise 3.2: Spin operators, disorder operators and parafermions.*

The aim of this exercise—and the following two—is to study some simple consequences of group symmetry in lattice statistical models.

Let us consider a lattice statistical model on a two dimensional square lattice $\Lambda := a^2\mathbb{Z}^2$ with spin variables s on each vertex of the lattice. These variables take discrete or continuous values, depending on the models. We consider neighbour spin interactions with a local hamiltonian $H(s, s')$ so that the Boltzmann weight of any given configuration $[c]$ is

$$W([c]) := \prod_{[i,j]=\text{edge}} w_{[i,j]}, \quad w_{[i,j]} = e^{-H(s_i,s_j)},$$

where, by convention, $[i, j]$ denotes the edge connecting the vertices i and j. Let $Z := \sum_{[c]} W([c])$ be the partition function.

Let us suppose that a group G is acting the spin variables. We denote by R the corresponding representation. Furthermore we assume that the interaction is invariant under this group action so that, by hypothesis,

$$H(R(g) \cdot s, R(g) \cdot s') = H(s, s'), \quad \forall g \in G.$$

(i) *Transfer matrix:* Define and construct the transfer matrix for these models.

(ii) *Spin operators:* Spin observables, which we denote $\sigma(i)$, are defined as the local insertions of the spin variables at the lattice site i. That is: $\sigma(i)$ is the function which to any configuration associate the variable s_i.
Write the expectations of the spin observables $\langle\sigma(i_1)\cdots\sigma(i_N)\rangle$ as a sum over configurations weighted by their Boltzmann weights.
Write the same correlation functions in terms of the transfer matrix.

(iii) *Disorder operators:* Disorder observables are defined on the dual lattice and are indexed by group elements. Let Γ be a closed anti-clockwise oriented contour on the square lattice $\tilde{\Lambda}$ dual to Λ –the vertices of $\tilde{\Lambda}$ are the center of the faces of Λ. Let ℓ denote an oriented edge of Γ. It crosses an edge of Λ and we denote by ℓ^- and ℓ^+ the vertices of this edge with ℓ^- inside the loop Γ. The disorder observable $\mu_g(\Gamma)$ for $g \in G$ is defined as

$$\mu_\Gamma(g) := \exp\big(\sum_{\ell\in\Gamma}(H(s_{\ell^-}, s_{\ell^+}) - H(s_{\ell^-}, R(g)s_{\ell^+})\big),$$

Inserting $\mu_\Gamma(g)$ in the Boltzmann sum amounts to introduce a defect by replacing the hamiltonian $H(s_{\ell^-}, s_{\ell^+})$ by its rotated version $H(s_{\ell^-}, R(g)s_{\ell^+})$ on all edges crossed by Γ.
Write the expectations of disorder observables in terms of the transfer matrix.

• *Exercise 3.3: Symmetries, conservation laws and lattice Ward identities*

The aim of this exercise is to understand some of the consequences of the presence of symmetries. The relations we shall obtain are the lattice analogue of the so-called Ward identities valid in field theory.

We consider the same two dimensional lattice model as in previous exercise. We recall that we assume the Bolztmann weight to be invariant under a symmetry group G in the sense that

$$H(R(g)\cdot s, R(g)\cdot s') = H(s,s'), \quad \forall g\in G.$$

(i) Let i_k be points on the lattice Λ and Γ a contour as in previous exercise.. Show that the group invariance implies that

$$\langle \mu_\Gamma(g)\prod_k \sigma(i_k)\rangle = \prod_{i_k \text{ inside } \Gamma} R_{i_k}(g)\cdot\langle\prod_k\sigma(i_k)\rangle,$$

where $R_{i_k}(g)$ denote the group representation R acting on the spins at site i_k.

Show that $\mu_g(\Gamma)$ is invariant under any smooth continuous deformation of Γ as long as the deformation does not cross points of spin insertions (it is homotopically invariant).

We now look at the consequences of these relations for infinitesimal transformations. Suppose that G is a Lie group and Lie(G) its Lie algebra. Let us give a name to small variations of H by defining $\partial_X H$. For $g = 1+\epsilon\, X+\cdots$ with $X\in$ Lie(G), we set

$$H(s,R(g)s') - H(s,s') =: \epsilon\,\partial_X H(s,s') + \cdots .$$

For $\ell = [\ell^-,\ell^+]$ an oriented edge of Γ as in previous exercise and $X\in$ Lie(G), we let

$$*J^X_\ell := \partial_X H(s_{\ell^-}, s_{\ell^+}),$$

They are specific observables, called *currents*, whose correlation functions are defined as usual via insertion into the Bolztmann sums.

(ii) Show that the following equality holds:

$$\langle\sum_{\ell\in\Gamma} *J^X_\ell \cdot \prod_i \sigma(i)\rangle = \langle\Big(\sum_{i_k \text{ inside } \Gamma} R_{i_k}(X)\Big)\cdot\prod_i\sigma(i)\rangle,$$

if some spin observables are inserted inside Γ.

(iii) Deduce that, if there is no observables inserted inside Γ, then the following equality holds inside any expectation values:

$$\sum_{\ell\in\Gamma} *J^X_\ell = 0,$$

That is: The second of these two equations is a conservation law (i.e. it is the analogue of the fact that $\int *J = 0$ if $*J$ is a closed form, or equivalently, if J is a conserved current), the first tells about the consequences of this conservation law when insertion of observables are taken into account. It is analogous to the Gauss law in electrodynamics. They are called Ward identities in field theory.

1.3 Chapter 4: From statistical models to field theories

- *Exercise 4.1: Mean field from a variational ansatz*

The aim of this exercise is to derive the Ising mean field approximation form a variational ansatz. We consider the Ising in homogeneous external field h_i so that the configuration energy is $E[s] = -\sum_{i,j} J_{ij} s_i s_j - \sum_i h_i s_i$, with J_{ij} proportional to the lattice adjacency matrix. The Ising spins take values $s_i = \pm$. Let $Z[h]$ be its partition function. (Note that we introduce the external magnetic field with a minus sign).

As an ansatz we consider the model of independent spins in an effective inhomogeneous external field h_i^o with ansatz energy $E^o[s] = -\sum_i h_i^o s_i$, so that the ansatz Boltzmann weights are $Z_0^{-1}\, e^{\beta \sum_i h_i^o s_i}$ with Z_0 the ansatz partition function.

(i) Show that $Z_0 = \prod_i [2\cosh(\beta h_i^o)]$.

(ii) Using a convexity argument, show that $\mathbb{E}_0[e^{-X}] \geq e^{-\mathbb{E}_0[X]}$ for any probability measure $\mathbb{E}_0$ and measurable variable X.

(iii) Choose to be $\mathbb{E}_0$ the ansatz measure and $X = \beta(E[s] - E^o[s])$ to prove that

$$Z[h] \geq Z_0\, e^{-\beta \mathbb{E}_0[E[s] - \beta \mathbb{E}_0[E^o[s]]},$$

or equivalently, $F[h] \leq F_0 - \mathbb{E}_0[E^o[s] - E[s]]$, with $F[h]$ and F_0 the Ising and ansatz free energy respectively.

The best variational ansatz is that which minimizes $F_0 - \mathbb{E}_0[E^o[s] - E[s]]$.

(iv) Compute F_0, $\mathbb{E}_0[E^o[s]]$ and $\mathbb{E}_0[E[s]]$ and show that the quantity to minimize is

$$F_0[h^o] + \sum_i h_i^o \bar{m}_i - \sum_{ij} J_{ij} \bar{m}_i \bar{m}_j - \sum_i h_i \bar{m}_i,$$

where $\bar{m}_i = -\frac{\partial F_0[h^o]}{\partial h_i^o} = \tanh(\beta h_i^o)$ is the local mean magnetization evaluated with the ansatz measure. Show that this minimization problem reduces to the Ising mean field equations.

- *Exercise 4.2: Thermodynamic functions and thermodynamic potentials*

The aim of this exercise is to recall a few basic fact about generating functions, thermodynamic functions and their Legendre transforms.

Let us consider a (generic) spin model and let $E[\{s\}]$ be the energy of a spin configuration $\{s\}$ with local spin s_i. We measure the energy in unit of the temperature so that the Boltzmann weights are $e^{-\beta\, E[\{s\}]}$. Let $Z[0] = \sum_{\{s\}} e^{-\beta\, E[\{s\}]}$ be the partition function. In the following we set $\beta = 1$ (or alternatively include the β-dependence in the other dimensionfull parameter).

(i) Give the expression of the energy $E_h[\{s\}]$ in presence an external inhomogeneous external field h.

Show that the generating function for this spin correlation functions can written as (with $(s,h)=\sum_i s_i h_i$)

$$\mathbb{E}[e^{(s,h)}] = \frac{Z[h]}{Z[0]},$$

Explain why the partition function $Z[h]$ is the generating function for spin correlations.

(ii) Let $F[h]$ be the free energy and let $W[h] = -\big(F[h]-F[0]\big)$. Verify that

$$\log \mathbb{E}[e^{(s,h)}] = W[h].$$

(iii) Let $\Gamma(m)$ be the thermodynamic potential defined as the Legendre transform of $W[h]$. Recall that

$$\Gamma(m) = (m,h_*) - W[h_*], \quad \text{with } \frac{\partial W}{\partial h}[h_*] = m.$$

Verify that this transformation is inverted by writing

$$W[h] = (m_*,h) - \Gamma[m_*], \quad \text{with } \frac{\partial \Gamma}{\partial m}[m_*] = h.$$

- *Exercise 4.3: An alternative representation of the Ising partition function.*

The aim of this exercise is to explicitly do the computation leading to the representation of the Ising partition function in terms of a bosonic field. It uses a trick—representing the interaction terms via a Gaussian integral over auxiliary varaibles—which find echoes in many other problems.

(i) Prove the following representation of the Ising partition function given in the text (without looking at its derivation given there):

$$Z = \int [\prod_k d\phi_k]\ e^{-S[\phi;h]},$$

with the action

$$S[\phi;h] = -\frac{1}{4}\sum_{ij}\phi_i J_{ij}\phi_j + \sum_i \log[\cosh(h_i + \sum_j J_{ij}\phi_j)].$$

(ii) Deduce what is the representation of the Ising spin variables s_i in terms of the bosonic variables ϕ_i.

- *Exercise 4.4: Mean field vector models*

We consider a theory with a vector order parameter $\vec{m}$ of dimension d. We denote by D the dimension of the space. This theory is described by the Landau action

$$S[\vec{m}] = \int d^D r \left\{ \frac{1}{2} \sum_{i=1}^{d} (\vec{\partial}_r m_i)^2 + \frac{a}{2} \sum_{i=1}^{d} m_i^2 + \frac{b}{4} \left(\sum_{i=1}^{d} m_i^2 \right)^2 \right\} .$$

We suppose that the parameters take the form

$$\begin{aligned} a &= a_0 t + O(t^2) \,, \qquad a_0 > 0 \,, \\ b &= b_0 + O(t) \,, \qquad b_0 > 0 \,, \end{aligned}$$

where $t = (T - T_\mathrm{c})/T_\mathrm{c}$ denotes the reduced temperature.

(i) What is the norm m of the system's spontaneous magnetisation? We write $\vec{m} = m\vec{e}$, where $\vec{e}$ is the direction of the magnetisation.

(ii) We define the susceptibility—or correlation function—by

$$G_{ij}(r - r') = \left. \frac{\delta m_i(r)}{\delta h_j(r')} \right|_{h=0} .$$

Show that G is the inverse matrix of the Hessian ($\equiv$ the matrix of second derivatives) of the action (in the d-dimensional space of components of the order parameter). Compute the Fourier transform $g^{-1}(k)$ of $G^{-1}(r - r')$.

(iii) We introduce the longitudinal projector

$$P^L_{ij} = e_i e_j$$

and the transverse projector

$$P^T_{ij} = \delta_{i,j} - e_i e_j \,.$$

Invert the matrix $g^{-1}(k)$ to obtain $g(k)$. Deduce an expression for $G_{ij}(r - r')$. What are the correlation lengths of the two different modes?

(iv) <u>Optional:</u>What happens in the presence of an external field?

1.4 Chapter 5: The renormalisation group and universality

- *Exercise 5.1: Real-space renormalisation: Ising model on the triangular lattice*

We consider the Hamiltonian of the two-dimensional Ising model

$$\beta H(\{S_i\}) = -N\ J_0 - J_1 \sum_{\langle i,j\rangle}^{N} S_i S_j - J_2 \sum_{i=1}^{N} S_i, \tag{3}$$

where the symbol $\langle i,j\rangle$ represents the pairs of nearest neighbours on the triangular lattice, shown in Figure 1. The mesh size of the lattice is a.

In this exercise we consider a transformation of the renormalisation group in real space, which consists of creating blocks of spins σ_α, where α indexes the block. The spins at the three vertices of a black triangle in Figure 1, such as S_1, S_2, S_3, form one block spin according to the rule

$$\sigma_\alpha = \text{sign}\big(S_1 + S_2 + S_3\big)\,. \tag{4}$$

It is seen that each spin S belongs to one and only one block α.

(i) What is the mesh size of the new triangular lattice formed by the blocks α? To each configuration $\{\sigma_\alpha\}$ we associate all the configuration $C(\{\sigma_\alpha\})$ of the spins $\{S_i\}$ that verify the definition (4). What is the number of elements in C?

(ii) We define a Hamiltonian $\mathcal{H}$ on the block spins $\{\sigma_\alpha\}$ by decimating the spins $\{S_i\}$ that belong to $C(\{\sigma_\alpha\})$:

$$\mathcal{H}(\{\sigma_\alpha\}) = -\frac{1}{\beta}\ \log\left[\sum_{\{S_i\}\in C(\{\sigma_\alpha\})} \exp\left(-\beta\ H(\{S_i\})\right)\right].$$

Show that the Hamiltonian $\mathcal{H}(\sigma_1,\sigma_2,\sigma_3)$ for the lattice of $N=9$ spins S_i (with $i=1,2,\ldots,9$) shown in Figure 1 can be written in the form

$$\begin{aligned} -\beta\mathcal{H}(\sigma_1,\sigma_2,\sigma_3) &= A_{123} + A_{12}\sigma_3 + A_{23}\sigma_1 + A_{13}\sigma_2 \\ &+ A_1\sigma_2\sigma_3 + A_2\sigma_1\sigma_3 + A_3\sigma_1\sigma_2 + A\sigma_1\sigma_2\sigma_3\,, \end{aligned}$$

where the $A_{\ldots}$ are constants.

(iii) Show next that the $A_{\ldots}$ are calculable starting from $H(\{S_i\})$ — at least in principle, that is, by performing sums over a large number of terms. Argue that all the $A_{\ldots}$ are non zero; in particular, the renormalised Hamiltonian $\mathcal{H}(\{\sigma_\alpha\})$ contains interactions which are not present in $H(\{S_i\})$. One may corroborate these arguments by explicit computations using symbolic algebra software such as MATHEMATICA or MAPLE.

(iv) We define the Hamiltonian

$$\beta H_0(\{S_i\}) = -N\ J_0 - J_1 \sum_{\alpha=1}^{N/3} \sum_{<i,j>\in\alpha} S_i S_j\,, \tag{5}$$

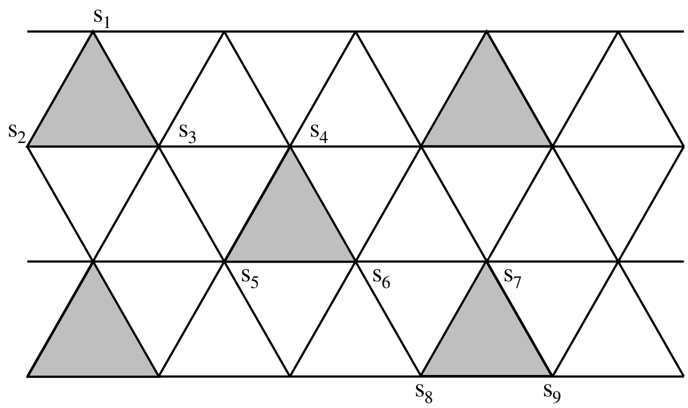

Figure 1: Block spins on the triangular lattice.

where the second sum is over pairs of nearest neighbour sites belonging to the same block α. We also denote by $\langle A\rangle_{\tilde{0}}$ the mean value of the observable A with the Hamiltonian H_0 for a fixed configuration $\{\sigma_\alpha\}$:

$$\begin{aligned}
\langle A\rangle_{\tilde{0}} &= \frac{1}{Z_{\tilde{0}}} \sum_{\{S_i\}\in C(\{\sigma_\alpha\})} A(\{S_i\})\ \exp\left(-\beta H_0(\{S_i\})\right) \\
Z_{\tilde{0}} &= \sum_{\{S_i\}\in C(\{\sigma_\alpha\})} \exp\left(-\beta H_0(\{S_i\})\right) .
\end{aligned}$$

Show that

$$\exp\left(-\beta\mathcal{H}(\{\sigma_\alpha\})\right) = Z_{\tilde{0}}\ \langle \exp\left(-\beta(H-H_0)\right)\rangle_{\tilde{0}}$$

Use the convexity of the exponential to deduce the following inequality

$$\beta\mathcal{H}(\{\sigma_\alpha\}) \le -\log Z_{\tilde{0}} + \beta\langle H-H_0\rangle_{\tilde{0}} \,. \tag{6}$$

(v) Compute $Z_{\tilde{0}}$. Show that

$$\langle S_i\rangle_{\tilde{0}} = \frac{e^{3J_1}+e^{-J_1}}{e^{3J_1}+3\,e^{-J_1}}\,\sigma_\alpha\,,$$

where α denotes the block containing the site i. Deduce from this the value of $\langle H-H_0\rangle_{\tilde{0}}$.

(vi) Establish the real-space renormalisation group transformation

$$\begin{aligned}
J_1' &= 2\ J_1 \left(\frac{e^{3J_1}+e^{-J_1}}{e^{3J_1}+3\,e^{-J_1}}\right)^2 , && (7)\\
J_2' &= 3\ J_2 \left(\frac{e^{3J_1}+e^{-J_1}}{e^{3J_1}+3\,e^{-J_1}}\right) , && (8)\\
J_0' &= 3\ J_0 + \log(e^{3J_1}+3\,e^{-J_1})\,, && (9)
\end{aligned}$$

by approximating $\mathcal{H}$ by the upper bound (**??**). What are the fixed points (J_1^*, J_2^*) of the flow in the space of the two coupling constants? Study their stability.

(vii) Compute the critical exponents at the non-trivial fixed point determined in the preceding question. One proceeds by linearisation of the renormalisation group flows around the fixed point.

• *Exercise 5.2: Correction to scaling.*

The aim of this exercise is to understand how the irrelevant variables induce sub-leading corrections to scaling behaviours. To simplify matter, let us suppose that the critical system possesses only one relevant scaling variable, say u_t with RG eigen-value $y_t > 0$, and one irrelevant variable, say $u_{\rm irr}$ with RG eigen-value $y_{\rm irr} < 0$. (Of course generic physical systems have an infinite number of irrelevant variables but considering only one will be enough to understand their roles).

(i) By iterating RG transformations as in the main text, show that the singular part of the free energy can be written as

$$f_{\rm sing} = |u_t|^{D/y_t}\, \varphi_\pm(u^0_{\rm irr}\, |u_t|^{|y_{\rm irr}|/y_t}),$$

where $\varphi_\pm$ are functions possibly different for $u_t > 0$ or $u_t < 0$, and $u^0_{\rm irr}$ is the initial value (before RG transformations) of the irrelevant coupling.

(ii) Argue (without formal proof) that the functions $\varphi_\pm$ may raisonably be expected to be smooth. Under this assumption, prove that

$$f_{\rm sing} = |u_t|^{D/y_t}\, \big(A_0 + A_1\, u^0_{\rm irr}\, |u_t|^{|y_{\rm irr}|/y_t} + \cdots\big),$$

where A_0 and A_1 are non-universal constants.

• *Exercise 5.3: Change of variables and covariance of RG equations.*

Let us consider a theory with a finite number of relevant coupling constants that we generically denote $\{g^i\}$. Let us write the corresponding beta functions as (no summation in the first term)

$$\beta^i(g) = y_i g^i - \frac{1}{2}\sum_{jk} C^i_{jk} g^j g^k + \cdots .$$

(i) Prove that, if all y_i are non-vanishing, then there exist a change of variables from $\{g^i\}$ to $\{u^i\}$, with $u^i = g^i + O(g^2)$, which diagonalizes the beta functions, up to two loops, i.e. such that $\beta^i(u) = y_i u^i + O(u^3)$.

(ii) Prove that, if all y_i are zero, then the second and third Taylor coefficient are invariant under a change of variables from $\{g^i\}$ to $\{u^i\}$, with $u^i = g^i + O(g^2)$.
That is: For marginal perturbation, the second and third loop beta function coefficients are independent on the renormalization scheme (alias on the choice of coordinate in the coupling constant space).

(iii) Let expand the beta functions to all orders in the coupling constants:

$$\beta^i(g) = y_i g^i - \sum_{n>0} \sum_{j_1,\cdots,j_n} C^i_{j_1,\cdots,j_n} g^{j_1} \cdots g^{j_n}.$$

Prove that, if there is no integers p_i, p_j such that $p_i y_i - p_j y_j \in \mathbb{Z}$, for $i \neq j$ (in such cases, one says they that there is non resonances), then there exists a change of variables from $\{g^i\}$ to $\{u^i\}$, with u^i a formal power series in the g^i's, with $u^i = g^i + O(g^2)$, which diagonalizes the beta functions as a formal power series in the u^i's. That is: There exist scaling variables, at least as formal power series.

1.5 Chapter 6: Free field theory

- *Exercise 6.1: Translation invariance and the stress-tensor*

The aim of this exercise is to see some aspect of the relation between translation invariance and the stress-tensor. Let us consider classical scalar field theory with Lagrangian $\mathcal{L}[\phi, \partial\phi]$ and action $S[\phi] = \int d^D x\, \mathcal{L}[\phi, \partial\phi]$. Recall that maps extremalizing this action are said to be solution of the classical equations of motion, which reads

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)(x)} \Big) = \frac{\partial \mathcal{L}}{\partial \phi(x)}.$$

These equations are the Euler-Lagrange equations.

(i) Consider an infinitesimal field transformation $\phi(x) \to \phi(x) + \epsilon(\delta\phi)(x)$. Suppose that, under such transformation the Lagrangian variation is $\delta\mathcal{L}[\phi, \partial\phi] = \epsilon \partial_\mu G^\mu$ so that the action is invariant. Show that the following Noether current

$$J^\mu = (\delta\phi)\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - G^\mu,$$

is conserved on solutions of the equations of motion.

(ii) Let us look at translations $x \to x - \varepsilon\, a$. How does a scalar field ϕ transforms under such translation? Argue that if the Lagrangian density is a scalar, then $\delta\mathcal{L} = \varepsilon\, a^\mu \partial_\mu \mathcal{L}$. Deduce that the action is then translation invariant and that associated conserved Noether current is $J^\mu_a = T^\mu_\nu a^\nu$ with

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, (\partial_\nu \phi) - \delta^\mu_\nu\, \mathcal{L}.$$

This tensor is called the stress-tensor. It is conserved: $\partial_\mu T^\mu_\nu = 0$.

(iii) Find the expression of the stress-tensor T^μ_ν for a scalar field theory with action $S[\phi] = \int d^D x\, \big(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big)$.

- *Exercise 6.2: Lattice scalar field and lattice Green function*

Recall that lattice scalar free theory is defined by the action

$$S[\phi] = \frac{a^{D-2}}{2} \sum_x \phi_x \left[(-\Delta^{\rm dis} + a^2 m^2) \right] \phi_x,$$

where ϕ_x are the value of the field at point x on the lattice and $\Delta_{\rm dis}$ discrete Laplacian on that lattice. We here consider only D-dimensional square lattice of mesh size a, i.e. $a\mathbb{Z}^D$. Let us also recall that the Fourier transforms in $a\mathbb{Z}^D$ are defined by

$$\hat{\phi}_k = a^D \sum_{\mathbf{n}} e^{-ix\cdot k}\, \phi_x, \quad \phi_x = \int_{\rm BZ} \frac{d^D k}{(2\pi)^D}\, e^{ix\cdot k}\, \hat{\phi}_k$$

where the integration is over the Brillouin zone, which is the hyper-cube $\mathrm{BZ} \equiv [-\frac{\pi}{a}, \frac{\pi}{a}]^D$.

(i) Verify that the Laplacian acts diagonally in the Fourier basis, with

$$(-\Delta^{\rm dis} + a^2m^2)_k = 2\sum_\alpha \big(\eta - \cos(a\,k_\alpha)\big),$$

with $\eta = 1 + \frac{a^2m^2}{2D}$ and k_α the component of the momentum k in the direction α.

(ii) Verify that in the Fourier basis the free field action reads

$$S[\phi] = \frac{1}{2}\int_{\rm BZ} \frac{d^Dk}{(2\pi/a)^D}\,\hat\phi_{-k}(-\Delta^{\rm dis}+m^2)_k\,\hat\phi_k.$$

(iii) Deduce that in Fourier space, a scalar free field is thus equivalent to a collection of i.i.d. Gaussian variables, indexed by the momentum k, with mean and covariance

$$\langle\hat\phi_k\rangle = 0,\quad \langle\hat\phi_k\hat\phi_p\rangle = \frac{1}{(-a^2\Delta^{\rm dis}+m^2)_k}\,(2\pi)^D\,\delta(k+p).$$

- *Exercise 6.4: Fractal dimension of free paths*

The fractal dimension $D_{\rm frac}$ of a set embedded in a metric space may be defined through the minimal number $\mathcal{N}_\epsilon$ of boxes of radius ϵ need to cover it by $D_{\rm frac} = \lim_{\epsilon\to 0}\log\mathcal{N}_\epsilon/\log(1/\epsilon)$.

(i) Prove that the fractal dimension of free paths is $D_{\rm frac} = 2$ using the fact that the composite operator ϕ^2, with ϕ a (massless) Gaussian free field, is the operator conditioning on two paths emerging from its insertion point.

- *Exercise 6.6: Two ways to compute the free energy*

The aim of this exercise is to compute the free energy, or the partition function, of a massless free boson in space dimension $d=1$ at temperature $T=1/\beta$. Let $D=d+1$. Recall that the partition function is defined as $Z = {\rm Tr}(e^{-\beta H})$ where the trace is over the quantum Hilbert space with H the hamiltonian. Let us suppose that the quantum theory is define dover an interval $\mathbb{I}$ of length L. We shall be interested in the large L limit.

(i) Argue (see Chapter 3) that the partition function is given by the Euclidean path integral on the cylinder $\mathbb{I}\times\mathbb{S}_1$ with a radius β:

$$Z = \int_{\phi(\mathbf{x},\beta)=\phi(\mathbf{x},0)} D\phi\, e^{-S[\phi]}.$$

We shall compute the partition function by quantizing the theory along two different channels (see Figure):
(a) either taking the direction $\mathbb{S}_1$ as time, this Euclidean time is then period with period β;
(b) or taking the direction $\mathbb{I}$ as time, this time then runs from 0 to L with $L\to\infty$. Global rotation invariance implies that this to way of computing gives identical result. Let us check. On the way this will give us a nice relation about the Riemann ζ-function.

(ii) Explain why the first computation gives $Z = e^{-\beta L \mathcal{F}(\beta)}$, where $\mathcal{F}$ the free energy.

(iii) Explain why the second computation gives $Z = e^{-LE_0(\beta,A)}$ with $E_0(\beta) = \beta\,\mathcal{E}_0(\beta)$ where E_0 is the vacuum energy and $\mathcal{E}_0$ is the vacuum energy density (this is the Casimir effect).

(iv) Show that the free energy density of a massless boson in one dimension is:

$$\mathcal{F} = \frac{1}{\beta}\int \frac{dk}{2\pi}\log(1-e^{-\beta|k|}) = \frac{1}{\beta^2}\int_0^\infty \frac{dx}{\pi}\log(1-e^{-x}).$$

(v) Compute the integral to write this free energy density as

$$\mathcal{F} = -\frac{1}{\pi\beta^2}\zeta(2).$$

We have introduce the so-called *zeta*-regularisation. Let $\zeta(s) := \sum_{n>0}\frac{1}{n^s}$. This function was introduced by Euler. This series is convergent for $\Re s > 2$. It is defined by analytic continuation for other value of s via an integral representation.

(vi) Show that the vacuum energy density is $\mathcal{E}_0(\beta) = \frac{1}{\beta}\sum_n \frac{1}{2}\left|\frac{2n\pi}{\beta}\right|$.

(vii) This is divergent. Argue that a regularization based on analytic continuation gives

$$\mathcal{E}_0(\beta) = \frac{2\pi}{\beta^2}\zeta(-1).$$

(viii) Conclusion: A remarkable fact is that $\zeta(2) = \frac{\pi^2}{6}$ and that the analytic continuation of ζ gives $\zeta(-1) = -\frac{1}{12}$. Thus

$$\mathcal{F}(\beta) = \mathcal{E}_0(\beta) = -\frac{\pi}{6\beta^2}.$$

Actually, we could reverse the logic: physics tells us that $\zeta(-1)$ has to be equal to $-\frac{1}{12}$ because $\mathcal{E}_0$ has to be equal to $\mathcal{F}$.

- *Exercise 6.7: Radial quantization (at least in 2D).*

  [... To be completed...]

- *Exercise 6.8: Spanning trees of a graph*

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The edges $e \in E$ are equipped with an arbitrary but fixed orientation. An example is shown in Fig. 2.

The *discrete Laplacian* of G is a matrix M of size $|V| \times |V|$ with elements m_{ij}. For $i \neq j$, $m_{ij} = -k$ if there are $k = 0, 1, 2, \ldots$ edges between the vertices i and j; and for $i = j$, m_{ii} is the number of edges incident on the vertex i.

The *incidence matrix* of G is a matrix A_0 of size $|V| \times |E|$ with elements a_{ij}. These are $a_{ij} = 1$ if the edge j goes out of the vertex i; $a_{ij} = -1$ if the edge j goes into the vertex i; and $a_{ij} = 0$ otherwise (i.e., if the edge j is not incident on the vertex i).

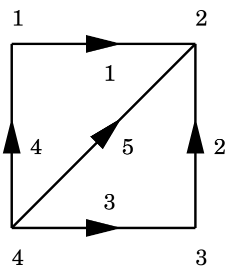

Figure 2: A graph G with 4 vertices and 5 oriented edges.

(i) Write A_0 and M for the example in Fig. 2.

(ii) Show that, in the general case, $M = A_0 \cdot A_0^{\mathrm{T}}$.

(iii) Show that the rank of A_0 is $|V| - \mathcal{C}$, where $\mathcal{C}$ denotes the number of connected components of G [Kirchhoff 1847].

We henceforth suppose that G is a connected graph, $\mathcal{C} = 1$. The *reduced incidence matrix* A is obtained from A_0 by erasing its last row.

Define a *spanning tree* of $G = (V, E)$ to be a sub-graph $G' = (V, E')$ with $E' \subseteq E$, so that G' is connected and has no cycles (i.e., the edges E' generate no closed loop).

(iv) Show that if B is a square sub-matrix of A, either B is singular, or $\det(B) = \pm 1$ [Poincaré 1901].

(v) Show that if the size of B is maximal (i.e., B is a $(|V|-1) \times (|V|-1)$ matrix), then B is non-singular if and only if the edges corresponding to its columns generate a spanning tree of G [Chuard 1922].

We recall the *Binet-Cauchy theorem*:
Let R be a $p \times q$ matrix and S a $q \times p$ matrix, with $p \leq q$. Let R' and S' be $p \times p$ sub-matrices of R and S respectively. Then,

$$\det(R \cdot S) = \sum \det(R') \cdot \det(S') \,,$$

where the sum is over all possible ways of forming sub-matrices R' and S'.

(vi) Prove the *matrix-tree theorem*: If A is the reduced incidence matrix of a graph G, then $\det(A \cdot A^{\mathrm{T}})$ equals the number of spanning trees of G.

(vii) Check this result for the example of Fig. 2.

One introduces a pair of fermionic fields (Grassmann variables) $\eta_1(i)$, $\eta_2(i)$ per vertex of G. By definition, and two of these variables anticommute ($\eta\tilde{\eta} + \tilde{\eta}\eta = 0$) and one integrates over them using the definitions $\int \mathrm{d}\eta \, 1 = 0$ and $\int \mathrm{d}\eta \, \eta = 1$. To lighten the notation, we shall denote, for $k = 1, 2$, $\mathrm{d}\eta_k \equiv \prod_{i=1}^{|V|} \mathrm{d}\eta_k(i)$ and $\eta_k \equiv [\eta_k(1), \eta_k(2), \ldots, \eta_k(|V|)]$.

(viii) Show that

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \, \mathrm{e}^{\eta_1 \cdot M \cdot \eta_2} = \det(M) = 0.$$

Deduce that

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \, \eta_1(|V|)\eta_2(|V|) \mathrm{e}^{\eta_1 \cdot M \cdot \eta_2} \tag{10}$$

is the number of spanning trees of G.

1.6 Chapter 7: Interacting field theory: basics

• *Exercise 7.1: The effective potential and magnetization distribution functions*

The aim of this exercise to probability distribution function of the total magnetization is governed by the effective potential — and this gives a simple interpretation of the effective potential.

Let $M_\phi := \int d^Dx\, \phi(x)$ be the total magnetization. It is suppose to be typically extensive so let m_ϕ be the spatial mean magnetization, $m_\phi = \text{Vol.}^{-1}\, M_\phi$.

(i) Find the expression of the generating function of the total magnetization, $\mathbb{E}[e^{zM_\phi}]$, in terms of the generating function $W[\cdot]$ of connected correlation functions.
Recall that if the source $J(x)$ is uniform, i.e. $J(x) = j$ independent of x, then $W[J]$ is extensive in the volume: $W[J(x) = j] = \text{Vol.}\, w(j)$.

(ii) Let $P(m)dm$ be the probability density for the random variable m_ϕ. Show that at large volume, we have

$$P(m) \simeq e^{-\text{Vol.}\, V_{\text{eff}}(m)},$$

with $V_{\text{eff}}(m)$ the effective potential, defined as the Legendre transformed of $w(j)$.

This has important consequence, in particular the most probable mean magnetization is at the minimum of the effective potential, and phase transition occurs when this minimum changes value.

• *Exercise 7.2: Two-point correlation and vertex functions*

Prove that the two-point connected correlation function and the two-point vertex function are inverse one from the other, that is:

$$\hat{\Gamma}^{(2)}(k)\, \hat{G}_c^{(2)}(k) = 1,$$

as mentioned in the text.

• *Exercise 7.2bis: Ward identities for the stress-tensor*

The aim of this exercise is to derive the Ward identities associated to translation symmetry. This will allows us to make contact with the stress tensor.

We consider a scalar field ϕ in D-dimensional Euclidean flat space with action

$$S[\phi] = \int d^Dx\, \big[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big].$$

Translations act on the field as $\phi(x) \to \phi(x-a)$ for any vector a. The infinitesimal transformation is $\phi(x) \to \phi(x) - \epsilon a^\mu (\partial_\mu \phi)(x)$.

(i) Let us consider an infinitesimal transformation $\phi(x) \to \phi(x) - \epsilon^\mu(x)(\partial_\mu\phi)(x)$ with the space dependent vector fields $\epsilon(x)$.

Prove that the variation of the action is (assuming that the boundary terms do not contribute)

$$\delta S[\phi] = -\int d^D x\, (\partial^\mu \epsilon^\sigma)(x)\, T_{\mu\sigma}(x) = \int d^D x\, \epsilon^\sigma(x)\, (\partial^\mu T_{\mu\sigma})(x),$$

with $T_{\mu\sigma}(x)$ the so-called stress-tensor ($g_{\mu\sigma}$ is the Euclidean flat metric):

$$T_{\mu\sigma}(x) = \partial_\mu \phi\, \partial_\sigma \phi - g_{\mu\sigma}\big[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big].$$

(ii) Prove that the stress tensor is conserved, that is: $\partial_\mu T_{\mu\nu}(x) = 0$ inside any correlation functions away from operator insertions.

(iii) Prove the following Ward identities (here we use the notation $\partial_j^\nu = \partial/\partial_{y_j^\nu}$):

$$\langle (\partial_\mu T^\nu_\mu)(x)\, \phi(y_1)\cdots\phi(y_p)\rangle = \sum_j \delta(x-y_j)\, \partial_j^\nu\, \langle \phi(y_1)\cdots\phi(y_p)\rangle,$$

in presence of scalar field insertion of the form $\phi(y_1)\cdots\phi(y_p)$.

(iv) Do the same construction but for rotation symmetry.

- *Exercise 7.3: Generating functions $Z[J]$, $W[J]$ and $\Gamma[\varphi]$ for a ϕ^3-theory in $D=0$.*

We consider a very simple theory in dimension $D=0$ with action

$$S[\phi] = \frac{1}{2}\phi^2 + \frac{g}{3!}\,\phi^3. \tag{11}$$

The partition function, with an external source J, is defined by

$$Z[J] = \int d\phi\, \exp\left[-\frac{1}{\hbar}\big(S[\phi] - J\phi\big)\right]. \tag{12}$$

The parameter $\hbar$ is an expansion parameter from the classical solution obtained in the limit $\hbar \to 0$. This theory has a meaning only in perturbation theory (because the potential ϕ^3 is unbounded from below). We are going to study it perturbatively. By convention we assume $g > 0$.

(i) We aim at calculating $Z[J]$ at one loop. We set $\phi = \phi_c(J) + \sqrt{\hbar}\,\chi$ where $\phi_c(J)$ minimizes the action $S[\phi; J] = S[\phi] - J\phi$ in presence of an external source. Compute $\phi_c(J)$ and the corresponding action $S[\phi_c; J]$.

(ii) Compute $Z[J]$ at leading order up to $O(\hbar)$.

(iii) Compute $W[J] = \hbar\, \log Z[J]$ up to $O(\hbar^{3/2})$ and expand in power of J up to order J^4 included. We set:

$$W[J] = \sum_{n=0}^{\infty} \frac{1}{n!}\, W^{(n)}\, J^n.$$

Determine $W^{(n)}$ for $n = 0, 1, 2, 3, 4$.

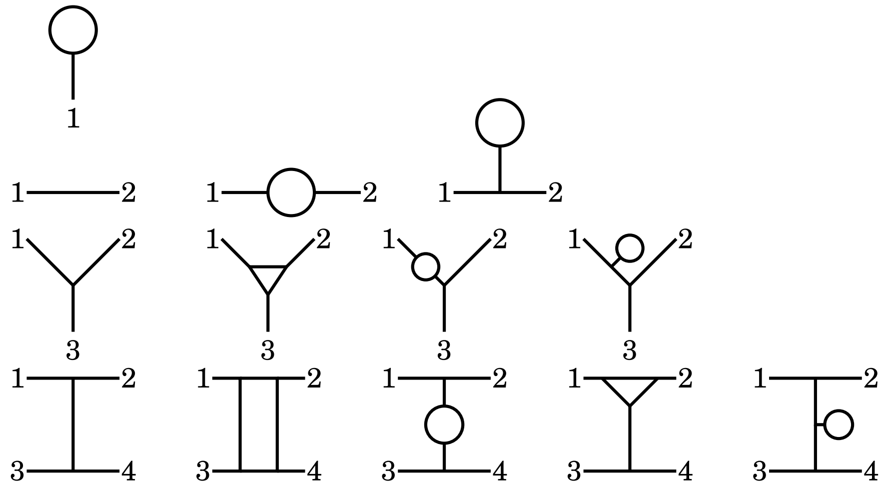

Figure 3: Feynman diagrams with 1, 2, 3 et 4 external lines.

(iv) Compare the previous results with a direct computation via (connected) Feynman diagrams up to order g^4.

(v) We now define the effective action $\Gamma[\varphi]$ via the Legendre transform:

$$\Gamma[\varphi] = J\varphi - W[J], \quad \text{with} \quad \varphi = \frac{\partial W[J]}{\partial J}.$$

Compute φ up to order J^3 included and neglecting terms of order $\hbar^2$ (i.e. up to two loop diagrams). Invert this relation to get J as a series in ρ with

$$\rho = \varphi + \frac{1}{2} g\hbar,$$

up to order ρ^3 included. To do this series expansion, assume that both J and ρ are small.

(vi) Show that the definition of Γ implies $\frac{\partial \Gamma}{\partial \varphi} = \frac{\partial \Gamma}{\partial \rho} = J$. Compute $\Gamma[\rho]$, up to terms of order ρ^5 or $\hbar^{3/2}$, by integrating $J[\rho]$ with to respect to ρ.
Let

$$\Gamma[\rho] = \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma^{(n)} \rho^n.$$

Determine $\Gamma^{(n)}$ for $n = 1, 2, 3, 4$.
Show that these results coincide with the one particle irreducible (1-PI) diagrams.

• *Exercise 7.4: Effective action and one-particle irreducible diagrams.*

The aim of this exercise is to prove the equality between the effective action and the generating function of 1-PI diagrams. To simplify matter, we consider a 'field' made of N ($N \gg 1$) components ϕ^j, $j = 1, \cdots, N$. We view ϕ^j as random variables.

Let us define a 'partition function' $Z_\epsilon[J]$ by

$$Z_\epsilon[J] = \int D\phi\, e^{-\epsilon^{-1}\left[\Gamma[\phi]-(J,\phi)\right]}, \quad \text{with } D\phi = \prod_j \frac{d\phi^j}{\sqrt{2\pi\epsilon}}.$$

with J a source $(J,\phi) = J_j\phi^j$, and $\Gamma[\phi]$ an action which we define via its (formal) series expansion (summation over repeated indices is implicit):

$$\Gamma[\phi] = \frac{1}{2}\Gamma^{(2)}_{jk}\phi^j\phi^k - \sum_{n\geq 3}\frac{1}{n!}\Gamma^{(n)}_{j_1\cdots j_n}\phi^{j_1}\cdots\phi^{j_n}.$$

We shall compute this partition function in two different ways: via a saddle point approximation or via a perturbation expansion.

(i) Justify that this integral can be evaluating the integral via a saddle-point when $\epsilon \to 0$.
Prove that
$$\log Z_\epsilon = \frac{1}{\epsilon}\, W[J]\,\big(1 + O(\epsilon)\big),$$
where $W[J]$ is the Legendre transform of the action Γ: $W[J] = (J,\phi_*) - \Gamma(\phi_*)$ with ϕ_* determined via $\frac{\partial\Gamma}{\partial\phi^j}(\phi_*) = J_j$.
Hint : Do the computation formally which amounts to assume that the integral converges and that there is only one saddle point.

Let us now compute $Z_\epsilon[J]$ in perturbation theory. Let us decompose the action as the sum of its Gaussian part plus the rest that we view as the interaction part: $\Gamma[\phi] = \frac{1}{2}\Gamma^{(2)}_{jk}\phi^j\phi^k - \hat{\Gamma}[\phi]$.

(ii) Write
$$Z_\epsilon[J] = \int D\phi\, e^{-\frac{1}{2\epsilon}\Gamma^{(2)}_{jk}\phi^j\phi^k}\, e^{\epsilon^{-1}\hat{\Gamma}[\phi]}\, e^{\epsilon^{-1}(J,\phi)}.$$
We view J/ϵ as source, and we aim at computing the connected correlation function using Feynman diagrams perturbative expansion.
Show that the propagator is $\epsilon\, G^{jk}$ with $G = (\Gamma^{(2)})^{-1}$ and the vertices are $\epsilon^{-1}\,\Gamma^{(n)}_{j_1\cdots j_n}$ with $n \geq 3$.

(iii) Compute the two-, three- and four-point connected correlations $G_{(n)}$, $n = 1,2,3$, at the level tree, defined by
$$G^{j_1\cdots j_n}_{(n)} = \frac{\partial^n}{\partial J_{j_1}\cdots\partial J_{j_n}} \log Z_\epsilon[J]\Big|_{\text{tree}}.$$
Show that they are of order ϵ^{-1}. Draw their diagrammatic representations (in terms of propagators and vertices) and compare those with the representations of the connected correlation functions in terms of 1-PI diagrams.

(iv) Prove that, when $\epsilon \to 0$, the leading contribution comes only from the planar tree diagrams and that all these diagrams scale like $1/\epsilon$. That is:

$$\log Z_\epsilon[J] = \frac{1}{\epsilon}\Big(\text{planar tree diagrams} + O(\epsilon)\Big).$$

Hint : Recall that, for a connected graph drawn on a surface of genus g (i.e. with g handles, $g > 0$), one has $V - E + L + 1 = 2 - g$ with V its number of vertices, E its number of edges and L its numbers of loops (this is called the Euler characteristics). Then, argue that each Feynman graph contributing to the N point connected functions is weighted by (symbolically) $(\epsilon\, G)^E\, (-\epsilon^{-1}\, \Gamma^{(n)})^{V_{\text{int}}}\, (\epsilon^{-1}\, J)^N$ with $V_{\text{int}} + N$ total number of vertices.

(v) By inverting the Legendre transform, deduce the claim that the effective action is the generating function of 1-PI diagrams.

- *Exercise 7.5: Computation of the one-loop effective potential*

Prove the formula for the one-loop effective potential of the ϕ^4-theory given in the text. Namely

$$V^{\text{eff}}_{1-\text{loop}}(\varphi) = \frac{1}{2!} A_\Lambda\, \varphi^2 + \frac{1}{4!}\, B_\Lambda\, \varphi^4 + \frac{\hbar}{(8\pi)^2}\big(V''(\varphi)\big)^2 \log[\frac{V''(\varphi)}{\mu^2}],$$

with

$$\begin{aligned} A_\Lambda &= m_0^2 + \frac{\hbar g_0}{2}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m_0^2}{(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2})\Big) + O((\hbar g_0)^2), \\ B_\Lambda &= g_0 - \hbar g_0^2 \frac{3}{2(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2}) + O(g_0(\hbar g_0)^2) \end{aligned}$$

with μ^2 an arbitrary scale that we introduced by dimensional analysis.

Analyse this potential and conclude.

- *Exercise 7.6: Computation of one-loop Feynman diagrams*

[...To be completed...]

- *Exercise 7.7: The $O(N)$ vector model with $N \to \infty$ in $D = 3$*

We study a model of N-component spins $\vec{\Phi}$ governed by the Hamiltonian

$$H[\vec{\Phi}] = \frac{1}{2}\int d\vec{x}\, \left\{ \sum_{\alpha=1}^{N}\left(\frac{\partial \Phi_\alpha}{\partial \vec{x}}\right)^2 + r_0 \sum_{\alpha=1}^{N}(\Phi_\alpha)^2 + \frac{u}{12N}\left(\sum_{\alpha=1}^{N}(\Phi_\alpha)^2\right)^2 \right\}.$$

The dimension of the embedding space is fixed as $D = 3$. In this exercise we shall compute the critical exponents for large N, and more precisely, first for infinite N and then the corrections to order $1/N$.

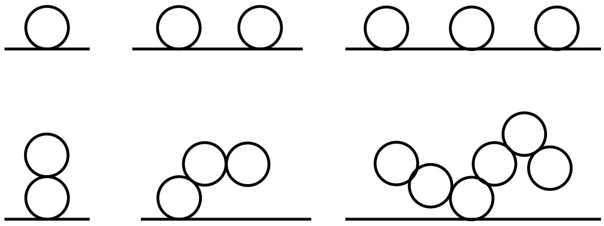

Figure 4: Some examples of "cactus" diagrams.

(i) Write the propagator and interaction vertex in Fourier space. Which diagrams contribute to the correlation function

$$G(k) = \langle \tilde{\Phi}_\alpha(-\vec{k}) \tilde{\Phi}_\alpha(\vec{k}) \rangle \ ?$$

Show that in the limit $N \to \infty$, the only surviving diagrams are of the "cactus" type (see Figure 4), where the solid line represents the bare (free) propagator.

(ii) Deduce the implicit equation

$$\frac{1}{G_\Phi^\infty(k)} = k^2 + r_0 + \frac{u}{6} \int_{q<\Lambda} \frac{d^3q}{(2\pi)^3} \, G_\Phi^\infty(q)$$

satisfied by the dressed propagator $G_\Phi^\infty(k)$, where Λ denotes an ultraviolet cut-off. (*Hint*: Formally sum up subclasses of diagrams in a geometric series.)

(iii) Interpret the identity (1.6). Use it to compute first the critical temperature and next the exponents η^∞ and ν^∞ in the limit $\Lambda \to \infty$. (*Hint*: The critical temperature is such that the renormalised mass vanishes.)

(iv) We wish to recover this result by the saddle point method. By introducing a new scalar field $\sigma(\vec{x})$, show that one may rewrite the partition function of the above model in the form

$$Z = \int D\vec{\Phi}(\vec{x}) \, D\sigma(\vec{x}) \, \exp\left(-H[\vec{\Phi}, \sigma]\right) ,$$

where

$$H[\vec{\Phi}, \sigma] = \frac{1}{2} \int d\vec{x} \, \left\{ \sum_{\alpha=1}^{N} \left(\frac{\partial \Phi_\alpha}{\partial \vec{x}} \right)^2 + \left(r_0 + i \sqrt{\frac{u}{3N}} \, \sigma \right) \sum_{\alpha=1}^{N} (\Phi_\alpha)^2 + \sigma^2 \right\} .$$

(v) Integrate over the fields $\vec{\Phi}$. Which effective action for the field σ does one arrive at?

(vi) In the limit $N \to \infty$ one can obtain Z by computing the saddle point of the preceeding action. One supposes that this saddle point is uniform (that is, independent of x). Show that we hence recover the implicit equation (1.6).

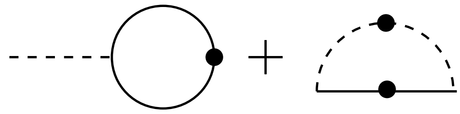

Figure 5: Development of the self-energy.

(vii) Verify that the classical solution obtained is indeed a local minimum of the action.

(viii) We now use the action found in (v) as the starting point for computing the corrections of order $1/N$ to the critical exponents. Write down the bare propagators of the fields σ and $\vec{\Phi}$, as well as the interaction vertex.

In the following questions we focus on obtaining the propagator of σ in the $N \to \infty$ limit. This step is necessary in order to go to the next order in the computation of the Φ-propagator.

(ix) Show that in the limit $N \to \infty$ the dressed propagator $G_\sigma^\infty(k)$ of σ satisfies the implicit equation illustrated in Figure **??**. (Here the solid line represents the propagator of Φ and the dashed line that of σ. The presence of a point on a propagator means that it is dressed.) Could this equation have been anticipated from the answer to question (iv)?

(x) Compute the integral

$$I = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(\mathbf{1}-\mathbf{q})^2} \cdot \frac{1}{\mathbf{q}^2} = \frac{1}{8} \,,$$

where $\mathbf{1}$ represents a unit vector. (*Hint*: use polar coordinates.)

(xi) Deduce that at the critical temperature we have

$$G_\sigma^\infty(k) \simeq \frac{48}{u}\, k \qquad (k \to 0)\,.$$

The final stage of the exercise is now to obtain the propagator of Φ to order $1/N$.

(xii) Show that the self-energy $\Sigma_\Phi^N(k)$ of the dressed propagator $G_\Phi^N(k)$ of Φ is given by Figure 5, where the first term represents the finite contribution for $N \to \infty$ that has been studied in question (iv), while the second term is the sought-for contribution at order $1/N$.

(xiii) Infer that

$$\Sigma_\Phi^N(k) - \Sigma_\Phi^N(0) \simeq \frac{8}{3\pi^2 N}\, k^2 \ln k \qquad (k \to 0)\,.$$

(xiv) Deduce the value of the exponent η to order $1/N$.

1.7 Chapter 8: Conformal field theory: basics

• *Exercise 8.1: Conformal mappings in 2D.*

(i) Verify that the map $z \to w = \frac{z-i}{z+i}$ is holomorphic map from the upper half plane $\mathbb{H} = \{z \in \mathbb{C},\ \mathrm{Im} z > 0\}$ to the unit disc $\mathbb{D} = \{w \in \mathbb{C},\ |w| < 1\}$ centred at the origin 0.

(ii) Similarly verify that the map $w \to z = e^{w/\beta}$ is a holomorphic map from the cylinder with radius β to the complex z-plane with the origin and the point at infinity removed.

• *Exercise 8.2: The group of conformal transformations.*

The aim of this exercise is to fill the missing steps in determining all infinitesimal conformal transformations in the flat Euclidean space $\mathbb{R}^D$.

Let us recall a few basic facts from the lectures. A diffeomorphism $x \to y$ is called conformal if it changes the metric by a space-dependent factor:

$$\hat{g}_{\mu\nu}(x) = \left(\frac{\partial y^\sigma}{\partial x^\nu}\right)\left(\frac{\partial y^\rho}{\partial x^\nu}\right) g_{\sigma\rho}(y(x)) := \mathrm{e}^{2\phi(x)} g_{\mu\nu}(x)\,. \tag{13}$$

Here ϕ is called the conformal factor. Now apply this to an infinitesimal transformation $x^\mu \to x^\mu + \epsilon\xi^\mu(x) + \cdots$, where $\xi^\mu(x)$ is the vector field generating conformal transformations. By developing the left- and right-hand sides of (13) to first order in the small parameter ϵ and comparing we obtain

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2(\delta\phi)\delta_{\mu\nu}\,. \tag{14}$$

Another useful relation is obtained by multiplying this by $\delta^{\nu\mu}$ on both sides. We then get $\partial^\nu \xi_\nu + \partial^\mu \xi_\mu = 2(\delta\phi)\delta^\nu_\nu = 2D(\delta\phi)$, or in other words

$$D(\delta\phi) = (\partial_\mu \xi^\mu)\,. \tag{15}$$

We denote $\partial \cdot \xi := \partial_\mu \xi^\mu$ and the Euclidean Laplacian $\Delta := \partial^\mu \partial_\mu$, with the summation convention throughout.

(i) Take derivatives of the previous equation to deduce that $D\,\Delta\xi_\nu = (2-D)\partial_\nu(\partial\cdot\xi)$, with Δ the Euclidean Laplacian.

(ii) Take further derivatives, either w.r.t ∂_ν or w.r.t. ∂_μ, to get two new equations: $(D-1)\Delta(\partial\cdot\xi) = 0$, and $2(2-D)\,\partial_\mu\partial_\nu(\partial\cdot\xi) = D\,\Delta(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)$.

(iii) Deduce that $(2-D)\partial_\mu\partial_\nu(\partial\cdot\xi) = 0$, and hence that, in dimension $D > 2$, the conformal factor $\delta\varphi(x)$ is linear in x.

Let us write $\delta\varphi(x) = k + b_\nu x^\nu$ with k and b_ν integration constantes. We thus have

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 2(k + b_\sigma x^\sigma)\,\delta_{\mu\nu}.$$

A way to determine ξ consists in getting information on the difference $\partial_\mu\xi_\nu - \partial_\nu\xi_\mu$.

(iv) By taking derivates of the previous equation w.r.t ∂_σ and permuting the indices, deduce that $\partial_\nu(\partial_\sigma\xi_\mu - \partial_\mu\xi_\sigma) = 2(b_\sigma\delta_{\mu\nu} - b_\mu\delta_{\nu\sigma})$, and hence, by integration, that

$$\partial_\sigma\xi_\mu - \partial_\mu\xi_\sigma = 2(b_\sigma x_\mu - b_\mu x_\sigma) + 2\theta_{\mu\sigma},$$

where $\theta_{\sigma\mu} = -\theta_{\mu\sigma}$ are new integration constants.

(v) Integrate the last equations to prove that

$$\xi_\nu(x) = a_\nu + kx_\nu + \theta_{\nu\sigma}x^\sigma + [(b\cdot x)x_\nu - \frac{1}{2}(x\cdot x)b_\nu],$$

where a_ν are new, but last, integration constants.

(vi) Find the explicit formula for all finite –not infinitesimal– conformal transformations in dimension D.
Hint: It is advantageous to consider (a) the flow generated by the above vector fields $\xi(x)$, i.e. to consider the one parameter family of transformations $x \to y_t$ such that $\partial_t y_t = \xi(y_t)$ with initial condition $y_{t=0} = x$, and (b) to change coordinate to $Y_t := \frac{y_t}{(y_t\cdot y_t)}$.

(vii) *Optional:* Verify that the Lie algebra of the group of conformal transformation in dimension D is isomorphic to $so(D+1,1)$.

- *Exercise 8.3: The two- and three-point conformal correlation functions.*

The aim of this exercise is to fill the missing steps in determining the two and three point function of conformal fields in conformal field theory. Let $G^{(2)}(x_1,x_2) = \langle\Phi_1(x_1)\Phi_2(x_2)\rangle$ be the two point function of to scalar conformal fields of scaling dimension h_1 and h_2 respectively.

(i) Prove that translation and rotation invariance implies that $G^{(2)}$ is a function of the distance $r = |x_1 - x_2|$ only.

(ii) Prove that dilatation invariance of the 2-point function demands that

$$[h_1 + x_1\cdot\partial_1 + h_2 + x_2\cdot\partial_2]G_2(x_1,x_2) = 0.$$

Deduce that $G_2(x_1,x_2) = \text{const.}\, r^{-(h_1+h_2)}$.

(iii) Prove that invariance under special conformal transformations (also called inversions) implies that

$$\sum_{j=1,2}\big[h_j(b\cdot x_j) + [(b\cdot x_j)x_j^\nu - \frac{1}{2}(x_j\cdot x_j)b^\nu]\partial_{x_j^\nu}\big]G_2(x_1,x_2) = 0.$$

Deduce that $G^{(2)}(x_1,x_2)$ vanishes unless $h_1 = h_2$.

Let us now look at the three point functions of scalar conformal fields. Let $G^{(3)}(x_1,x_2,x_3) = \langle\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle$, be their correlation functions.

(iv) Prove that invariance under infinitesimal conformal transformations demands that

$$\sum_{j=1,2,3}\big[h_j\, D^{-1}(\partial.\xi)(x_j) + \xi^\mu(x_j)\partial_{x_j^\mu}\big]G^{(3)}(x_1,x_2,x_3) = 0,$$

for any conformal vector $\xi^\mu(x)$. See previous exercise.

(v) Integrate this set of differential equations to determine the explicit expression of $G^{(3)}(x_1,x_2,x_3)$ up to constant.

• *Exercise 8.4: Diff $\mathbb{S}^1$ and its central extension.*

The aim of this exercise is to study the Lie algebra Diff $\mathbb{S}^1$ of vector fields in the circle and its central extension the Virasoro algebra. Let $z = e^{i\theta}$ coordinate on the unit circle. A diffeomorphism is on application $\theta \to f(\theta)$ from $\mathbb{S}^1$ onto $\mathbb{S}^1$. Using the coordinate z, we can write it as $z \to f(z)$ so that it is, at least locally, identified with a holomorphic map (again locally holomorphic). They act on functions $\phi(z)$ by composition: $\phi(z) \to (f\cdot\phi)(z) = \phi(f^{-1}(z))$. For an infinitesimal transformation, $f(z) = z + \epsilon\, v(z) + \cdots$ avec $\epsilon \ll 1$, the transformed function is

$$(f\cdot\phi)(z) = \phi(z) + \epsilon\,\delta_v\phi(z) + \cdots, \quad \text{with } \delta_v\,\phi(z) = -v(z)\,\partial_z\phi(z).$$

(i) Take $v(z) = z^{n+1}$, with n integer. Verify that $\delta_v\phi(z) = \ell_n\phi(z)$ with $\ell_n \equiv -z^{n+1}\partial_z$. Show t

$$\big[\ell_n, \ell_m\big] = (n-m)\,\ell_{n+m}.$$

This Lie algebra is called the Witt algebra.

(ii) Let us consider the (central) extension of the Witt algebra, generated by the ℓ_n and the central element c, with the following commutation relations

$$\big[\ell_n, \ell_m\big] = (n-m)\ell_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m;0}, \quad [c, \ell_n] = 0.$$

Verify that this set of relation satisfy the Jacobi identity. This algebra is called the Virasoro algebra.

(iii) Prove that this is the unique central extension of the Witt algebra.

• *Exercise 8.5: The stress-tensor OPE in 2D CFT*

Let ϕ be a massless Gaussian free field in 2D with two point function $\langle\phi(z,\bar z)\phi(w,\bar w)\rangle = -\log(|z-w|^2/R^2)$. Recall that the (chiral component of the) stress-tensor of a massless 2D Gaussian field is $T(z) = -\frac{1}{2} : (\partial_z\phi)^2(z) :$.

(i) Prove, using Wick's theorem, that it satisfies the OPE

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \big[\frac{2}{(z_1-z_2)^2} + \frac{1}{(z_1-z_2)}\partial_z\big]T(z_2) + \text{reg}.$$

• *Exercise 8.6: Transformation of the stress-tensor in 2D CFT.*

Under a conformal transformation $z \to w = w(z)$, the transformation rules for the stress tensor in two-dimensional CFT is

$$T(z) \to \hat{T}(w) = \left[z'(w)\right]^2 T(z(w)) + \frac{c}{12} S(z;w)\,, \tag{16}$$

where $z'(w)$ is the derivative of z with respect to w, while $S(z;w)$ denotes the Schwarzian derivative:

$$S(z;w) = \left[\frac{z'''(w)}{z'(w)}\right] - \frac{3}{2}\left[\frac{z''(w)}{z'(w)}\right]^2 . \tag{17}$$

(i) Let us consider two conformal transformations $z \to w = w(z)$ and $w \to \xi = \xi(w)$ and their composition $z \to \xi = \xi(z)$. Prove that consistency of the stress-tensor transformation rules demands that:

$$S(z;\xi) = S(w;\xi) + [\xi'(w)]^2\, S(z,w).$$

Verify this relation from the definition of $S(z;w)$.

(ii) Use this formula to compute the stress-tensor expectation for a CFT defined over a infinite cylinder of radius R. Show that

$$\langle T(z)\rangle_{\text{cylinder}} = -c\,\frac{\pi}{12\,R^2}.$$

• *Exercise 8.7: Regularization of vertex operators*

In the text, we use the connection with lattice model to argue for the anomalous transformation of vertex operators in gaussian conformal field theory. The aim of this exercise is to derive (more rigorously) this transformation within field theory (without making connection with lattice models).

Let $\phi(z,\bar{z})$ a Gaussian free field normalized by $\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\log(|z-w|^2/R^2)$ with R the IR cut-off tending to infinity. In order to regularized the field we introduce a smeared version ϕ_ϵ of ϕ defined by integrating it around a small circle, of radius ϵ, centred at z:

$$\phi_\epsilon(z,\bar{z}) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, \phi(z_\epsilon(\theta), \bar{z}_\epsilon(\theta)),$$

with $z_\epsilon(\theta)$ be point on this circle, $0 < \theta < 2\pi$. The small radius ϵ play the role of *UV* cutoff.

(i) Prove that (notice that we consider the smeared at the same central position z but with two different cutoff ϵ and ϵ')

$$\langle\phi_\epsilon(z,\bar{z})\phi_{\epsilon'}(z,\bar{z})\rangle = \min(\log(R/\epsilon)^2, \log(R/\epsilon')^2).$$

In particular $\langle\phi_\epsilon(z,\bar{z})^2\rangle = \log(R/\epsilon)^2$.

(ii) Verify that $\langle e^{i\alpha\phi_\epsilon(z,\bar{z})}\rangle = (\epsilon/R)^{\alpha^2}$, for α real. Let us define the vertex operator by

$$V_\alpha(z,\bar{z}) = \lim_{\epsilon\to 0} \epsilon^{-\alpha^2}\, e^{i\alpha\phi_\epsilon(z,\bar{z})}.$$

Argue that this limit exists within any expectation values.

(iii) Let us now consider a conformal transformation $z \to w = w(z)$ or inversely $w \to z = z(w)$. Show that a small circle of radius $\hat{\epsilon}$, centred at point w, in the w-plane is deformed into a small close curve in the z-plane which approximate a circle of radius $\epsilon = |z'(w)|\,\hat{\epsilon}$, centred at $z(w)$.
Deduce that under such conformal transformation the vertex operator transforms as follows:

$$\hat{V}_\alpha(w,\bar{w}) = |z'(w)|^{\alpha^2}\, V_\alpha(z,\bar{z}).$$

That is: the anomalous scaling transformation of the vertex operator arises from the fact that the regularization scheme/geometry is not preserved by the conformal transformations.

1.8 Chapter 9: Scaling limits and the field theory renormalisation group

- *Exercise 9.1: Explicit RG flows*

The aim of this exercise is to study simple, but important, examples beta functions and solutions of the Callan-Symanzik equation.

(i) Consider a field theory with only one relevant coupling constant g and suppose that its beta function is $\beta(g) = yg$.
Show that the RG flow, solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ is $g(\lambda) = g_1\, \lambda^y$.
Show that the RG mass scale, solution of $\beta(g)\partial_g m(g) = m(g)$ is $m(g) = m_*\, g^{1/y}$.
Consider the two point function $G(r; g)$ of a scaling field Φ of scaling dimension Δ, i.e. $G(r, g) = \langle\Phi(r)\Phi(0)\rangle_g$. Prove (using the Callan-Symanzik equation) that

$$G(r; g) = r^{-2\Delta}\, F(m(g)r),$$

with $m(g)$ the RG mass scale defined above.

(ii) Consider a field theory with only one marginal coupling constant g and suppose that its beta function is $\beta(g) = cg^2$ ($c > 0$ corresponds to marginally relevant, $c < 0$ to marginally irrelevant).
Prove that the RG flow, solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ is $g(\lambda) = g_\mu/(1-cg_\mu \log(\lambda/\mu))$.
Notice that $g_\lambda \to 0^+$, if $c < 0$, while g_λ flows up if $c > 0$, as $\lambda \to \infty$ (with $g_\mu > 0$ initially). Prove that the RG mass scale, solution of $\beta(g)\partial_g m(g) = m(g)$ is $m(g) = m_*\, e^{-1/cg}$.
Notice that this mass scale is non perturbative in the coupling constant.
Consider the two point function $G(r; g)$ of a scaling field Φ whose matrix of anomalous dimension is $\gamma(g) = \Delta + \gamma_0 g$. Prove (using the Callan-Symanzik equation) that $G(r/\lambda; g(\lambda)) = Z(\lambda)^2 G(r, g)$ with

$$Z(\lambda) = \text{const.}\, \lambda^{\Delta}\, [g(\lambda)]^{\gamma_0/c}.$$

Deduce from this that, in the case marginally irrelevant perturbation (i.e. $c < 0$) and asymptotically for r large,

$$G(r; g_a) \simeq \text{const.}\, r^{-2\Delta}\, [\log(r/a)]^{-2\gamma_0/c}.$$

This codes for logarithmic corrections to scaling.

- *Exercise 9.2: Anomalous dimensions and beta functions*

(i) Prove the relation $\gamma_\alpha^\sigma(g) = D\delta_\alpha^\sigma - \partial_\alpha\beta^\sigma(g)$ between the matrix of anomalous dimensions and the beta functions.

(ii) Give two proofs of the formula $\gamma_\alpha^\sigma(g) = \Delta_\alpha\delta_\alpha^\sigma + S_D \sum_i g^i C_{i\alpha}^\sigma$ for the matrix of anomalous dimensions to first order in perturbation theory (Here g^i are the perturbative coupling constant and S_D the volume of the D-dimensional unit sphere): one proof comes from using the previous result, the second proof comes from analysing the perturbative expansion of the correlation functions.

• *Exercise 9.3: Renormalisation of ϕ^3 in $D = 6$: One-particle irreducible functions*

In this exercise and the following, we consider the ϕ^3 action of the scalar field ϕ defined by

$$S[\phi] = \int d^d x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{6} g m^{\epsilon/2} \phi^3 \right],$$

where $\epsilon = 6 - d$.

In this first part, we shall compute the one-particle irreducible functions $\Gamma^{(n)}$ for $n = 1, 2, 3$.

(i) What is the dimension of ϕ and of the coupling constant g? Determine the superficial degree of (ultra-violet) divergence of $\Gamma^{(n)}$ to L loops. For which values of d the theory is renormalisable, super-renormalisible, non-renormalisable?

(ii) We first work in $d = 6$ dimensions. Which Feynman diagrams are superficially divergent? Is their number finite or infinite? Same question for one-particle irreducible diagrams.

(iii) Compute $\Gamma^{(1)}$, $\Gamma^{(2)}(p, -p)$ and $\Gamma^{(3)}(p_1, p_2, -p_1 - p_2)$ to one-loop order. To this end, use dimensional regularisation and the formulae

$$\begin{aligned} \frac{1}{a_1\, a_2} &= \int_0^1 dx \frac{1}{[a_1 x + a_2(1-x)]^2} \\ \frac{1}{a_1\, a_2\, a_3} &= 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[a_1 x + a_2 y + a_3(1-x-y)]^3}, \end{aligned}$$

as well as

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + 2\vec{q}.\vec{k} + p^2)^n} = \frac{\Gamma(n - \frac{d}{2})}{(4\pi)^{d/2} \Gamma(n)} (p^2 - k^2)^{\frac{d}{2} - n}.$$

(iv) Give expressions for $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$, neglecting terms of order ϵ. To this end, use the following propreties of the Euler Γ function:

$$\begin{aligned} \Gamma(x+1) &= x\, \Gamma(x) \\ \Gamma(x) &= \frac{1}{x} + \psi(1) + O(x) \qquad (x \to 0) \\ \psi(x) &= \frac{d}{dx} \log \Gamma(x). \end{aligned}$$

Express the results in terms of the two functions

$$\begin{aligned} f_1(u) &= \int_0^1 dx [1 + ux(1-x)] \log[1 + ux(1-x)] \\ f_2(u, v, w) &= \int_0^1 dx \int_0^{1-x} dy \log[1 + ux(1-x) + vy(1-y) + 2wxy]. \end{aligned}$$

(v) Show that the divergence of $\Gamma^{(3)}$ to one-loop order can be formally eliminated by redefining the coupling constant as follows:

$$g = \tilde{g} \left(1 - \frac{\tilde{g}^2}{(4\pi)^3 \epsilon} \right).$$

Verify that by replacing $1/\epsilon$ by $\log(\Lambda/m)$ in the above formula, one recovers the divergent part corresponding to a regularisation of the theory by an ultra-violet cut-off Λ.

- *Exercise 9.4: Current-current perturbations and applications.*

  [...To be completed...]

- *Exercise 9.5: Disordered random bound 2D Ising model.*

  [...To be completed...]

1.9 Chapter 10: Miscellaneous applications

• *Exercise 10.1: The XY model*

The XY model is a statistical spin model with spin variables $\vec{S}_i$, on each site i of the lattice Λ, which are two component unit vectors, $\vec{S}_i^2 = 1$. The energy of a configuration $[\vec{S}]$ is defined as $E[\vec{S}] = -\sum_{[ij]} \vec{S}_i \cdot \vec{S}_j$ where the sum runs over neighboor points on Λ. Parametrising the unit spin vectors $\vec{S}_i$ by an angle Θ_i defined modulo 2π, we write the configuration energy as

$$E[\vec{S}] = -\sum_{[ij]} \cos(\Theta_i - \Theta_j).$$

The partition function is $Z = \int [\prod_i \frac{d\Theta_i}{2\pi}] \exp\Big(\beta \sum_{[i,j]} \cos(\Theta_i - \Theta_j)\Big)$ with $\beta = 1/k_B T$ the inverse temperature.

Here is the solution of the problem on the XY model given in Section 9.1.

IA- The XY model on a lattice: High temperature expansion

The aim of this section is to study the high temperature ($\beta \ll 1$) behavior of the XY model. It is based on rewriting the Boltzmann sums in terms of dual flow variables.

IA-1 Explain why we can expand $e^{\beta\cos\Theta}$ in series as $e^{\beta\cos\Theta} = I(\beta)\,\big(1+\sum_{n\neq 0} t_n(\beta) e^{in\Theta}\big)$, where $I(\beta)$ and $t_n(\beta)$ are some real β-dependent coefficients. We set $t_0(\beta) = 1$.

IA-2 By inserting this series in the defining expression of the partition function and by introducing integer variables $u_{[ij]}$ on each edge $[ij]$ of the lattice Λ, show that the partition function can be written as $Z = I(\beta)^{N_\mathrm{e}} \cdot \hat{Z}$ with N_e the number of edges and

$$\hat{Z} = \sum_{[u],\ [\partial u = 0]} \ \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the partition sum is over all configurations $[u]$ of integer edge variables $u_{[ij]}$ such that, for any vertex $i \in \Lambda$, the sum of these variables arriving at i vanishes, i.e. $\sum_j u_{[ij]} = 0$.
Remark: The variables u are attached to the edge of the lattice and may be thought of as 'flow variables'. The condition that their sum vanishes at any given vertex is a divergence free condition. The divergence at a vertex i of a configuration $[u]$ is defined as $(\partial u)_i := \sum_j u_{[ij]}$.

IA-3 Let i_1 and i_2 be two points of Λ and $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ be the two-point spin correlation function.
Explain why $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle = \mathrm{Re}\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle$.
Show that,

$$\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle = \frac{1}{\hat{Z}} \cdot \sum_{\substack{[u] \\ [\partial u = \delta_{\cdot;i_1} - \delta_{\cdot;i_2}]}} \ \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the sum is over all integer flow configurations such that their divergence is equal to $+1$ at point i_1, to -1 at point i_2, and vanishes at any other vertex.

IA-4 Show that $t_n(\beta) = t_{-n}(\beta) \simeq \frac{\beta^n}{2^n n!}$ as $\beta \to 0$.
Argue, using this asymptotic expression for the $t_n(\beta)$'s, that the leading contribution to

the spin correlation functions at high temperature comes from flow configurations with $u = 0$ or $u = \pm 1$ on each edge of the lattice.

IA-5 Deduce that, at high temperature, the correlation function $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ decreases exponentially with the distance between the two points i_1 and i_2.
Show that the correlation length behaves as $\xi \simeq a/\log(2/\beta)$ at high temperature.

IB- Low temperature expansion

The aim of this section is to study the low temperature ($\beta \gg 1$) behavior of the XY model. It consists in expanding the interaction energy $\cos(\Theta_i - \Theta_j)$ to lowest order in the angle variables so that we write the configuration energy as (up to an irrelevant additive constant)

$$E[\vec{S}] = \text{const.} + \frac{1}{2} \sum_{[i,j]} (\Theta_i - \Theta_j)^2 + \cdots .$$

This approximation neglects the 2π-periodicity of the angle variables.

IB-1 Argue that the higher order terms in this expansion, say the terms proportional to $\sum_{[i,j]} (\Theta_i - \Theta_j)^4$, are expected to be irrelevant and can be neglected.

IB-2 Write the expression of the partition function Z of the model within this approximation.
Explain why, in this approximation, the theory may be viewed as a Gaussian theory.

IB-3 Let $G_\beta(x)$ be the two-point function of this Gaussian theory. Show that $G_\beta(x) = \beta^{-1}\, G(x)$ with

$$G(x) = \int_{-\pi/a}^{+\pi/a} \frac{d^2p}{(2\pi/a)^2} \frac{e^{ip\cdot x}}{4 - 2(\cos ap_1 + \cos ap_2)},$$

with p_1, p_2 the two components of the momentum p and a the lattice mesh.

IB-4 Let i_1 and i_2 be two points on Λ and x_1 and x_2 be their respective Euclidean positions. Let $C_\alpha(x_1, x_2) = \langle e^{i\alpha(\Theta_{i_1} - \Theta_{i_2})} \rangle$ with α integer. Show that

$$C_\alpha(x_1, x_2) = e^{-\frac{\alpha^2}{\beta}\left(G(0) - G(x_1 - x_2)\right)}.$$

IB-5 Explain why $G(x)$ is actually IR divergent[1] and what is the origin of this divergence, but that $G(0) - G(x)$ is finite for all x. Show that

$$G(0) - G(x) = \frac{1}{2\pi} \log(|x|/a) + \text{const.} + O(1/|x|).$$

IB-6 Deduce that the correlation functions C_α decrease algebraically at large distance according to

$$C_\alpha(x_1, x_2) \simeq \text{const.}\, (a/|x_1 - x_2|)^{\alpha^2/2\pi\beta}.$$

Compare with the high temperature expansion.

II- The role of vortices in the XY field theory

[1] So that, when defining $G(x)$, we implicitly assumed the existence of an IR cut-off, say $|p| > 2\pi/L$ with L the linear size of the box on which the model is considered.

The previous computations show that the model is disordered at high temperature but critical at low temperature with temperature dependent exponents. The aim of this section is to explain the role of topological configurations, called vortices, in this transition.

We shall now study the model in continuous space, the Euclidean plane $\mathbb{R}^2$, but with an explicit short distance cut-off a. We shall consider the XY system in a disc of radius L.

In the continuous formulation, the spin configurations are then maps Θ from $\mathbb{R}^2$ to $[0, 2\pi]$ modulo 2π. The above Gaussian energy is mapped into the action

$$S_0[\Theta] = \frac{\kappa}{2} \int d^2x (\nabla\Theta)^2,$$

with a coefficient κ proportional to β.

II-1 Argue that the coefficient κ cannot be absorbed into a rescaling of the field variable Θ?

II-2 A vortex, centred at the origin, is a configuration such that $\Theta^{\pm}_{\mathrm{v}}(z) = \pm\mathrm{Arg}(z)$, with z the complex coordinate on $\mathbb{R}^2$, or in polar coordinates[2], $\Theta^{\pm}_{\mathrm{v}}(r, \phi) = \pm\phi$.
Show that $\Theta^{\pm}_{\mathrm{v}}$ is an extremum of S_0 in the sense that $\nabla^2\Theta^{\pm}_{\mathrm{v}} = 0$ away from the origin.
Show that $\oint_{C_0} d\Theta^{\pm}_{\mathrm{v}} = \pm 2\pi$ for C_0 a small contour around the origin.

II-3 Let a_0 be a small short distance cut-off and let $\mathbb{D}(a_0)$ be the complex plane with small discs of radius a_0 around the vortex positions cut out. Prove that, evaluated on $\Theta^{\pm}_{\mathrm{v}}$, the action S_0 integrated over $\mathbb{D}(a_0)$ (with an IR cut-off L) is

$$S^{(1)}_{\mathrm{vortex}} = \frac{\kappa}{2} \int_{\mathbb{D}(a_0)} d^2x\, (\nabla\Theta^{\pm}_{\mathrm{v}})^2 = \pi\kappa \log\big[L/a_0\big].$$

Give an interpretation of the divergence as $a_0 \to 0$.

II-4 What is the entropy of single vortex configurations? Show that the contribution of single vortex configurations to the free energy is

$$e^{-F^{(1)}_{\mathrm{vortex}}} \simeq \mathrm{const.}\, \big(\frac{L}{a_0}\big)^2\, e^{-\pi\kappa \log[L/a_0]}$$

Conclude that vortex configurations are irrelevant for $\pi\kappa > 2$ but relevant for $\pi\kappa < 2$.

<u>IIIA- The XY field theory: Mapping to the sine-Gordon theory</u>

This mapping comes about when considering a gas of pairs of vortices of opposite charges $\pm$, so that the vortex system is neutral ($\sum_a q_a = 0$). We denote x^+_j (resp. x^-_j) the positions of the vortices of charge $+$ (resp. $-$).

The vortex gas is defined by considering all possible vortex pair configurations (with arbitrary number of pairs) and fluctuations around those configurations. We set $\Theta = \Theta^{(2n)}_{\mathrm{v}} + \theta_{\mathrm{sw}}$ and associate to each such configuration a statistical weights e^{-S} with action given by

$$S = S^{(2n)}_{\mathrm{vortex}}[x^+_j, x^-_j] + S_0[\theta_{\mathrm{sw}}],$$

[2] We recall the expression of the gradient in polar coordinates: $\nabla\Theta = (\partial_r\Theta, \frac{1}{r}\partial_\phi\Theta)$. The Laplacian is $\nabla^2 F = \frac{1}{r}\partial_r(r\partial_r)F + \frac{1}{r^2}\partial^2_\phi F$.

with $S_0[\theta_{\rm sw}]$ the Gaussian action $\frac{\kappa}{2}\int d^2x(\nabla\theta_{\rm sw})^2$. We still assume a short-distance cut-off a.

IIIA-1 Write the expression of the action $S^{(2n)}_{\rm vortex}[x_j^+,x_j^-]$ for a collection of n pairs of vortices at positions $x_j^{\pm}$, $j=1,\cdots,n$.

IIIA-2 Argue that the partition function of the gas of vortex pairs is given by the product $Z=Z_{\rm sw}\times Z_{\rm vortex}$ with $Z_{\rm sw}$ the partition function for the Gaussian free field $\theta_{\rm sw}$ and

$$Z_{\rm vortex}=\sum_{n\geq 0}\frac{\mu^{2n}}{n!\cdot n!}\times\int(\prod_{j=1}^n d^2x_j^+\prod_{j=1}^n d^2x_j^-)\,\frac{\prod_{i<j}(|x_i^+-x_j^+|/a)^{2\pi\kappa}(|x_i^- - x_j^-|/a)^{2\pi\kappa}}{\prod_{i,j}(|x_i^+-x_j^-|/a)^{2\pi\kappa}},$$

with $\mu=\big(\frac{a_0}{a}\big)^{\pi\kappa}\,e^{-\beta\epsilon_c}$.

IIIA-3 The aim of the following questions is to express $Z_{\rm vortex}$ as a path integral over an auxiliary bosonic field φ. Let $\tilde S_\kappa[\varphi]=\frac{1}{2\kappa}\int d^2x(\nabla\varphi)^2$ be a Gaussian action. Show that, computed with this Gaussian action,

$$\langle e^{i2\pi\varphi(x)}e^{-i2\pi\varphi(y)}\rangle_{\tilde S_\kappa}=\frac{1}{|x-y|^{2\pi\kappa}}.$$

Hint: The Green function associated to the action $\tilde S_\kappa[\varphi]$ is $G(x,y)=-\frac{\kappa}{2\pi}\log\big(|x-y|/a\big)$.

IIIA-4 What is the scaling dimension (computed with the Gaussian action $\tilde S_\kappa[\varphi]$) of the operators $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$?
Deduce that the perturbation $\cos(2\pi\varphi)$ is relevant for $\pi\kappa<2$ and irrelevant for $\pi\kappa>2$.
Is the the perturbation $(\nabla\varphi)^2$ relevant or irrelevant?

IIIA-5 Show that $Z_{\rm vortex}$ can be written as the partition function of Gaussian bosonic field with action $S_{sG}[\varphi]$,

$$Z_{\rm vortex}=\int[D\varphi]\,e^{-S_{sG}[\varphi]},$$

where the action S_{sG} is defined as

$$S_{sG}[\varphi]=\int d^2x\Big[\frac{1}{2\kappa}(\nabla\varphi)^2-2\mu\cos(2\pi\varphi)\Big].$$

This is called the sine-Gordon action.
Hint: Compute perturbatively the above partition function as a series in μ while paying attention to combinatorial factors.

IIIB- The XY field theory: The renormalization group analysis

IIIB-1 We now study the renormalization group flow of the action S_{sG} for κ close to the critical value $\kappa_c=2/\pi$. We let $\kappa^{-1}=\kappa_c^{-1}-\delta\kappa$ and write

$$S_{sG}[\varphi]=\tilde S_{\kappa_c}[\varphi]-\int d^2x\Big[\frac{1}{2}(\delta\kappa)(\nabla\varphi)^2+2\mu\cos(2\pi\varphi)\Big]$$

Show that, to lowest order, the renormalization group equations for the coupling constants $\delta\kappa$ and μ are of the following form:

$$\begin{aligned}\dot{(\delta\kappa)}=\ell\partial_\ell\,(\delta\kappa)\ &=\ b\,\mu^2+\cdots\\ \dot\mu=\ell\partial_\ell\,\mu\ &=\ a\,(\delta\kappa)\,\mu+\cdots\end{aligned}$$

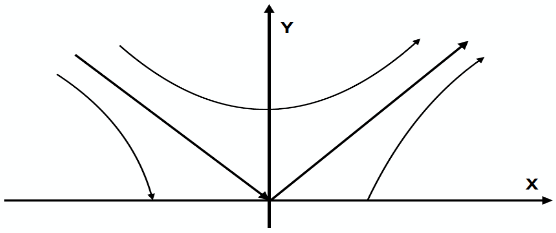

Figure 6: The XY RG flow.

with a and b some positive numerical constants.
Hint: It may be useful to first evaluate the OPE of the fields $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$.

IIIB-2 We redefine the coupling constants and set $X = a\,(\delta\kappa)$ and $Y = \sqrt{ab}\,\mu$ such that the RG equations now reads $\dot{X} = Y^2$ and $\dot{Y} = XY$.
Show that $Y^2 - X^2$ is an invariant of this RG flow.
Draw the RG flow lines in the upper half plane $Y > 0$ near the origin.

IIIB-3 We look at the flow with initial condition $X_I < 0$ and Y_I.
Show that if $Y_I^2 - X_I^2 < 0$ and $X_I < 0$, then the flow converges toward a point on the line $Y = 0$.
Deduce that for such initial condition the long distance theory is critical. Compare with section I-B.

IIIB-4 Show that if $Y_I^2 - X_I^2 > 0$ and $X_I < 0$, the flow drives X and Y to large values. Let $Y_0^2 = Y_I^2 - X_I^2$ with $Y_0 > 0$. Show that the solution of the RG equations are

$$\log\big(\frac{\ell}{a}\big) = \frac{1}{Y_0}\Big[\arctan\big(\frac{X(\ell)}{Y_0}\big) - \arctan\big(\frac{X_I}{Y_0}\big)\Big].$$

IIIB-5 The initial condition X_I and Y_I are smooth functions of the temperature T of the XY model. The critical temperature T_c is such that $X_I + Y_I = 0$. We take the initial condition to be near the critical line $X_I + Y_I = 0$ with $X_I < 0$. We let $X_I = -Y_I(1+\tau)$ in which $\tau \ll 1$ is interpreted at the distance from the critical temperature: $\tau \propto (T - T_c)$. For $\tau > 0$, we define the correlation length as the length ξ at which $X(\ell)$ is of order 1.
Why is this a good definition?
Show that

$$\xi/a \simeq \text{const.}\ e^{\text{const.}/\sqrt{\tau}}.$$