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Statistical Field Theory and Applications : An Introduction for (and by) Amateurs

Exercice Book - 2020

(with corrections)

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1 Exercise corrections

1.1 Chapter 2: Brownian motions and random paths

• *Exercise 2.1: Random variables and generating functions.*

Let X be a real random variable. Let its characteristic function (also called generating function) be defined by

$$\Phi(z) := \mathbb{E}[\mathrm{e}^{\mathrm{i}zX}] .$$

We assume henceforth $z \in \mathbb{R}$.

(i) Show that $\Phi(z)$ is always well defined for $X \in \mathbb{R}$ and $z \in \mathbb{R}$.

(ii) Define also the function

$$W(z) := \log \Phi(z)$$

or conversely $\Phi(z) = \mathrm{e}^{W(z)}$.

Expand Φ and W in powers of z and identify the first few Taylor coefficients.

(iii) Suppose that X is an integer-valued discrete random variable having the Poisson distribution

$$\mathbb{P}[X = n] = \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}$$

for $n \in \mathbb{N}$, with parameter λ.

What are its mean, its covariance and its generating function?

(iv) Suppose now that X is a Gaussian variable with probability distribution density

$$\mathbb{P}\left[X \in [x, x + \mathrm{d}x]\right] = \frac{\mathrm{d}x}{\sqrt{2\pi\sigma}}\mathrm{e}^{-\frac{x^2}{2\sigma}} .$$

Verify that $\mathbb{P}$ is correctly normalised and compute its characteristic function.

Correction :

(i) For convenience we suppose that X is a continuous random variable, but the argument for a discrete variable is similar. Let $\mathbb{P}[X]$ denote the probability density function of X. We have then by definition

$$\int \mathbb{P}[x]\,\mathrm{d}X = 1 .$$

This implies that the integral

$$\Phi(z) = \int \mathrm{e}^{\mathrm{i}zX}\mathbb{P}[X]\,\mathrm{d}X$$

is absolutely convergent (since $|\mathrm{e}^{\mathrm{i}zX}| = 1$) and hence simply convergent. Notice that the properties of $\Phi(z)$ under analytical continuation depend on the distribution.

(ii) We have

$$\Phi(z) = 1 + \mathrm{i}z\mathbb{E}[X] - \frac{z^2}{2}\mathbb{E}[X^2] + \ldots = \sum_{n=0}^{\infty} \frac{(\mathrm{i}z)^n}{n!}\mathbb{E}[X^n]\,,$$
$$W(z) = \mathrm{i}z\mathbb{E}[X] - \frac{z^2}{2}\left(\mathbb{E}[X^2] - \mathbb{E}[X]^2\right) + \ldots = \sum_{n=1}^{\infty} \frac{(\mathrm{i}z)^n}{n!}\mathrm{Cum}_n[X]\,.$$

We see that $\Phi(z)$ is the moment generating function of X, while $W(z)$ is the cumulant generating function. The quantity $\mathrm{Cum}_n[X]$ is called the n'th cumulant of X.

(iii) Note that $\mathbb{P}[X]$ is normalised:

$$\sum_{n=0}^{\infty} \mathbb{P}[X = n] = 1\,.$$

We find

$$\Phi(z) = \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}zn}\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda} = \mathrm{e}^{-\lambda}\mathrm{e}^{\lambda \mathrm{e}^{\mathrm{i}z}} = \mathrm{e}^{\lambda(\mathrm{e}^{\mathrm{i}z}-1)}$$

and thus

$$W(z) = \lambda(\mathrm{e}^{\mathrm{i}z} - 1) = \lambda \sum_{n=1}^{\infty} \frac{(\mathrm{i}z)^n}{n!}\,.$$

From $W(z)$ we read off that $\mathrm{Cum}_n[X] = \lambda$ for any $n \geq 1$. In particular

$$\mathbb{E}[X] = \lambda\,, \qquad \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda\,.$$

(iv) The normalisation is a standard integral. Otherwise, consider two independent Gaussians such that

$$\mathbb{P}\left[X \in [x, x + \mathrm{d}x] \times [y, y + \mathrm{d}y]\right] = \frac{\mathrm{d}x\,\mathrm{d}y}{2\pi\sigma}\,\mathrm{e}^{-\frac{x^2+y^2}{2\sigma}}\,.$$

Change from Carthesian coordinates (x, y) to polar coordinates (r, θ). Then $x^2 + y^2 = r^2$. Moreover the area element is $r\,\mathrm{d}r\,\mathrm{d}\theta$, so that we obtain the correct normalisation:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi\sigma}}\,\mathrm{e}^{-\frac{x^2+y^2}{2\sigma}} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \int_0^{\infty} \frac{r\,\mathrm{d}r}{\sigma}\,\mathrm{e}^{-\frac{r^2}{2\sigma}} = \left[-\mathrm{e}^{-\frac{r^2}{2\sigma}}\right]_0^{\infty} = 1\,.$$

To compute $\Phi(z)$ we complete the square and change variables:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}zX}] = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi\sigma}}\,\mathrm{e}^{-\frac{x^2}{2\sigma}+izx} = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi\sigma}}\,\mathrm{e}^{-\frac{1}{2\sigma}(x-\mathrm{i}\sigma z)^2}\,\mathrm{e}^{-\frac{\sigma z^2}{2}} = \mathrm{e}^{-\frac{\sigma z^2}{2}}\,.$$

In particular $W(z) = -\frac{z^2}{2}\sigma$, so that $\mathrm{Cum}_2[X] = \sigma$ is the only non-zero cumulant.

• *Exercise 2.2: Random Gaussian vectors.*

Let $\vec{X}$ be an N-dimensional Gaussian random vector with real coordinates X^i, for $i = 1, \ldots, N$. By definition its probability distribution is

$$\mathbb{P}(X)\mathrm{d}^N X = \mathrm{d}^N X \left(\frac{\det G}{(2\pi)^N}\right)^{1/2} \exp\left(-\frac{1}{2}\langle X|G|X\rangle\right)\,,$$

with $\langle X|G|X\rangle := \sum_{ij} X^i G_{ij} X_j$, where the real symmetric form G_{ij} is supposed to be non-degenerate. Denote by $\hat{G}$ its inverse: $\sum_j G_{ij}\hat{G}^{jk} = \delta_i^k$.

(i) Verify that this distribution is normalised, that is:

$$\int \frac{\mathrm{d}^N X}{(2\pi)^{N/2}}\, \mathrm{e}^{-\frac{1}{2}\langle X|G|X\rangle} = (\det G)^{-1/2}\,.$$

(ii) For a vector U living in the dual space with respect to X, we define $\langle U|X\rangle = \sum_i U_i X^i$. Show that the corresponding generating function is

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle U|X\rangle}] = \mathrm{e}^{-\frac{1}{2}\langle U|\hat{G}|U\rangle}\,.$$

(iii) Show that the mean $\mathbb{E}[X^i] = 0$ and the covariance $\mathbb{E}[X^i X^j] = \hat{G}^{ij}$.

Correction :

(i) G is real and symmetric, so it can be diagonalised by an orthogonal matrix $\mathcal{O}$:

$$G = \mathcal{O}^{-1} D \mathcal{O} = \mathcal{O}^{\mathrm{T}} D \mathcal{O}\,,$$

where $D = \mathrm{diag}(d_i)$ denotes a diagonal matrix and d_i are the eigenvalues of G.
Now change variables $X \to Y := \mathcal{O}^{\mathrm{T}} X$. The Jacobian of this transformation is $|\det \mathcal{O}| = 1$, which follows from $\mathcal{O}^{\mathrm{T}} = \mathcal{O}^{-1}$ and the fact that $\det(A^{-1}) = (\det A)^{-1}$. Using

$$\langle X|G|X\rangle = \langle X\mathcal{O}|\mathcal{O}^{\mathrm{T}} G \mathcal{O}|\mathcal{O}^{\mathrm{T}} X\rangle = \langle Y|D|Y\rangle$$

the required integral becomes

$$\prod_{i=1}^{N} \left(\int \frac{\mathrm{d}Y_i}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2} d_i Y_i^2} \right) = \prod_{i=1}^{N} (d_i)^{-1/2} = (\det G)^{-1/2}$$

as required.

(ii) Let us examine the argument of the exponential in the corresponding Gaussian integral. We have

$$\frac{1}{2}\langle X|G|X\rangle - \mathrm{i}\langle U|X\rangle = \frac{1}{2}\langle X - \mathrm{i}V|G|X - \mathrm{i}V\rangle + \frac{1}{2}\langle V|G|V\rangle\,,$$

where we have set $GV = U$. The first piece corresponds to the completion of the square in the one-dimensional case, and it will integrate to unity thanks to the normalisation (after the change of variables with trivial Jacobian). Since $V = \hat{G}U$ we can rewrite the second piece as $\frac{1}{2}\langle U|\hat{G}|U\rangle$. Therefore

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle U|X\rangle}] = \mathrm{e}^{-\frac{1}{2}\langle U|\hat{G}|U\rangle}$$

proving the result.

(iii) This follows directly from $W(z) = \log \Phi(z) = -\frac{1}{2}\langle U|\hat{G}|U\rangle$ and its expansion in terms of cumulant (similar to the one-dimensional computation).

- *Exercise 2.3: The law of large number and the central limit theorem.*

The aim of this exercise is to prove (a simplified version of) the central limit theorem. Let ϵ_k, with $k = 1, \ldots, n$, be independent identically distributed (ii.d) variables. Each $\epsilon_k = \pm 1$ with equal probabilities.

In this case, the central limit theorem states that the sum $\hat{S}_n = \frac{1}{\sqrt{n}} \sum_k \epsilon_k$ converges (to be precise, in law) in the $n \to \infty$ limit to a Gaussian variable.

(i) Prove that

$$\mathbb{E}[e^{iz\hat{S}_n}] = \Big[\cos\big(\frac{z}{\sqrt{n}}\big)\Big]^n \overset{n\to\infty}{\longrightarrow} e^{-\frac{z^2}{2}},$$

and conclude.
Hint: Recall the Taylor expansion $\cos(\frac{z}{\sqrt{n}}) = 1 - \frac{z^2}{2n} + o(\frac{1}{n})$ and use $\lim_{n\to\infty}[1 - \frac{y}{n} + o(\frac{1}{n})]^n = e^{-y}$ (which can be proved by taking the logarithm).

Correction :

(i) First note that

$$e^{\frac{iz}{\sqrt{n}}\sum_k \epsilon_k} = \prod_{k=1}^{n} e^{\frac{iz}{\sqrt{n}\epsilon_k}}.$$

Taking expectations values we obtain

$$\mathbb{E}[e^{iz\hat{S}_n}] = \prod_{k=1}^{n}\mathbb{E}[e^{\frac{iz}{\sqrt{n}}\epsilon_k}] = \prod_{k=1}^{n}\cos\left(\frac{z}{\sqrt{n}}\right) = \left[\cos\left(\frac{z}{\sqrt{n}}\right)\right]^n.$$

We replace the cosine by the first two terms in its Taylor expansion and take the limit:

$$\lim_{n\to\infty}\left(1 - \frac{z^2}{2n}\right)^n = e^{-\frac{z^2}{2}}.$$

- *Exercise 2.4: Free random paths.*

The scaling limit of free random paths has been treated in section 2.2 of the main text. We recall that such paths are defined on a hypercubic lattice in D dimensions. Each step can be written $\pm a\mathbf{e}_j$, where $\mathbf{e}_j$ for $j = 1, \ldots, D$ is a basis of orthonormal vectors in $\mathbb{R}^D$, and is associated with a Boltzmann weight (fugacity) μ. The partition function for paths going from 0 to $\mathbf{x}$ is

$$Z(\mathbf{x}) = \sum_{\Gamma:0\to\mathbf{x}} \mu^{|\Gamma|},$$

where $|\Gamma|$ denotes the length of the path Γ. It satisfies the difference equation

$$Z(x) = \delta_{x;0} + \mu\sum_{j=1}^{D}\big(Z(x + a\mathfrak{e}_j) + Z(x - a\mathfrak{e}_j)\big). \tag{1}$$

(i) Compute the Fourier transform of $Z(x)$ and prove that

$$Z(x) = \int_{\mathrm{BZ}} \frac{d^D\mathbf{k}}{(2\pi/a)^D}\,\frac{e^{i\mathbf{k}\cdot x}}{1 - 2\mu\sum_j \cos(a\mathbf{k}\cdot\mathbf{e}_j)}, \tag{2}$$

where $\mathrm{BZ} = \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]^D$ is the Brillouin zone of the square lattice.

(ii) Let $\Delta^{\text{dis.}}$ be the discrete Laplacian and write $\Delta^{\text{dis.}} = \Theta - 2D\,\mathbb{I}$ with Θ the lattice adjacency matrix and $\mathbb{I}$ the identity matrix. We view Θ as acting on functions via $(\Theta \cdot f)(x) = \sum_{j=1}^{D} \big(f(x + a\mathfrak{e}_j) + f(x - a\mathfrak{e}_j)\big)$. Show that:

$$Z(x) = \langle x| \frac{1}{\mathbb{I} - \mu\,\Theta} |0\rangle,$$

with $|x\rangle$ the δ-function peaked at x, i.e. $\langle y|x\rangle = \delta_{y;x}$.
Give an expression of $W_N^{\text{free}}(\mathbf{x}$ as matrix elements of powers of the matrix Θ and give a geometrical interpretation of this formula.

(iii) Deduce from this formula that $Z(x)$ converges for $|\mu| < \mu_c$ with $\mu_c = 1/2D$.

(iv) Prove the formula for the Green function $G(x)$ given in the main text.

Correction :

(i) Let $\hat{Z}(\mathbf{k})$ denote the Fourier transform of $Z(\mathbf{x})$. By definition, the two are related by

$$Z(\mathbf{x}) = \int_{\text{BZ}} \frac{\mathrm{d}^D\mathbf{k}}{(2\pi/a)^D}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \hat{Z}(\mathbf{k})\,.$$

The strategy is to Fourier transform the difference equation (1). The Kronecker delta obviously transforms as

$$\delta_{\mathbf{x},0} = \int_{\text{BZ}} \frac{\mathrm{d}^D\mathbf{k}}{(2\pi/a)^D}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,.$$

The shifted partition functions $Z(\mathbf{x} \pm a\mathbf{e}_j)$ will provide extra factors of $\mathrm{e}^{\pm \mathrm{i}a\mathbf{k}\cdot\mathbf{e})j}$, and moving this to the left-hand side we obtain:

$$\Big(1 - 2\mu \sum_j \cos(a\mathbf{k}\cdot\mathbf{e}_j)\Big)\hat{Z}(\mathbf{k}) = 1\,.$$

Isolating $\hat{Z}(\mathbf{k})$ we apply the inverse Fourier transform on both sides and obtain indeed

$$Z(\mathbf{x}) = \int_{\text{BZ}} \frac{\mathrm{d}^D\mathbf{k}}{(2\pi/a)^D} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{1 - 2\mu\sum_j \cos(a\mathbf{k}\cdot\mathbf{e}_j)}\,.$$

(ii) The difference equation (1) can be written as

$$Z(\mathbf{x}) = \delta_{\mathbf{x},0} + \mu(\Theta \circ Z)(\mathbf{x})\,.$$

Hence $(\mathbb{I} - \mu\Theta)\circ Z(\mathbf{x}) = \delta_{\mathbf{x},0} = \langle 0|x\rangle$. Multiplying both sides by the inverse of $\mathbb{I} - \mu\Theta$ produces the result.
An alternative derivation is obtained by expanding $(\mathbb{I} - \mu\Theta)^{-1}$ as a formal geometric series:

$$\big\langle 0\,\big|(\mathbb{I} - \mu\Theta)^{-1}\big|\,x\big\rangle = \sum_{N=0}^{\infty} \mu^N \langle 0|\Theta^N|x\rangle\,.$$

On the right-hand side Θ^N means producing an N-step walk, so its matrix element $\langle 0|\Theta^N|x\rangle = W_N^{\text{free}}(\mathbf{x})$, the number of N-step paths $\Gamma : 0 \to \mathbf{x}$. Recalling that

$$Z(\mathbf{x}) = \delta_{\mathbf{x},0} + \sum_{N>0} \mu^N W_N^{\text{free}}(\mathbf{x})$$

we obtain once again

$$\big\langle 0\,\big|(\mathbb{I} - \mu\Theta)^{-1}\big|\,x\big\rangle = Z(\mathbf{x})\,.$$

(iii) It is easy to see that Δ^{dis} is a non-positive operator (all its eigenvalues are non-positive). Moreover, the sum of each row (or column) is zero, so it has a zero eigenvalue. It follows from $\Delta^{\text{dis}} = \Theta - 2D\mathbb{I}$ that the spectral radius of Θ is $\rho(\Theta) = 2D$. (Recall that the spectral radius is defined as the supremum among the absolute values of the elements in its spectrum.) It follows that the series $\sum_{N=0}^{\infty} \mu^N \Theta^N$ is convergent for $|\mu| < \frac{1}{2D}$. By the above expressions the same is then true for $Z(\mathbf{x})$.

Remark: For self-avoiding walks the existence of a finite μ_c can be proved by subadditivity arguments, but the exact value of μ_c is only known for a few specific models. In particular, μ_c for self-avoiding walks on a $D = 2$ hexagonal lattice is known exactly ($\mu^{-1} = \sqrt{2+\sqrt{2}}$) and was derived non-rigorously by Nienhuis in 1982 and proved by Duminil-Copin and Smirnov in 2010. For the $D = 2$ square lattice we have only very precise estimates, the latest in date (2016) being $\mu^{-1} = 2.638\,158\,530\,327\,90 \pm 0.000\,000\,000\,000\,03$.

(iv) If we take the lattice spacing $a \to 0$, the difference equation (1) reads $(1 - 2D\mu)Z(\mathbf{x}) = \delta_{\mathbf{x},0}$. Since $a^{-D}\delta_{\mathbf{x},0} \to \delta(\mathbf{x})$ we obtain

$$\lim_{a\to 0} a^{-D} Z(\mathbf{x}) = \left(1 - \frac{\mu}{\mu_c}\right)^{-1} \delta(\mathbf{x})\,.$$

The naive conclusion of this is that $Z(\mathbf{x})$ is proportional to $\delta(\mathbf{x})$, implying that all random walks remain concentrated at the origin. The less naive conclusion is that this can be avoided if we simultaneously take the limit $\mu \to \mu_c$, because then the prefactor will diverge. Obviously, then, we need to take the $a \to 0$ limit more carefully to obtain non-trivial results. Let us rewrite the difference equation as

$$\left(1 - \frac{\mu}{\mu_c}\right) Z(\mathbf{x}) = \delta_{\mathbf{x},0} + \mu \left(\Delta^{\text{dis}} \circ Z\right)(\mathbf{x})\,.$$

As discussed in the main text, the correct limit of the discrete Laplacian is $a^{-2}\Delta^{\text{dis}} \to \Delta_{\mathbf{x}}$. Hence, multiplying the above equation by $\mu_c a^{-D}$ gives

$$\mu\left(\mu^{-1} - \mu_c^{-1}\right) a^{-2} G_a(\mathbf{x}) = \mu_c \delta(\mathbf{x}) + \mu \Delta_{\mathbf{x}} G_a(\mathbf{x})\,,$$

where we have introduced the Green's function $G_a(\mathbf{x}) = \mu_c a^{2-D} Z(\mathbf{x})$. Recall that $\mu_c = \frac{1}{2D}$. Let us now set

$$m^2 = a^{-2}\left(\mu^{-1} - \mu_c^{-1}\right)\,.$$

The idea is to take the limit $a \to 0$ and $\mu \to \mu_c$ simultaneously, whilst keeping the "mass" m finite. Inserting and rearranging we obtain

$$\frac{\mu}{\mu_c}\left(m^2 - \Delta_{\mathbf{x}}\right) G(\mathbf{x}) = \delta(\mathbf{x})\,,$$

where $G(\mathbf{x}) = \lim_{a\to 0} G_a(\mathbf{x})$. Since $\mu \sim \mu_c$ we obtain the desired equation for the Green's function:

$$\left(-\Delta_{\mathbf{x}} + m^2\right) G(\mathbf{x}) = \delta(\mathbf{x})\,.$$

This can be solved for $G(\mathbf{x})$ in various fashions. One option is to proceed by Fourier transform, obtaining

$$G(\mathbf{x}) = \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{m^2 + \mathbf{k}^2}\,.$$

Another way is to insert $2\sum_j \cos(a\mathbf{k}\cdot\mathbf{e}_j) = 2D + a^2\mathbf{k}^2 + \cdots$ into the Fourier representation (2) of $Z(\mathbf{x})$. This gives

$$\hat{Z}(\mathbf{k}) = \frac{\mu^{-1}}{\mu^{-1} - \mu_c^{-1} + a^2\mathbf{k}^2 + \cdots} = a^{-2}\frac{\mu_c^{-1}}{m^2 + \mathbf{k}^2} + \cdots$$

The result follows from inverse Fourier transform of this expression, recalling the definition of the Green's function.

- *Exercise 2.5: Computation of a path integral Jacobian determinants.*

The aim of this exercise is to compute the determinant $\mathrm{Det}[\partial_t - A(t)]$ of the linear map acting functions $f(t)$ as follows $f(t) \to (Jf)(t) = f'(t) - A(t)f(t)$ with $A(t)$ a given function. Instead of computing directly this determinant we factorize the derivation operator and we write $\mathrm{Det}[\partial_t - A(t)] := \mathrm{Det}[\partial_t] \times \mathrm{Det}[1 - K]$. The operator K is defined by integration as follows:

$$K : f(t) \to (Kf)(t) = \int_0^t ds A(s) f(s),$$

for any function f defined on the finite interval $[0, T]$. The aim of this exercise is thus to compute the determinant $\mathrm{Det}[1 - K]$ and to prove that

$$\mathrm{Det}[1 - K] = e^{-\alpha \int_0^T ds\, A(s)},$$

with α a parameter depending on the regularization procedure ($\alpha = 0$ for Itô and $\alpha = \frac{1}{2}$ for Stratonovich conventions). This illustrates possible strategy to define and compute functional –infinite dimensional– determinants.

To define the determinant $\mathrm{Det}[1 - K]$ we need to discretize it by representing the integral of any function by a Riemann sum. Let us divide the interval $[0, T]$ in N sub-interval $[n\delta, (n+1)\delta]$ with $n = 0, \cdots N-1$ and $\delta = T/N$. We will then take the limit $N \to 0$. To simplify notation we denote $f_n := f(n\delta)$. There are many possible discretizations but we shall only consider two of them (which correspond to the Itô and Stratonovich conventions):

$$\text{Ito} \quad : \quad \int_0^t f(t)dt := \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} f_k,$$

$$\text{Stratonovich} \quad : \quad \int_0^t f(t)dt := \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} \frac{1}{2}(f_k + f_{k+1}).$$

(i) Write the regularized action of the operator K on function f by writing the expression of $(Kf)_n$.

(ii) Show that the operator $1 - K$ is lower triangular and determine the diagonal entries (which are convention dependent).

(iii) Deduce, by taking the large N limit, the formula for the determinant:

$$\text{Ito} \quad : \quad \mathrm{Det}[1 - K] = 1,$$

$$\text{Stratonovich} \quad : \quad \mathrm{Det}[1 - K] = e^{-\frac{1}{2}\int_0^T ds\, A(s)}.$$

More general discretization are defined by sampling differently the Riemann sum as follows: $\int_0^t f(t)dt = \lim_{N\to\infty} \delta \sum_{k=0}^{n-1}((1-\alpha)f_k + \alpha f_{k+1})$. Following the same strategy as above, it is then clear that $\mathrm{Det}[1 - K] = e^{-\alpha \int_0^T ds\, A(s)}$.

Correction :

(i) If f is a function with $f_k = f(k\delta)$ for $k = 1, \cdots, N$ and $\delta = T/N$, we have:

$$\text{Ito}: \qquad (Kf)_n = \lim_{N\to\infty} \delta \sum_{k=0}^{n-1} A_k f_k,$$

$$\text{Strato}: \qquad (Kf)_n = \lim_{N\to\infty} \frac{\delta}{2} \sum_{k=0}^{n-1} (A_k f_k + A_{k+1} f_{k+1}),$$

with $A_k = A(k\delta)$ for $k = 1, \cdots, N$.

(ii) From the above expressions we see that the operator $(\mathbb{I} - K)$ is triangular and the diagonal matrix elements are $\text{Diag}((\mathbb{I} - K)f)_n = 1$ with the Itô convention and $\text{Diag}((\mathbb{I} - K)f)_n = (1 - \frac{\delta}{2} A_n) f_n$ with the Stratonovich convention.

(iii) In both cases the determinant si product of the diagonal elements. Thus

$$\text{Ito}: \qquad \text{Det}(\mathbb{I} - K) = \lim_{N\to\infty} \prod_{k=0}^{N} 1 = 1,$$

$$\text{Strato}: \qquad \text{Det}(\mathbb{I} - K) = \lim_{N\to\infty} \prod_{k=0}^{N} (1 - \frac{\delta}{2} A_k) = \lim_{N\to\infty} e^{\sum_{k=0}^{N} \log(1 - \frac{T}{2N} A_k)} = e^{-\frac{1}{2}\int_0^T ds A(s)},$$

This construction can easily be generalized if we sample the Riemann sum differently (by weighting differently the end points of the elementary intervals).

- *Exercise 2.6: Levy's construction of the Brownian motion.*

The path integral representation is actually closely related to an older (!) construction of the Brownian motion due to P. Levy. The aim of this exercise is to present the main point of Levy's approach which constructs the Brownian paths by recursive dichotomy.

We aim at constructing the Brownian curves on the time interval $[0, T]$ starting point x_0. The construction is recursive:

(a) First, pick the end point x_T with the Gaussian probability density $\frac{dx_T}{\sqrt{2\pi T}} e^{-(x_T - x_0)^2/2T}$ and draw (provisionally) a straight line from x_0 to x_T.

(b) Second, construct the intermediate middle point $x_{T/2}$ at time $T/2$ by picking it randomly from the Gaussian distribution centered around the middle of the segment joining x_0 to x_T, and with the appropriate covariance to be determined. Then, draw (provisionally) two straight lines from x_0 to $x_{T/2}$ and from $x_{T/2}$ to x_T.

(c) Next, iterate by picking the intermediate points at times $T/4$ and $3T/4$, respectively, from the Gaussian distribution centered around the middle point of the two segments drawn between x_0 and $x_{T/2}$ and between $x_{T/2}$ and x_T, respectively, and with the appropriate covariance. Then draw (provisionally) all four segments joining the successive points x_0, $x_{T/4}$, $x_{T/2}$, $x_{3T/4}$ and $x_{T/2}$.

(d) Iterate ad infinitum...

Show that this construction yields curves sampled with the Brownian measure.
Hint: This construction works thanks to the relation

$$\frac{(x_i - x)^2}{2(t/2)} + \frac{(x - x_f)^2}{2(t/2)} = \frac{(x_i - x_f)^2}{2t} + \frac{(x - \frac{x_i + x_f}{2})^2}{2(t/4)}$$

Correction :

Follow the instructions given in the text. The construction is such that to ensures that

$$d\mathbb{P}(x_T|x_0)\, d\mathbb{P}(x_{T/2}|x_0; x_T) = \mathbb{P}(x_{T/2}|x_0)dx_T\, \mathbb{P}(x_T|x_{T/2})dx_{T/2}.$$

• *Exercise 2.7: The over-damped limit of the noisy Newtonian particle.*

Consider Newton's equation for a particle of mass m subject to a friction and random forcing (white noise in time). That is, consider the SDEs:

$$dX_t = \frac{P_t}{m}\, dt, \quad dP_t = -\gamma\, dX_t + dB_t,$$

with X_t the position and P_t the momentum. We are interested in the limit $m \to 0$ (or equivalently γ large). Let us set $m = \epsilon^2$ to match the Brownian scaling. Then show that:

(i) the process γX_t^ϵ converges to a Brownian motion B_t;

(ii) $Y_t^\epsilon := \epsilon\, \dot{X}_t^\epsilon$ converges to a finite random variable with Gaussian distribution.

That is: Introducing the mass, or ϵ, is a way to regularize the Brownian curves in the sense that X_t^ϵ admits a time derivative contrary to the Brownian motion. But quantities such as Y_t^ϵ, which are naively expected to vanish in the limit $\epsilon \to 0$, actually do not disappear because the smallness of ϵ is compensated by the irregularities in $\dot{X}_t^\epsilon$ as $\epsilon \to 0$. For instance $\mathbb{E}[\frac{1}{2}m\dot{X}_t^2]$ is finite in the limit $m \to 0$. Such phenomena—the existence of naively zero but nevertheless finite quantities due to the emergence of irregular structures in absence of regularizing—are common in statistical field theory, and are (sometimes) called 'anomaly'.

Correction :

We let $m = \epsilon^2$. The SDE are then $dX_t = \frac{P_t}{\epsilon^2}dt$ and $dP_t = -\gamma dX_t + dB_t$. We denote by $(X_t^\epsilon, P_t^\epsilon)$ a solution of this SDE for a given ϵ.

(i) The SDE for P_t can be written as $dP_t = -\frac{\gamma}{2}P_t dt + dB_t$, which is an Ornstein-Uhlenbeck process. The unique typical time scale is this of order ϵ^2, by dimensional analysis. This means that at time of order $\epsilon^2 \to 0$ the variable P is distribued according to the invariant measure of the SDE. This invariant measure is proportional to $e^{-\gamma P^2/\epsilon^2}dP$.It goes to the Dirac measure $\delta(0)$ centred at 0 as $\epsilon \to 0$. Hence, $P_t = 0$ with probability one as $\epsilon \to 0$. Since $dP_t = d(B_t - \gamma X_t)$, this implies that $\gamma X_t = B_t$ with probability one.

(ii) This is proved by making the previous time change explicit. Let $s = t/\epsilon^2$ and $W_s = B_t/\epsilon$. Then W_s is a normalized Brownian motion, with $dW_s^2 = ds$. Let $Z_s = P_t\epsilon$ then $dZ_s = -\frac{\gamma}{\epsilon}\frac{P_t}{\epsilon^2}dt + \frac{1}{\epsilon}dB_t$ or equivalently,

$$dZ_s = -\gamma Z_s ds + dW_s.$$

This is again an Ornstein-Uhlenbeck process. At fixed t, $s = t/\epsilon^2 \to \infty$ as $\epsilon \to 0$. Hence $Z_s \to Z_\infty$ which is a random variable distributed according to the invariant measure of the SDE $dZ_s = -\gamma Z_s ds + dW_s$. This measure is proportional to the Gaussian measure $e^{-\gamma^2 z^2} dz$.

Finally, let $Y_t = \epsilon \dot{X}_t$. Using $dX_t = \frac{P_t}{\epsilon^2}dt$, we have $Y_t = P_t/\epsilon$ or equivalently $Y_t = \epsilon \dot{X}_t = Z_s$. This proves that, for t fixed, $(\epsilon \dot{X}_t)$ is a finite Gaussian random variable in the limit $\epsilon \to 0$. Notice that there is a compensation between ϵ and the irregularity of $\dot{X}_t$ which increases as ϵ.

- *Exercise 2.8: SDEs with 'multiplicative' noise.*

Generalize the results described above for a more general SDE of the form

$$dX_t = a(X_t)dt + b(X_t)dB_t$$

with $a(x)$ and $b(x)$ smooth non constant functions. To deal with the small noise limit one may introduce a small parametr ϵ by rescaling $b(x)$ via $b(x) \to \epsilon\, b(x)$.

(i) Prove that the Fokker-Planck operator for SDEs reads

$$\mathcal{H} = \partial_x\big(\frac{1}{2}\partial_x\, b^2(x) - a(x)\big)$$

(ii) Verify that the invariant measure (if normalizable) is

$$\mathbb{P}_{\rm inv}(x)\, dx = b^{-2}(x)\, e^{-2s(x)}\, dx, \quad s(x) := -\int^x dy \frac{a(y)}{b^2(y)}.$$

What is the invariant measure if the later is not normalizable?
What is then the physical interpretation of this new measure?

(iii) Show that the action of the path integral representation of these SDEs is

$$S = \frac{1}{2}\int_0^T ds\, \frac{(\dot{x}_s - a(x_s))^2}{b^2(x_s)}.$$

in the small noise limit $\epsilon \ll 1$. Verify (by going back to the discret formulation) that this way of writing the action is still valid away from the small noise limit provided that one carefully defined the integrals.

Correction :

A simple adaptation of the construction explained in the lecture notes in the case $b(x) = 0$.

- *Exercise 2.9: Multivariable SDEs*

Generalize all these results for multivariable SDEs of the form $dX^i = a^i(X)\, dt + b^i_j(X)\, dB^j_t$ where B^j are Brownian motions with covariance $\mathbb{E}[B^i_t B^j_s] = \delta^{ij}\min(t,s)$.

1.2 Chapter 3: Statistical lattice models

- *Exercise 3.1: Fermionic representation of the 2D Ising model*

The aim of this exercise is to complete the study of the 2D Ising model presented in the lecture notes. Recall the definition of the 2D Ising model given in the text.

(i) Prove—or argue—that the Ising model is described by the Hamiltonian

$$\mathcal{H} = -\gamma \sum_{x=1}^{N_x} \tau_x^3 - \beta \sum_{x=1}^{N_x} \tau_x^1 \tau_{x+1}^1 \,,$$

where $\gamma = \mathrm{e}^{-2K_y}$ and $\beta = K_x$ are related to the anisotropic coupling constants (K_x, K_y) in the completely anisotropic limit, $K_x \ll 1$ and $K_y \gg 1$.

(ii) Recall the Jordan-Wigner transformations given in the main text which construct fermionic operators in terms of Pauli matrices via

$$a_x = e^{i\pi \sum_{y=1}^{x-1} \tau_y^- \tau_y^+}\, \tau_x^+ , \quad a_x^\dagger = e^{-i\pi \sum_{y=1}^{x-1} \tau_y^- \tau_y^+}\, \tau_x^- .$$

Show that we may alternatively write

$$a_x = \big(\prod_{y=1}^{x-1} \tau_y^z \big)\, \tau_x^+ , \quad a_x^\dagger = \big(\prod_{y=1}^{x-1} \tau_y^z \big)\, \tau_x^- .$$

Verify that they satisfy the canonical fermionic relation $a_x^\dagger a_y + a_y a_x^\dagger = \delta_{x,y}$.

(iii) Show that the Hamiltonian becomes

$$\begin{aligned} \mathcal{H} &= -\gamma \sum_{x=1}^{N_x} \tau_x^3 - \beta \sum_{x=1}^{N_x} \tau_x^1 \tau_{x+1}^1 \\ &= \gamma \sum_{x=1}^{N_x} \left(a_x^\dagger a_x - a_x a_x^\dagger \right) - \beta \sum_{x=1}^{N_x} \left(a_x^\dagger - a_x \right) \left(a_{x+1}^\dagger + a_{x+1} \right) . \end{aligned}$$

(iv) Complete the proof of the diagonalisation of the Ising hamiltonian and its spectrum. Proof that, after an appropriate Bogoliubov transformation on the fermion operators, the Ising hamiltonian can be written in the final form given in the main text, which we recall here,

$$\mathcal{H} = \sum_{k>0} h_k \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger \right) ,$$

with single particle spectrum $h_k = \left[(\gamma - \beta)^2 + 4\gamma\beta \sin^2(k/2) \right]^{1/2}$.

Correction :

(i) The detailed argument appears in the lecture notes. One can also quickly check that this expression is reasonable, in a more loose sense. The second, diagonal term obviously gives the K_x interaction inside a fixed time slice. The first, non-diagonal term is related with the K_y interaction that propagates from one time slice to the next. From the point of view of the transfer matrix, $\mathcal{T} = \mathrm{e}^{K_y N_x} \mathrm{e}^{-\mathcal{H}}$, this term contains a (large, since $K_y \gg 1$) component along the identity corresponding to aligned spins. Taking the logarithm of $\mathcal{T}$, the other small component present in $\mathcal{H}$ hence corresponds to the spin at site x being flipped. This is precisely what τ^3_x does.

(ii) We have, for any x, $[\tau^1_x, \tau^2_x] = 2i\tau^3_x$, and the raising and lowering operators are $\tau^\pm_x = \frac{1}{2}(\tau^1_x \pm i\tau^2_x)$. This implies of course

$$(\tau^+_x)^2 = (\tau^-_x)^2 = 0\,, \quad \tau^+_x \tau^-_x + \tau^-_x \tau^+_x = 1\,.$$

Note that $[\tau^i_x, \tau^j_y] = 0$ for $x \neq y$; the Jordan-Wigner transformation is designed to recover the fermionic relations also when $x \neq y$.
We may write

$$\exp\left(i\pi \sum_{y=1}^{x-1} \tau^-_y \tau^+_y \right) = \prod_{y=1}^{x-1} \exp\left(i\pi \tau^-_y \tau^+_y \right)\,,$$

and note that $\exp\left(i\pi\tau^-_y\tau^+_y\right)$ just gives a sign $+$ to $|\uparrow\rangle_y$, and $-$ to $|\downarrow\rangle_y$. Individual factors may be moved around freely, as long as the act on different tensorands. For example,

$$\begin{aligned} a_x a^\dagger_x &= \exp\left(i\pi \sum_{y=1}^{x-1} \tau^-_y \tau^+_y \right) \tau^+_x \exp\left(-i\pi \sum_{y=1}^{x-1} \tau^-_y \tau^+_y \right) \tau^-_x \\ &= \exp\left(i\pi \sum_{y=1}^{x-1} \tau^-_y \tau^+_y \right) \exp\left(-i\pi \sum_{y=1}^{x-1} \tau^-_y \tau^+_y \right) \tau^+_x \tau^-_x \\ &= \tau^+_x \tau^-_x\,. \end{aligned}$$

In particular $a^\dagger_x a_x + a_x a^\dagger_x = 1$ as required.
Similarly we obtain, for $y < x$,

$$a_x a^\dagger_y + a^\dagger_y a_x = \left(\prod_{k=y}^{x-1} \mathrm{e}^{i\pi\tau^-_k\tau^+_k} \right) \tau^+_k \tau^-_y + \tau^-_y \tau^+_x \left(\prod_{k=y}^{x-1} \mathrm{e}^{i\pi\tau^-_k\tau^+_k} \right)\,.$$

By acting explicitly on the basis states $|\uparrow\rangle_y$ and $|\downarrow\rangle_y$ we see that

$$\tau^-_y \mathrm{e}^{i\pi\tau^-_y\tau^+_y} = -\mathrm{e}^{i\pi\tau^-_y\tau^+_y} \tau^-_y\,,$$

so that

$$a_x a^\dagger_y + a^\dagger_y a_x = \left(\prod_{k=y}^{x-1} \mathrm{e}^{i\pi\tau^-_k\tau^+_k} \right) \left(\tau^+_x \tau^-_y - \tau^-_y \tau^+_x \right) = 0$$

as required.

(iii) The first term is straightforward:

$$\tau^3_x = \tau^-_x \tau^+_x - \tau^+_x \tau^-_x = a^\dagger_x a_x - a_x a^\dagger_x\,,$$

by inverting the JW transformation and noticing that the phases cancel.
For the second term, we first remark that

$$\tau^1_x \tau^1_{x+1} = \left(\tau^+_x + \tau^-_x\right) \left(\tau^+_{x+1} + \tau^-_{x+1}\right)$$

flips both spins at positions x and $x+1$. Writing out $(a_x^\dagger - a_x)\left(a_{x+1}^\dagger + a_{x+1}\right)$ in terms of the τ, it is easy to verify that this also flips the spins at positions x and $x+1$ while leaving all other spins unchanged. This concludes the demonstration.

(iv) As shown in the lecture notes, one easily gets

$$\mathcal{H}_k = 2 \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \gamma - \beta \cos k & -i\beta \sin k \\ i\beta \sin k & -\gamma + \beta \cos k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}.$$

It suffices here to show that the Fourier transformation preserves the fermionic relations. This latter form can be diagonalised by successively applying

$$a_{-k}^\dagger = i\tilde{a}_{-k}^\dagger\,, \qquad a_{-k} = -i\tilde{a}_{-k}$$

and the Bogoliubov transformation (orthogonal rotation)

$$\begin{pmatrix} a_k \\ \tilde{a}_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos\phi_k & \sin\phi_k \\ -\sin\phi_k & \cos\phi_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}.$$

The first transformation gives easily

$$\mathcal{H}_k = 2 \begin{pmatrix} a_k^\dagger & \tilde{a}_{-k} \end{pmatrix} \begin{pmatrix} \gamma - \beta \cos k & \beta \sin k \\ \beta \sin k & -\gamma + \beta \cos k \end{pmatrix} \begin{pmatrix} a_k \\ \tilde{a}_{-k}^\dagger \end{pmatrix}.$$

As for the Bogoliubov transformation, the crucial point is to make sure that c_k and $c_k^\dagger$ are still fermionic. To show this, it is easiest to go the other way around: supposing anticommutation relations for the c, we show that the same is true for the a. For example

$$\begin{aligned} \left\{a_k, a_k^\dagger\right\} &= \left\{\cos(\phi_k)c_k + \sin(\phi_k)c_{-k}^\dagger, \sin(\phi_k)c_{-k} + \cos(\phi_k)c_k^\dagger\right\} \\ &= \cos^2(\phi_k) + \sin^2(\phi_k) = 1\,, \end{aligned} \tag{3}$$

whereas

$$\begin{aligned} \{a_k, \tilde{a}_{-k}\} &= \left\{\cos(\phi_k)c_k + \sin(\phi_k)c_{-k}^\dagger, \cos(\phi_k)c_{-k} - \sin(\phi_k)c_k^\dagger\right\} \\ &= -\cos(\phi_k)\sin(\phi_k) + \cos(\phi_k)\sin(\phi_k) = 0\,. \end{aligned}$$

In terms of the rotation matrix U we now have

$$\mathcal{H}_k = 2 \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} U^\dagger \begin{pmatrix} \gamma - \beta \cos k & \beta \sin k \\ \beta \sin k & -\gamma + \beta \cos k \end{pmatrix} U \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}.$$

It is in fact obvious that U can diagonalise the 2×2 matrix, so it suffices to find its eigenvalues without working out the actual value of ϕ_k. The eigenvalues are $\pm h_k$ with

$$h_k = 2\left[(\gamma-\beta)^2 + 4\gamma\beta \sin^2(k/2)\right]^{1/2},$$

so that

$$H_k = \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} h_k & 0 \\ 0 & -h_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}.$$

- *Exercise 3.2: Spin operators, disorder operators and parafermions.*

The aim of this exercise—and the following two—is to study some simple consequences of group symmetry in lattice statistical models.

Let us consider a lattice statistical model on a two dimensional square lattice $\Lambda := a^2\mathbb{Z}^2$ with spin variables s on each vertex of the lattice. These variables take discrete or continuous values, depending on the models. We consider neighbour spin interactions with a local hamiltonian $H(s,s')$ so that the Boltzmann weight of any given configuration $[c]$ is

$$W([c]) := \prod_{[i,j]=\text{edge}} w_{[i,j]}, \quad w_{[i,j]} = e^{-H(s_i,s_j)},$$

where, by convention, $[i,j]$ denotes the edge connecting the vertices i and j. Let $Z := \sum_{[c]} W([c])$ be the partition function.

Let us suppose that a group G is acting the spin variables. We denote by R the corresponding representation. Furthermore we assume that the interaction is invariant under this group action so that, by hypothesis,

$$H(R(g)\cdot s, R(g)\cdot s') = H(s,s'), \quad \forall g \in G.$$

(i) *Transfer matrix:* Define and construct the transfer matrix for these models.

(ii) *Spin operators:* Spin observables, which we denote $\sigma(i)$, are defined as the local insertions of the spin variables at the lattice site i. That is: $\sigma(i)$ is the function which to any configuration associate the variable s_i.
Write the expectations of the spin observables $\langle \sigma(i_1)\cdots\sigma(i_N)\rangle$ as a sum over configurations weighted by their Boltzmann weights.
Write the same correlation functions in terms of the transfer matrix.

(iii) *Disorder operators:* Disorder observables are defined on the dual lattice and are indexed by group elements. Let Γ be a closed anti-clockwise oriented contour on the square lattice $\tilde{\Lambda}$ dual to Λ –the vertices of $\tilde{\Lambda}$ are the center of the faces of Λ. Let ℓ denote an oriented edge of Γ. It crosses an edge of Λ and we denote by ℓ^- and ℓ^+ the vertices of this edge with ℓ^- inside the loop Γ. The disorder observable $\mu_g(\Gamma)$ for $g \in G$ is defined as

$$\mu_\Gamma(g) := \exp\big(\sum_{\ell\in\Gamma}(H(s_{\ell^-},s_{\ell^+}) - H(s_{\ell^-},R(g)s_{\ell^+})\big),$$

Inserting $\mu_\Gamma(g)$ in the Boltzmann sum amounts to introduce a defect by replacing the hamiltonian $H(s_{\ell^-},s_{\ell^+})$ by its rotated version $H(s_{\ell^-},R(g)s_{\ell^+})$ on all edges crossed by Γ.
Write the expectations of disorder observables in terms of the transfer matrix.

Correction :

(i) The row-to-row transfer matrix can be written as the product of two matrices, one coding for the spin interaction along the horizontal edges and the other for the spin interaction along the vertical edges:

$$T(\{s'\};\{s\}) = \prod_k e^{-H_v(s'_k,s_k)} \prod_k e^{-H_h(s'_k,s_{k+1})}.$$

(ii) By inserting the spin operators at the appropriate places and propagating from row to row in between these insertions with the transfer matrix.

(iii) It is simpler, but generic enough, to describe two special cases where the contour Γ is a portion of horizontal lime or a portion of vertical line. All contours Γ can be reconstructed by gluing such portions of contours.
If Γ is a horizontal segment from point n to point m, and for $g \in G$ it effects modify the transfer matrix which is now

$$\begin{aligned} T_{\Gamma-\text{defect}}(\{s'\};\{s\}) &= \big[\prod_{k<n} e^{-H_v(s'_k,s_k)} \prod_{k<n} e^{-H_h(s'_k,s_{k+1})}\big] \\ &\quad \big[\prod_{k=n}^{m} e^{-H_v(s'_k,gs_k)} \prod_{k=n}^{m} e^{-H_h(gs'_k,gs_{k+1})}\big] \\ &\quad \big[\prod_{k>m} e^{-H_v(s'_k,s_k)} \prod_{k>m} e^{-H_h(s'_k,s_{k+1})}\big]. \end{aligned}$$

If Γ is a vertical segment, it crosses the row on which the transfer matrix acts at a point n say, and for $g \in G$ it effects modify the transfer matrix which is now

$$\begin{aligned} T_{\Gamma-\text{defect}}(\{s'\};\{s\}) &= \big[\prod_{k<n-1} e^{-H_v(s'_k,s_k)} \prod_{k<n} e^{-H_h(s'_k,s_{k+1})}\big] \\ &\quad \big[e^{-H_v(s'_{n-1},s_{n-1})} e^{-H_h(s'_{n-1},gs_n)} e^{-H_v(s'_k,gs_n)}\, e^{-H_h(gs'_n,gs_{n+1})}\big] \\ &\quad \big[\prod_{k>n} e^{-H_v(s'_k,s_k)} \prod_{k>n} e^{-H_h(s'_k,s_{k+1})}\big]. \end{aligned}$$

These formulae are better explained by a drawing.

- *Exercise 3.3: Symmetries, conservation laws and lattice Ward identities*

The aim of this exercise is to understand some of the consequences of the presence of symmetries. The relations we shall obtain are the lattice analogue of the so-called Ward identities valid in field theory.

We consider the same two dimensional lattice model as in previous exercise. We recall that we assume the Bolztmann weight to be invariant under a symmetry group G in the sense that

$$H(R(g)\cdot s, R(g)\cdot s') = H(s,s'), \quad \forall g \in G.$$

(i) Let i_k be points on the lattice Λ and Γ a contour as in previous exercise.. Show that the group invariance implies that

$$\langle \mu_\Gamma(g) \prod_k \sigma(i_k)\rangle = \prod_{i_k \text{ inside } \Gamma} R_{i_k}(g) \cdot \langle \prod_k \sigma(i_k)\rangle,$$

where $R_{i_k}(g)$ denote the group representation R acting on the spins at site i_k.

Show that $\mu_g(\Gamma)$ is invariant under any smooth continuous deformation of Γ as long as the deformation does not cross points of spin insertions (it is homotopically invariant).

We now look at the consequences of these relations for infinitesimal transformations. Suppose that G is a Lie group and Lie(G) its Lie algebra. Let us give a name to small variations of H by defining $\partial_X H$. For $g = 1 + \epsilon\, X + \cdots$ with $X \in \mathrm{Lie}(G)$, we set

$$H(s, R(g)s') - H(s, s') =: \epsilon\, \partial_X H(s, s') + \cdots.$$

For $\ell = [\ell^-, \ell^+]$ an oriented edge of Γ as in previous exercise and $X \in \mathrm{Lie}(G)$, we let

$$*J_\ell^X := \partial_X H(s_{\ell^-}, s_{\ell^+}),$$

They are specific observables, called *currents*, whose correlation functions are defined as usual via insertion into the Bolztmann sums.

(ii) Show that the following equality holds:

$$\langle \sum_{\ell \in \Gamma} *J_\ell^X \cdot \prod_i \sigma(i) \rangle = \langle \big(\sum_{i_k \text{ inside } \Gamma} R_{i_k}(X) \big) \cdot \prod_i \sigma(i) \rangle,$$

if some spin observables are inserted inside Γ.

(iii) Deduce that, if there is no observables inserted inside Γ, then the following equality holds inside any expectation values:

$$\sum_{\ell \in \Gamma} *J_\ell^X = 0,$$

That is: The second of these two equations is a conservation law (i.e. it is the analogue of the fact that $\int *J = 0$ if $*J$ is a closed form, or equivalently, if J is a conserved current), the first tells about the consequences of this conservation law when insertion of observables are taken into account. It is analogous to the Gauss law in electrodynamics. They are called Ward identities in field theory.

Correction :

(i) Do a change of variable $s_i \to g s_i$ (which is assumed to be a bijection) on all spin inside the contour Γ. Thanks to the invariance of the interaction under the group G (i.e. the G-invariance of $H(s', s)$), this changes of variables eliminates the contour Γ and proves the claimed result.

(ii) Take $g = 1 + \epsilon X$ with $\epsilon \ll 1$ and $X \in Lie(G)$. Expand the previous result to first order in ϵ to get the result.

(iii) If there is no insertion inside the contour Γ, then the r.h.s of the previous equation is zero and we have:

$$\langle \sum_{\ell \in \Gamma} *J_\ell^X \cdot \prod_i \sigma(i) \rangle = 0.$$

This means that $\sum_{\ell \in \Gamma} *J_\ell^X = 0$ if there is no insertion inside the contour Γ.

1.3 Chapter 4: From statistical models to field theories

- *Exercise 4.1: Mean field from a variational ansatz*

The aim of this exercise is to derive the Ising mean field approximation form a variational ansatz. We consider the Ising in homogeneous external field h_i so that the configuration energy is $E[s] = -\sum_{i,j} J_{ij} s_i s_j - \sum_i h_i s_i$, with J_{ij} proportional to the lattice adjacency matrix. The Ising spins take values $s_i = \pm$. Let $Z[h]$ be its partition function. (Note that we introduce the external magnetic field with a minus sign).

As an ansatz we consider the model of independent spins in an effective inhomogeneous external field h_i^o with ansatz energy $E^o[s] = -\sum_i h_i^o s_i$, so that the ansatz Boltzmann weights are $Z_0^{-1}\, e^{\beta \sum_i h_i^o s_i}$ with Z_0 the ansatz partition function.

(i) Show that $Z_0 = \prod_i [2\cosh(\beta h_i^o)]$.

(ii) Using a convexity argument, show that $\mathbb{E}_0[e^{-X}] \geq e^{-\mathbb{E}_0[X]}$ for any probability measure $\mathbb{E}_0$ and measurable variable X.

(iii) Choose to be $\mathbb{E}_0$ the ansatz measure and $X = \beta(E[s] - E^o[s])$ to prove that

$$Z[h] \geq Z_0\, e^{-\beta \mathbb{E}_0[E[s] - \beta \mathbb{E}_0[E^o[s]]},$$

or equivalently, $F[h] \leq F_0 - \mathbb{E}_0[E^o[s] - E[s]]$, with $F[h]$ and F_0 the Ising and ansatz free energy respectively.

The best variational ansatz is that which minimizes $F_0 - \mathbb{E}_0[E^o[s] - E[s]]$.

(iv) Compute F_0, $\mathbb{E}_0[E^o[s]]$ and $\mathbb{E}_0[E[s]]$ and show that the quantity to minimize is

$$F_0[h^o] + \sum_i h_i^o \bar{m}_i - \sum_{ij} J_{ij} \bar{m}_i \bar{m}_j - \sum_i h_i \bar{m}_i,$$

where $\bar{m}_i = -\frac{\partial F_0[h^o]}{\partial h_i^o} = \tanh(\beta h_i^o)$ is the local mean magnetization evaluated with the ansatz measure. Show that this minimization problem reduces to the Ising mean field equations.

Correction :

(i) The energy E^o is that of independent spins, $E^o[s] = -\sum_i h_i^o s_i$, so that the partition function is

$$Z_0 = \sum_{\{s_i = \pm\}} e^{\beta \sum_i h_i^o s_i} = \prod_i \left(e^{\beta h_i^o} + e^{-\beta h_i^o}\right) = \prod_i [2\cosh(\beta h_i^o)].$$

We have $\beta F_0 = -\sum_i \log\left(2\cosh(\beta h_i^o)\right)$.

(ii) A function $f(x)$ is siad to be convex is $f(px_0 + (1-p)x_1) \leq pf(x_0) + (1-p)f(x_1)$ for any p such that $0 \leq p \leq 1$ and any pair of points x_0 and x_1. This generalizes (say be recursion) to any probability measure $\mathbb{E}$ over say a variable X so that for any convex function one has $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$. In particular for $f(x) = e^{-x}$, we get $\mathbb{E}_0[e^{-X}] \geq e^{-\mathbb{E}_0[X]}$.

(iii) We take $X = \beta(E[s] - E^o[s])$ as in the text. Then

$$\mathbb{E}_0[X] = \beta\mathbb{E}_0[E] - \beta\mathbb{E}_0[E^o],$$

and

$$\mathbb{E}_0[e^{-X}] = Z_0^{-1}\sum_{[s]} e^{-\beta E^o[s]}\, e^{-\beta(E[s]-E^o[s])} = Z/Z_0 = e^{-\beta(F-F_0)}.$$

(iv) Because of the inequality, the best choice of the variational anstaz $\mathbb{E}_o$ is that which minimizes $F_0 - \mathbb{E}_0[E^o[s] - E[s]]$. We compute successively the three terms. First, as derived above $\beta F_0 = -\sum_i \log\big(2\cosh(\beta h_i^o)\big)$. Second,

$$\mathbb{E}_0[E^o[s]] = -\sum_i h_i^o m_i,$$

with $m_i = \mathbb{E}_0[s_i] = \tanh(\beta h_i^o)$. Third, because the spins are independent under the ansatz measure $\mathbb{E}_o$, we have

$$\mathbb{E}_)[E[s]] = -\sum_{i,j} J_{ij} m_i m_j - \sum_i h_i m_i,$$

The minimizing condition is obtained by computing the derivative of $F_0 - \mathbb{E}_0[E^o[s] - E[s]]$ w.r.t. the variational parameters h_i^o. The variational condition then reads

$$h_i^o = h_i + 2\sum_j J_{ij} m_j,$$

with $m_i = -\frac{\partial F_0}{\partial h_i^o} = \tanh(\beta h_i^o)$. These are the Ising mean field equations.

- *Exercise 4.2: Thermodynamic functions and thermodynamic potentials*

The aim of this exercise is to recall a few basic fact about generating functions, thermodynamic functions and their Legendre transforms.

Let us consider a (generic) spin model and let $E[\{s\}]$ be the energy of a spin configuration $\{s\}$ with local spin s_i. We measure the energy in unit of the temperature so that the Boltzmann weights are $e^{-\beta\, E[\{s\}]}$. Let $Z[0] = \sum_{\{s\}} e^{-\beta\, E[\{s\}]}$ be the partition function. In the following we set $\beta = 1$ (or alternatively include the β-dependence in the other dimensionfull parameter).

(i) Give the expression of the energy $E_h[\{s\}]$ in presence an external inhomogeneous external field h.
Show that the generating function for this spin correlation functions can written as (with $(s,h) = \sum_i s_i h_i$)

$$\mathbb{E}[e^{(s,h)}] = \frac{Z[h]}{Z[0]},$$

Explain why the partition function $Z[h]$ is the generating function for spin correlations.

(ii) Let $F[h]$ be the free energy and let $W[h] = -\big(F[h] - F[0]\big)$. Verify that

$$\log \mathbb{E}[e^{(s,h)}] = W[h].$$

(iii) Let $\Gamma(m)$ be the thermodynamic potential defined as the Legendre transform of $W[h]$. Recall that

$$\Gamma(m) = (m, h_*) - W[h_*], \quad \text{with } \frac{\partial W}{\partial h}[h_*] = m.$$

Verify that this transformation is inverted by writing

$$W[h] = (m_*, h) - \Gamma[m_*], \quad \text{with } \frac{\partial \Gamma}{\partial m}[m_*] = h.$$

Correction :

(i) By definition $E_h[s] = E[s] - (s, h)$ with $(s, h) = \sum_i s_i h_i$. By definition the partition function $Z[h]$ reads (with the convention $\beta = 1$)

$$Z[h] = \sum_{\{s\}} e^{-E_h[s]} = \sum_{\{s\}} e^{-E[s]}\, e^{(s,h)} = Z_0\, \mathbb{E}[e^{(s,h)}],$$

with $\mathbb{E}$ the probability measure specified by the Bolztmann weight $e^{-\beta\, E[\{s\}]}$ (i.e. at zero external fields).

(ii) By definition $Z[h] = e^{-F[h]}$. Hence $\mathbb{E}[e^{(s,h)}] = e^{-F[h]+F[0]}$.

(iii) Usual formula for Legendre transform.

• *Exercise 4.3: An alternative representation of the Ising partition function.*

The aim of this exercise is to explicitly do the computation leading to the representation of the Ising partition function in terms of a bosonic field. It uses a trick—representing the interaction terms via a Gaussian integral over auxiliary varaibles—which find echoes in many other problems.

(i) Prove the following representation of the Ising partition function given in the text (without looking at its derivation given there):

$$Z = \int [\prod_k d\phi_k]\; e^{-S[\phi;h]},$$

with the action

$$S[\phi; h] = -\frac{1}{4} \sum_{ij} \phi_i J_{ij} \phi_j + \sum_i \log[\cosh(h_i + \sum_j J_{ij}\phi_j)].$$

(ii) Deduce what is the representation of the Ising spin variables s_i in terms of the bosonic variables ϕ_i.

Correction :

See the lecture notes.

- *Exercise 4.4: Mean field vector models*

We consider a theory with a vector order parameter $\vec{m}$ of dimension d. We denote by D the dimension of the space. This theory is described by the Landau action

$$S[\vec{m}] = \int d^D r \left\{ \frac{1}{2} \sum_{i=1}^{d} (\vec{\partial}_r m_i)^2 + \frac{a}{2} \sum_{i=1}^{d} m_i^2 + \frac{b}{4} \left(\sum_{i=1}^{d} m_i^2 \right)^2 \right\} .$$

We suppose that the parameters take the form

$$\begin{aligned} a &= a_0 t + O(t^2)\,, \qquad a_0 > 0\,, \\ b &= b_0 + O(t)\,, \qquad b_0 > 0\,, \end{aligned}$$

where $t = (T - T_\mathrm{c})/T_\mathrm{c}$ denotes the reduced temperature.

(i) What is the norm m of the system's spontaneous magnetisation? We write $\vec{m} = m\vec{e}$, where $\vec{e}$ is the direction of the magnetisation.

(ii) We define the susceptibility—or correlation function—by

$$G_{ij}(r - r') = \left. \frac{\delta m_i(r)}{\delta h_j(r')} \right|_{h=0} .$$

Show that G is the inverse matrix of the Hessian ($\equiv$ the matrix of second derivatives) of the action (in the d-dimensional space of components of the order parameter). Compute the Fourier transform $g^{-1}(k)$ of $G^{-1}(r - r')$.

(iii) We introduce the longitudinal projector

$$P^L_{ij} = e_i e_j$$

and the transverse projector

$$P^T_{ij} = \delta_{i,j} - e_i e_j\,.$$

Invert the matrix $g^{-1}(k)$ to obtain $g(k)$. Deduce an expression for $G_{ij}(r - r')$. What are the correlation lengths of the two different modes?

(iv) Optional:What happens in the presence of an external field?

Correction :

(i) The saddle point equation in the absence of an external field reads

$$\frac{\delta S}{\delta m_i} = a m_i + \frac{b}{4}\left(2\sum_{j=1}^{d} m_j^2\right)\cdot 2m_i = 0\,,$$

whence $m_i = 0$, or $\sum_{j=1}^{d} m_j^2 = -\frac{a}{b}$. We have thus

$$\begin{aligned} \vec{m}^2(t) &= 0, \qquad t>0 \\ \vec{m}^2(t) &= -\frac{a}{b}, \qquad t<0. \end{aligned}$$

(ii) We $h_j = \delta S/\delta m_j$ by the saddle point equation. Thus

$$G^{-1}_{ij}(r-r') = \left.\frac{\delta h_j(r')}{\delta m_i(r)}\right|_{h=0} = \left.\frac{\delta^2 S}{\delta m_i(r)\delta m_j(r')}\right|_{h=0}.$$

Note that this result implies that G and G^{-1} are symmetric under $i \leftrightarrow j$ as well as $r \leftrightarrow r'$. This was not obvious in advance.
We now compute G^{-1}. First

$$\begin{aligned} \frac{\delta}{\delta m_i(r)}\int \mathrm{d}^D r' \left(\sum_k m_k(r')^2\right)^2 &= \int \mathrm{d}^D r' \left[2\left(\sum_k m_k(r')^2\right)\cdot 2m_i(r')\delta(r-r')\right] \\ &= 4m_i(r)\left(\sum_k m_k(r)^2\right). \end{aligned}$$

A second derivation $\delta/\delta m_j(r')$ then produces

$$\frac{\delta^2}{\delta m_i(r)\delta m_j(r')}\int \mathrm{d}^D r' \left(m(r')^2\right)^2 = \left(4\delta_{ij}\sum_k m_k^2 + 8 m_i m_j\right)\delta(r-r').$$

Finally

$$G^{-1}_{ij}(r-r') = \left[\left(-\nabla^2 + a + bm^2\right)\delta_{ij} + 2b m_i m_j\right]\delta(r-r').$$

In the disordered phase, $T > T_c$, we have $m = 0$ and the inverse follows by Fourier transformation:

$$G_{ij}(r-r') = \delta_{ij}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\,\frac{\mathrm{e}^{-ik\cdot(r-r')}}{k^2+a}.$$

In fact, this result can be verified as follows:

$$\begin{aligned} (G^{-1}G)_{ij}(r-r') &= \int \mathrm{d}^D r'' \sum_k G^{-1}_{ik}(r-r'')G_{kj}(r''-r') \\ &= \int \mathrm{d}^D r'' \sum_k \left[\delta(r-r'')\delta_{ij}(-\nabla^2+a^2)\right]\cdot\left[\delta_{kj}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\,\frac{\mathrm{e}^{-ik\cdot(r''-r')}}{k^2+a^2}\right] \\ &= \delta_{ij}\int \frac{\mathrm{d}^D k}{(2\pi)^D}(-\nabla^2+a^2)\frac{\mathrm{e}^{-ik\cdot(r-r')}}{k^2+a^2} \\ &= \delta_{ij}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\mathrm{e}^{-ik\cdot(r-r')} = \delta_{ij}\delta(r-r'). \end{aligned}$$

We have thus shown that $G^{-1}G = I$.

(iii) In the ordered phase, $T < T_c$, we have instead $|\vec{m}| = \sqrt{-a/b} \equiv m_0$. Let $\vec{e}$ be the unit vector along $\vec{m}$: $\vec{m} = m_0 \vec{e}$. We first express things in terms of the orthogonal projectors on the longitudinal and transverse directions, $P^{\rm L} = |e\rangle\langle e|$ and $P^{\rm T} = I - |e\rangle\langle e|$:

$$\begin{aligned} G^{-1}_{ij}(r-r') &= \delta(r-r')\left[(-\nabla^2 + a + bm_0^2)P^{\rm T}_{ij} + (-\nabla^2 + a + bm_0^2 + 2bm_0^2)P^{\rm L}_{ij}\right] \\ &= \delta(r-r')\left[-\nabla^2 P^{\rm T}_{ij} + (-\nabla^2 - 2a)P^{\rm L}_{ij}\right], \end{aligned} \tag{4}$$

where we used $m_0^2 = -a/b$.
Then perform the Fourier transformation as before:

$$G^{\pm 1}_{ij}(r-r') = \int \frac{{\rm d}^D k}{(2\pi)^D} {\rm e}^{-ik(r-r')} g^{\pm 1}_{ij}(k)\,.$$

We find

$$g^{-1}_{ij}(k) = k^2 P^{\rm T}_{ij} + (k^2 - 2a)P^{\rm L}_{ij}.$$

To invert this, we consider the general situation of two operators

$$\begin{aligned} O^{-1} &= AP^{\rm L} + BP^{\rm T} \\ O &= CP^{\rm L} + DP^{\rm T}. \end{aligned}$$

As $(P^{\rm L})^2 = P^{\rm L}$, $(P^{\rm T})^2 = P^{\rm T}$ and $P^{\rm L}P^{\rm T} = P^{\rm T}P^{\rm L} = 0$, the product reads

$$O^{-1}O = ACP^{\rm L} + BDP^{\rm T} = I = P^{\rm L} + P^{\rm T},$$

whence $AC = 1$ and $BD = 1$.
In our case we thus have

$$g_{ij}(k) = \frac{1}{k^2} P^{\rm T}_{ij} + \frac{1}{k^2 - 2a} P^{\rm L}_{ij}.$$

Defining the components of G through

$$G_{ij}(r-r') = P^{\rm T}_{ij} G^{\rm T}(r-r') + P^{\rm L}_{ij} G^{\rm L}(r-r')$$

we finally find

$$\begin{aligned} G^{\rm T}(r-r') &= \int \frac{{\rm d}^D k}{(2\pi)^D} \frac{{\rm e}^{-ik(r-r')}}{k^2} \\ G^{\rm L}(r-r') &= \int \frac{{\rm d}^D k}{(2\pi)^D} \frac{{\rm e}^{-ik(r-r')}}{k^2 - 2a}. \end{aligned}$$

This implies that the logitudinal modes (those along $\vec{m}$) have a mass $\sqrt{-2a} \sim \sqrt{2a_0(-t)}$, corresponding to a correlation length $\xi = 1/\sqrt{-2a}$. Conversely, the transverse modes are still massless (even for $T \neq T_c$), and hence infinite correlation length: These are the so-called *Goldstone modes* originating from the spontaneous symmetry breaking.
Another interpretation is the following: The action consists of a kinetic term $\frac{1}{2}\nabla^2 m$ and the potential

$$V(\vec{m}) = \frac{a}{2} m^2 + \frac{b}{4}(m^2)^2.$$

For $t < 0$, we have $a < 0$, so $V(\vec{m})$ looks like a "Mexican hat". The minima are situated on a circle of radius $m_0 = \sqrt{-a/b}$. Changing the direction of $\vec{m}$, while remaining on this circle, does not cost any potential energy: Such a change corresponds to the massless transverse modes (Goldstone modes). On the other hand, a change of $\vec{m}$ in the radial (longitudinal) direction amounts to an increase in potential energy, and thus to a massive mode.

(iv) Let us set $\vec{h} = h\vec{e}$ (in a fixed arbitrary direction). For $a < 0$ the new magnetisation reads

$$\vec{\tilde{m}}_0 = [\sqrt{-a/b} + h]\vec{e},$$

or

$$\begin{aligned} \tilde{m_0^2} &= m_0^2 + 2m_0 h + \mathcal{O}(h^2) \\ &\simeq -\frac{a}{b} + 2m_0 h. \end{aligned}$$

We can then find the corresponding masses from (4):

$$\begin{aligned} (\mu_{\mathrm{T}})^2 = a + b\tilde{m}_0^2 &= 2m_0 h \\ (\mu_{\mathrm{L}})^2 = a + 3b\tilde{m}_0^2 &= (a + bm_0^2) + 2bm_0^2 + 6bm_0 h \\ &= 0 - 2a + 6bm_0 h. \end{aligned}$$

1.4 Chapter 5: The renormalisation group and universality

- *Exercise 5.1: Real-space renormalisation: Ising model on the triangular lattice*

We consider the Hamiltonian of the two-dimensional Ising model

$$\beta H(\{S_i\}) = -N\ J_0 - J_1 \sum_{\langle i,j \rangle}^{N} S_i S_j - J_2 \sum_{i=1}^{N} S_i, \tag{5}$$

where the symbol $\langle i,j \rangle$ represents the pairs of nearest neighbours on the triangular lattice, shown in Figure 1. The mesh size of the lattice is a.

In this exercise we consider a transformation of the renormalisation group in real space, which consists of creating blocks of spins σ_α, where α indexes the block. The spins at the three vertices of a black triangle in Figure 1, such as S_1, S_2, S_3, form one block spin according to the rule

$$\sigma_\alpha = \mathrm{sign}\big(S_1 + S_2 + S_3\big)\,. \tag{6}$$

It is seen that each spin S belongs to one and only one block α.

(i) What is the mesh size of the new triangular lattice formed by the blocks α? To each configuration $\{\sigma_\alpha\}$ we associate all the configuration $C(\{\sigma_\alpha\})$ of the spins $\{S_i\}$ that verify the definition (6). What is the number of elements in C?

(ii) We define a Hamiltonian $\mathcal{H}$ on the block spins $\{\sigma_\alpha\}$ by decimating the spins $\{S_i\}$ that belong to $C(\{\sigma_\alpha\})$:

$$\mathcal{H}(\{\sigma_\alpha\}) = -\frac{1}{\beta}\ \log\left[\sum_{\{S_i\}\in C(\{\sigma_\alpha\})} \exp\left(-\beta\ H(\{S_i\})\right)\right]\,.$$

Show that the Hamiltonian $\mathcal{H}(\sigma_1,\sigma_2,\sigma_3)$ for the lattice of $N = 9$ spins S_i (with $i = 1,2,\ldots,9$) shown in Figure 1 can be written in the form

$$\begin{aligned} -\beta\mathcal{H}(\sigma_1,\sigma_2,\sigma_3) &= A_{123} + A_{12}\sigma_3 + A_{23}\sigma_1 + A_{13}\sigma_2 \\ &+ A_1\sigma_2\sigma_3 + A_2\sigma_1\sigma_3 + A_3\sigma_1\sigma_2 + A\sigma_1\sigma_2\sigma_3\,, \end{aligned}$$

where the $A_{\ldots}$ are constants.

(iii) Show next that the $A_{\ldots}$ are calculable starting from $H(\{S_i\})$ — at least in principle, that is, by performing sums over a large number of terms. Argue that all the $A_{\ldots}$ are non zero; in particular, the renormalised Hamiltonian $\mathcal{H}(\{\sigma_\alpha\})$ contains interactions which are not present in $H(\{S_i\})$. One may corroborate these arguments by explicit computations using symbolic algebra software such as MATHEMATICA or MAPLE.

(iv) We define the Hamiltonian

$$\beta H_0(\{S_i\}) = -N\ J_0 - J_1 \sum_{\alpha=1}^{N/3} \sum_{<i,j>\in\alpha} S_i S_j\,, \tag{7}$$

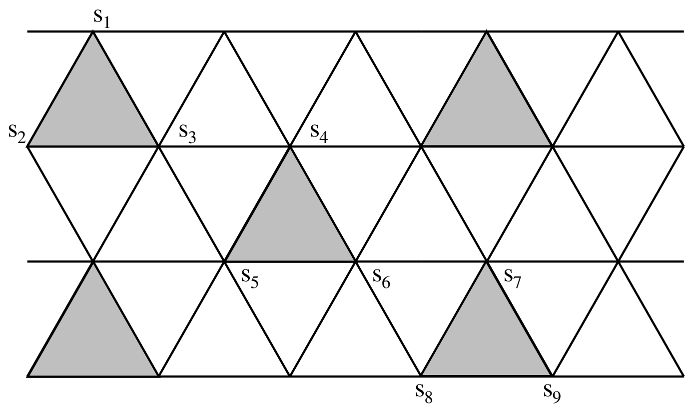

Figure 1: Block spins on the triangular lattice.

where the second sum is over pairs of nearest neighbour sites belonging to the same block α. We also denote by $\langle A\rangle_{\tilde{0}}$ the mean value of the observable A with the Hamiltonian H_0 for a fixed configuration $\{\sigma_\alpha\}$:

$$\begin{aligned}\langle A\rangle_{\tilde{0}} &= \frac{1}{Z_{\tilde{0}}} \sum_{\{S_i\}\in C(\{\sigma_\alpha\})} A(\{S_i\})\ \exp\left(-\beta H_0(\{S_i\})\right) \\ Z_{\tilde{0}} &= \sum_{\{S_i\}\in C(\{\sigma_\alpha\})} \exp\left(-\beta H_0(\{S_i\})\right) .\end{aligned}$$

Show that

$$\exp\left(-\beta\mathcal{H}(\{\sigma_\alpha\})\right) = Z_{\tilde{0}}\ \langle \exp\left(-\beta(H-H_0)\right)\rangle_{\tilde{0}}$$

Use the convexity of the exponential to deduce the following inequality

$$\beta\mathcal{H}(\{\sigma_\alpha\}) \le -\log Z_{\tilde{0}} + \beta\langle H - H_0\rangle_{\tilde{0}} \,. \tag{8}$$

(v) Compute $Z_{\tilde{0}}$. Show that

$$\langle S_i\rangle_{\tilde{0}} = \frac{e^{3J_1}+e^{-J_1}}{e^{3J_1}+3\,e^{-J_1}}\,\sigma_\alpha\,,$$

where α denotes the block containing the site i. Deduce from this the value of $\langle H - H_0\rangle_{\tilde{0}}$.

(vi) Establish the real-space renormalisation group transformation

$$\begin{aligned} J_1' &= 2\,J_1 \left(\frac{e^{3J_1}+e^{-J_1}}{e^{3J_1}+3\,e^{-J_1}}\right)^2 , & (9)\\ J_2' &= 3\,J_2 \left(\frac{e^{3J_1}+e^{-J_1}}{e^{3J_1}+3\,e^{-J_1}}\right) , & (10)\\ J_0' &= 3\,J_0 + \log(e^{3J_1}+3\,e^{-J_1})\,, & (11)\end{aligned}$$

by approximating $\mathcal{H}$ by the upper bound (??). What are the fixed points (J_1^*, J_2^*) of the flow in the space of the two coupling constants? Study their stability.

(vii) Compute the critical exponents at the non-trivial fixed point determined in the preceding question. One proceeds by linearisation of the renormalisation group flows around the fixed point.

Correction :

(i) One easily sees that the distance between two neighbouring blocks is twice the height of an equilateral triangle, that is $\sqrt{3}a$.
Formally we have

$$C(\{\sigma_\alpha\}) = \left\{ \{S_i\} \middle| \text{signe}\left(\sum_{i \in \text{bloc } \alpha} S_i \right) = \sigma_\alpha \right\}.$$

To obtain a block spin $\sigma_\alpha = +1$, we need either $S_1 + S_2 + S_3 = 3$ (one possibility), or $S_1 + S_2 + S_3 = 1$ (three possibilities).
Similarly, the four choices $(S_1, S_2, S_3) = (-1,-1,-1), (+1,-1,-1), (-1,+1,-1), (-1,-1,+1)$ all lead to $\sigma_\alpha = -1$. Therefore,

$$\text{card}(C) = 4^{N/3}, \tag{12}$$

since there are $N/3$ blocks α. One recovers of course the total number of spin configurations:

$$\mathcal{N} = \text{card}(C) \cdot 2^{N/3} = 8^{N/3} = 2^N.$$

(ii) For the block of nine spins shown in Figure 1, we have

$$\begin{aligned} -\beta H[S_1, \ldots, S_9] &= 9J_0 + J_1 \left[\sum_{i=1}^{8} S_i S_{i+1} + S_3(S_1 + S_5) + S_6(S_4 + S_8) + S_7 S_9 \right] \\ &+ J_2 \sum_{i=1}^{9} S_i, \end{aligned}$$

and

$$Z(\sigma_1, \sigma_2, \sigma_3) \equiv \mathrm{e}^{-\beta H(\sigma_1,\sigma_2,\sigma_3)} = \sum_{S_i \in C(\sigma_1,\sigma_2,\sigma_3)} \mathrm{e}^{-\beta H(\{S_i\})},$$

where the summation contains $4^3 = 64$ terms for each choice of the $\sigma_1, \sigma_2, \sigma_3$.

(iii) The result can be developed as

$$\begin{aligned} -\beta H(\sigma_1, \sigma_2, \sigma_3) &= \log Z(\sigma_1, \sigma_2, \sigma_3) \\ &= A_{123} + A_{12}\sigma_3 + A_{23}\sigma_1 + A_{13}\sigma_2 \\ &+ A_1\sigma_2\sigma_3 + A_2\sigma_1\sigma_3 + A_3\sigma_1\sigma_2 + A\sigma_1\sigma_2\sigma_3, \end{aligned}$$

by noticing that the matrix transforming the eight constants $\{A\}$ into the eight values $H(\sigma_1, \sigma_2, \sigma_3)$ is invertible. It is interesting to remark that we necessarily generate an interaction between *three* spins, each is this did not exist in the original Hamiltonian.
Using MATHEMATICA, it is simple to compute the constants $\{A\}$, for instance

$$A_2 = \frac{1}{8} \sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} \sigma_1\sigma_3 \log Z(\sigma_1, \sigma_2, \sigma_3).$$

With the notation

$$A_2 = \sum_{l=0}^{\infty} f_l(J_2) J_1^l$$

one finds $f_0(J_2) = 0$, $f_1(J_2) = 0$ and

$$f_2(J_2) = -32 \left(\frac{1 + \mathrm{e}^{2J_2}}{3 + 10\mathrm{e}^{2J_2} + 3\mathrm{e}^{4J_2}} \right)^4 (1 + 6\mathrm{e}^{2J_2} + 18\mathrm{e}^{4J_2} + 6\mathrm{e}^{6J_2} + \mathrm{e}^{8J_2}).$$

It is seen that $f_2(J_2) \neq 0$ for each J_2, and in particular $A_2 \neq 0$ if $J_1 \neq 0$. So there is an interaction between σ_1 and σ_3, even if those spins are seperated by twice the distance between block spins ($= 2\sqrt{3}a$).
One can perform the same development for the $\sigma_1\sigma_2\sigma_3$ term, leading to the result

$$A = -32 \frac{(1 + \mathrm{e}^{2J_2})^5 (-1 - 5\mathrm{e}^{2J_2} + 5\mathrm{e}^{4J_2} + \mathrm{e}^{6J_2})}{(3 + 10\mathrm{e}^{2J_2} + 3\mathrm{e}^{4J_2})^4} J_1^2 + \mathcal{O}(J_1^3).$$

In particulier, $A \neq 0$ if $J_1 \neq 0$ and $J_2 \neq 0$.
Notice that the block spin construction leads to interactions that are more general than those in the original Ising model. In particular, we obtain long-range interactions upon iterating the construction. To establish the renormalisation group equations, we will need to impose a troncation within this space of coupling constants.

(iv) Recall that

$$\mathrm{e}^{-\beta H(\{\sigma_\alpha\})} = \sum_{\{S_i\} \in C(\{\sigma_\alpha\})} \mathrm{e}^{-\beta H(\{S_i\})},$$

and inserting $H = H_0 + (H - H_0)$ one finds

$$\begin{aligned} \mathrm{e}^{-\beta H(\{\sigma_\alpha\})} &= \sum_{S_i \in C} \mathrm{e}^{-\beta H_0} \, \mathrm{e}^{-\beta (H - H_0)} \\ &= Z_{\tilde{0}} \langle \mathrm{e}^{-\beta(H-H_0)} \rangle_{\tilde{0}}. \end{aligned}$$

Use now the convexity of the exponential (a cord between two points is always above the curve), $\langle \mathrm{e}^x \rangle \geq \mathrm{e}^{\langle x \rangle}$, to deduce that

$$\langle \mathrm{e}^{-\beta(H-H_0)} \rangle_{\tilde{0}} \geq \mathrm{e}^{-\beta \langle H - H_0 \rangle_{\tilde{0}}},$$

from which

$$\beta H[\{\sigma_\alpha\}] \leq -\log Z_{\tilde{0}} + \beta \langle H - H_0 \rangle_{\tilde{0}}.$$

(v) We can now compute

$$\begin{aligned} Z_{\tilde{0}} &= \sum_{S_i \in C(\{\sigma_\alpha\})} \mathrm{e}^{N J_0 + J_1 \sum_{\alpha=1}^{N/3} \sum_{\langle i,j \rangle} S_i S_j} \qquad (13) \\ &= \mathrm{e}^{N J_0} \prod_{\alpha=1}^{N/3} \left\{ \sum_{\substack{S_1, S_2, S_3 = \pm 1 \\ \mathrm{sign}(S_1 + S_2 + S_3) = \sigma_\alpha}} \mathrm{e}^{J_1 (S_1 S_2 + S_1 S_3 + S_2 S_3)} \right\}. \end{aligned}$$

The quantity $\{\cdots\} = \mathrm{e}^{3J_1} + 3\mathrm{e}^{-J_1}$ is the same for the either case, $\sigma_\alpha = +1$ or $\sigma_\alpha = -1$. Thus,

$$Z_{\tilde{0}} = \mathrm{e}^{N J_0} \left(\mathrm{e}^{3J_1} + 3\mathrm{e}^{-J_1} \right)^{N/3},$$

which leads to the flow of J_0 (see (11)).
To compute $\langle S_i\rangle_{\tilde{0}}$ we use that the blocks are independent with respect to the interaction defined by H_0 :

$$\begin{aligned}\langle S_i\rangle_{\tilde{0}} &= \frac{\sum_{\text{sign}(S_1+S_2+S_3)=\sigma_\alpha} S_1 \mathrm{e}^{J_1(S_1S_2+S_1S_3+S_2S_3)}}{\mathrm{e}^{3J_1}+3\mathrm{e}^{-J_1}} \\ &= \sigma_\alpha \frac{1\cdot \mathrm{e}^{3J_1}+(1+1-1)\mathrm{e}^{-J_1}}{\mathrm{e}^{3J_1}+3\mathrm{e}^{-J_1}} = \sigma_\alpha \frac{\mathrm{e}^{3J_1}+\mathrm{e}^{-J_1}}{\mathrm{e}^{3J_1}+3\mathrm{e}^{-J_1}}\end{aligned}$$

By subtracting (5) and (7) we find that

$$\beta(H-H_0) = -J_2\sum_\alpha\sum_{i\in\alpha} S_i - J_1 \sum_{\substack{\langle i,j\rangle \\ i\in\alpha, j\in\beta}} S_iS_j,$$

where α and β are now two *different* blocks. Using again the independence of the blocks with respect to the interaction H_0, the mean value factorises:

$$\begin{aligned}\beta\langle H-H_0\rangle_{\tilde{0}} &= -J_2\sum_\alpha\sum_{i\in\alpha}\langle S_i\rangle_{\tilde{0}} - J_1\sum_{\langle\alpha,\beta\rangle}\sum_{\substack{\langle i,j\rangle \\ i\in\alpha, j\in\beta}}\langle S_i\rangle_{\tilde{0}}\langle S_j\rangle_{\tilde{0}} \\ &= -3J_2\left(\frac{\mathrm{e}^{3J_1}+\mathrm{e}^{-J_1}}{\mathrm{e}^{3J_1}+3\mathrm{e}^{-J_1}}\right)\sum_\alpha \sigma_\alpha - J_1\sum_{\langle\alpha,\beta\rangle}\mathcal{N}_{\alpha,\beta}\left(\frac{\mathrm{e}^{3J_1}+\mathrm{e}^{-J_1}}{\mathrm{e}^{3J_1}+3\mathrm{e}^{-J_1}}\right)^2\sigma_\alpha\sigma_\beta.\end{aligned}$$

Notice that

$$\mathcal{N}_{\alpha,\beta} = |\langle i,j\rangle : i\in\alpha, j\in\beta| = 2,$$

since there are two interactions between elementary spins for each choice of two neighbouring blocks (see Figure 1).
By approximating the inequality (8) by its upper bound, one deduces the flows of J_1 and J_2 (see (9) and (10)), since we have just shown that the effective interaction between the block spins takes the form of and Ising model defined on the σ_α.

(vi) We have already shown (9)–(11). The flow of J_0 is of little interest, since it is just a regular contribution to the partition function.
To study the fixed points (J_1^*, J_2^*), suppose first that $J_1^*\neq 0$ and that $J_2^*\neq 0$. One may then divide (9) by J_1^*, and (10) by J_2^*. The result is that $(\cdots)^2=\frac{1}{2}$ and $(\cdots)=\frac{1}{3}$, which is clearly inconsistent.
Suppose instead that $J_1^*=0$. Then (10) reads $J_2^* = 3J_2^*\cdot\frac{2}{4}$, yielding $J_2^*=0$. Thus,

$$(J_1^*, J_2^*) = (0,0)$$

is a “trivial” fixed point (at infinite temperature).
Finalement, suppose that $J_2^*=0$. Then (9) becomes

$$1 = 2\left(\frac{\mathrm{e}^{3J_1^*}+\mathrm{e}^{-J_1^*}}{\mathrm{e}^{3J_1^*}+3\mathrm{e}^{-J_1^*}}\right)^2,$$

or $(\mathrm{e}^{4J_1^*}+3)^2 = 2(\mathrm{e}^{4J_1^*}+1)^2$, with solution:

$$(J_1^*, J_2^*) = (\frac{1}{4}\log(1+2\sqrt{2}), 0).$$

This is the non-trivial fixed point.
Remark : It is well-known that the Ising model has a critical point at a non-trivial temperature T_c and zero field, $H = 0$. This is what we have retrieved here. Using a self-duality argument one can establish the exact value of the critical coupling on the triangular lattice, $J_{1,c} = \frac{1}{4}\ln 3 \simeq 0.2746$. Here, we have found instead a larger value, $J_1^* \simeq 0.3356$. It is consistent that our approximate value is too larger, since in reality the spins are more correlated than in our treatment (since we have neglected the effective long-range correlations).
In the vicinity of the fixed points, one may linearise the flow:

$$\begin{aligned} \delta J_1' \equiv J_1' - J_1^* &= \left.\frac{\partial J_1'}{\partial J_1}\right|_{J^*} \delta J_1 + \left.\frac{\partial J_1'}{\partial J_2}\right|_{J^*} \delta J_2 \\ \delta J_2' \equiv J_2' - J_2^* &= \left.\frac{\partial J_2'}{\partial J_1}\right|_{J^*} \delta J_1 + \left.\frac{\partial J_2'}{\partial J_2}\right|_{J^*} \delta J_2 \end{aligned}$$

with the result

$$\begin{aligned} \delta J_1' &= \left[2\left(\frac{e^{3J_1^*} + e^{-J_1^*}}{e^{3J_1^*} + 3e^{-J_1^*}}\right)^2 + 32 J_1^* \left(\frac{e^{3J_1^*} + e^{-J_1^*}}{e^{3J_1^*} + 3e^{-J_1^*}}\right) \frac{e^{2J_1^*}}{(e^{3J_1^*} + 3e^{-J_1^*})^2}\right] \delta J_1 \\ \delta J_2' &= 3\left(\frac{e^{3J_1^*} + e^{-J_1^*}}{e^{3J_1^*} + 3e^{-J_1^*}}\right)^2 \delta J_2 + 24 J_2^* \frac{e^{2J_1^*}}{(e^{3J_1^*} + 3e^{-J_1^*})^2} \delta J_1. \end{aligned}$$

This is more conveniently rewritten in terms of a matrix $M_{\alpha\beta}$ through

$$\delta J_\alpha' = \sum_\beta M_{\alpha\beta} \delta J_\beta. \tag{14}$$

At the trivial fixed point it takes the value

$$M_{\alpha\beta} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

Having an eigenvalue > 1 (resp. < 1) means that the corresponding coupling is relevant (resp. irrelevant). We see here that the temperature (J_1 coupling) is irrelevant, whilst the magnetic field (J_2 coupling) is relevant. The physical interpretation is that at high temperature the spins are independent and only follow the local magnetic field (paramagnetic phase).
Meanwhile, for the non-trivial fixed point we have

$$M_{\alpha\beta} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with

$$\begin{aligned} \lambda_1 &= 1 + \frac{8 - 5\sqrt{2}}{2} \log(1 + 2\sqrt{2}) \simeq 1.624 \\ \lambda_2 &= \frac{3}{\sqrt{2}} \simeq 2.121. \end{aligned} \tag{15}$$

So here both couplings are relevant. One often uses the notation $\lambda_i = 3^{y_i}$ (where 3 is the scale factor between the original and renormalised system). Numerically, $y_1 \simeq 0.441$ and $y_2 \simeq 0.685$.
Remark : The fact that $M_{\alpha\beta}$ is a diagonal matrix in both cases is non-trivial. This means that both temperature and magnetic field are "eigenoperators" with respect to the RG flow. In a more general situation, one would have at this stage to diagonalise $M_{\alpha\beta}$, hence requiring knowledge of the proper "eigenoperators".

(vii) The following analysis is a classic of the RG literature.
The singular contribution to the free energy (i.e., without the regular contribution from J_0) at the critical point satisfies

$$\frac{N}{3} f(J_1', J_2') = N f(J_1, J_2)$$

with $J_1 \simeq J_1^*$ and $J_2 \simeq J_2^*$. Let us set $f_1(x_1, x_2) = f(J_1^* + x_1, J_2^* + x_2)$, so that

$$\frac{1}{3} f_1(\delta J_1', \delta J_2') = \frac{1}{3} f_1(3^{y_1} \delta J_1, 3^{y_2} \delta J_2) = f_1(\delta J_1, \delta J_2).$$

Repeating the renormalisation transformation l times, this becomes

$$3^{-l} f_1(3^{l y_1} \delta J_1, 3^{l y_2} \delta J_2) = f_1(\delta J_1, \delta J_2).$$

One has the liberty to choose l at will. In particular, the choice $3^{y_2 l} \delta J_2 = 1$ is convenient for studying the system at the critical temperature, while the choice $3^{y_1 l} \delta J_1 = 1$ is nice for studying the propreties in zero magnetic field.
Let us begin by studying the first case. We have $l = -\log(\delta J_2)/(y_2 \log 3)$, meaning that

$$3^{\log(\delta J_2)/(y_2 \log 3)} f_1(\lambda_1^{-\log(\delta J_2)/(y_2 \log 3)} \delta J_1, 1) = f_1(\delta J_1, \delta J_2),$$

or

$$f_1(\delta J_1, \delta J_2) = (\delta J_2)^{1/y_2} f_1\left((\delta J_2)^{-y_1/y_2} \delta J_1, 1\right).$$

We remark that

$$\delta J_2 \equiv \beta h - \beta^* h^* = \beta h \simeq \beta_c h$$

and that

$$\delta J_1 \equiv \beta J - \beta^* J = \frac{J}{T} - \frac{J}{T_c} \simeq \frac{J}{T_c} \frac{T_c - T}{T_c} \equiv \frac{J}{T_c} t,$$

where t measures the deviation from the critical temperature. Let us change notations once more:

$$f_1\left(\frac{J}{T_c} t, \beta_c h\right) \equiv f_2(t, h).$$

Then

$$f_2(t, h) = (\beta_c h)^{1/y_2} f_2\left((\beta_c h)^{-y_1/y_2} t, 1\right),$$

and in this form it is convenient to set $T = T_c$ (so that $\delta t = J_1 = 0$):

$$f_2(t, h) = (\beta_c h)^{1/y_2} f_2(0, 1).$$

We can now compute a critical exponent

$$m = \frac{\partial f}{\partial h} \sim h^{1/y_2 - 1} \sim h^{1/\delta},$$

whence

$$\delta = \frac{y_2}{1 - y_2} \simeq 2.170.$$

The exact solution of the 2D Ising model gives instead $\delta = 15$.
We next study the second case (with $3^{y_1 l} \delta J_1 = 1$), where we find

$$f_3(t, h) = \left(\frac{J}{T_c} t\right)^{1/y_1} f_3\left(1, \left(\frac{J}{T_c} t\right)^{-y_2/y_1} h\right).$$

With $h = 0$, this gives $f_3 \sim t^{1/y_1}$, and

$$C_V \sim \frac{\partial^2 f}{\partial t^2} \sim \frac{\partial^2 f_3}{\partial t^2} \sim t^{1/y_1 - 2} \sim t^{-\alpha}$$

whence

$$\alpha = 2 - \frac{1}{y_1} \simeq -0.267.$$

The exact value in 2D is here $\alpha = 0$ (but with a logarithmic divergence of C_V instead).
Furthermore

$$m|_{h=0} = \frac{\partial f_3}{\partial h}\Big|_{h=0} = \left(\frac{J}{T_c}t\right)^{1/y_1} \left(\frac{J}{T_c}t\right)^{-y_2/y_1} \partial_2 f_3(1,0),$$

and since $m \sim t^\beta$ we find

$$\beta = \frac{1 - y_2}{y_1} \simeq 0.715,$$

to be compared with the exact 2D value $\beta = \frac{1}{8}$.
Finally,

$$\chi = \frac{\partial m}{\partial h}\Big|_{h=0} = \left(\frac{J}{T_c}t\right)^{1/y_1} \left(\frac{J}{T_c}t\right)^{-2y_2/y_1} (\partial_2)^2 f_3(1,0),$$

and since $\chi \sim t^{-\gamma} \sim (-t)^{-\gamma'}$ we obtain

$$\gamma = \gamma' = \frac{2y_2 - 1}{y_1} \simeq 0.837,$$

to be compared with the exact 2D values $\gamma = \gamma' = \frac{7}{4}$.
Notice that we have the scaling laws

$$\begin{aligned} 1 + \frac{\gamma}{\beta} &= \delta \\ \alpha + 2\beta + \gamma &= 2. \end{aligned}$$

Using also

$$\begin{aligned} \alpha &= 2 - \nu d \qquad (16) \\ \gamma &= \nu(2 - \eta) \end{aligned}$$

we find the further exponents

$$\nu = \frac{1}{2y_1} \simeq 1.134 \qquad (17)$$

$$\eta = 4(1 - y_2) \simeq 1.262 \qquad (18)$$

to be compared with the exact 2D values $\nu = 1$ and $\eta = \frac{1}{4}$.
It is seen that the four scaling laws are arise from the fact that the true critical exponents are in fact just the RG eigenvalues y_1 et y_2, which in turn are linked to the existence of two relevant couplings, t and h. In certain situations, notably when the critical points is at $T_c = 0$, one may identify a third relevant coupling, which leads to the failure of one of the scaling laws, namely the so-called "hyper-scaling" relation (16). This happens for instance for the random-field Ising model.

- *Exercise 5.2: Correction to scaling.*

The aim of this exercise is to understand how the irrelevant variables induce subleading corrections to scaling behaviours. To simplify matter, let us suppose that the critical system possesses only one relevant scaling variable, say u_t with RG eigen-value $y_t > 0$, and one irrelevant variable, say $u_{\rm irr}$ with RG eigen-value $y_{\rm irr} < 0$. (Of course generic physical systems have an infinite number of irrelevant variables but considering only one will be enough to understand their roles).

(i) By iterating RG transformations as in the main text, show that the singular part of the free energy can be written as

$$f_{\rm sing} = |u_t|^{D/y_t}\,\varphi_\pm(u^0_{\rm irr}\,|u_t|^{|y_{\rm irr}|/y_t}),$$

where $\varphi_\pm$ are functions possibly different for $u_t > 0$ or $u_t < 0$, and $u^0_{\rm irr}$ is the initial value (before RG transformations) of the irrelevant coupling.

(ii) Argue (without formal proof) that the functions $\varphi_\pm$ may raisonably be expected to be smooth. Under this assumption, prove that

$$f_{\rm sing} = |u_t|^{D/y_t}\left(A_0 + A_1\,u^0_{\rm irr}\,|u_t|^{|y_{\rm irr}|/y_t} + \cdots\right),$$

where A_0 and A_1 are non-universal constants.

Correction :

(i) Recall the RG transformation law of the singular part of the free energy (see lecture notes): $f_{\rm sing}(u_t, u_{\rm irr}) = \lambda^{-D} f_{\rm sing}(\lambda u_t, \lambda u_{\rm irr})$. Choose $\lambda = |u_t|^{-1/y_t}$ to get (using $y_{\rm irr} = -|y_{\rm irr}|$ because $y_{\rm irr} < 0$)

$$f_{\rm sing}(u_t, u_{\rm irr}) = |u_t|^{D/y_t}\,f_{\rm sing}(\pm 1, u_{\rm irr}\,|u_t|^{|y_{\rm irr}|/y_t}),$$

which coincides with the formula in the text with $\varphi_\pm(x) = f_{\rm sing}(\pm 1, x)$.

(ii) The x-dependence of $\varphi_\pm(x) = f_{\rm sing}(\pm 1, x)$ is that of irrelevent degree of freedom, so that we don't expect non-smoothness there. Hence, we assume that Taylor expansion for $\varphi_\pm(x)$ is valid. We write $\varphi_\pm(x) = A_0 + A_1\,x + \cdots)$ to get

$$f_{\rm sing}(u_t, u_{\rm irr}) = |u_t|^{D/y_t}\left(A_0 + A_1\,u^0_{\rm irr}\,|u_t|^{|y_{\rm irr}|/y_t} + \cdots\right).$$

- *Exercise 5.3: Change of variables and covariance of RG equations.*

Let us consider a theory with a finite number of relevant coupling constants that we generically denote $\{g^i\}$. Let us write the corresponding beta functions as (no summation in the first term)

$$\beta^i(g) = y_i g^i - \frac{1}{2}\sum_{jk} C^i_{jk} g^j g^k + \cdots .$$

(i) Prove that, if all y_i are non-vanishing, then there exist a change of variables from $\{g^i\}$ to $\{u^i\}$, with $u^i = g^i + O(g^2)$, which diagonalizes the beta functions, up to two loops, i.e. such that $\beta^i(u) = y_i u^i + O(u^3)$.

(ii) Prove that, if all y_i are zero, then the second and third Taylor coefficient are invariant under a change of variables from $\{g^i\}$ to $\{u^i\}$, with $u^i = g^i + O(g^2)$.
That is: For marginal perturbation, the second and third loop beta function coefficients are independent on the renormalization scheme (alias on the choice of coordinate in the coupling constant space).

(iii) Let expand the beta functions to all orders in the coupling constants:

$$\beta^i(g) = y_i g^i - \sum_{n>0} \sum_{j_1,\cdots,j_n} C^i_{j_1,\cdots,j_n} g^{j_1} \cdots g^{j_n}.$$

Prove that, if there is no integers p_i, p_j such that $p_i y_i - p_j y_j \in \mathbb{Z}$, for $i \neq j$ (in such cases, one says they that there is non resonances), then there exists a change of variables from $\{g^i\}$ to $\{u^i\}$, with u^i a formal power series in the g^i's, with $u^i = g^i + O(g^2)$, which diagonalizes the beta functions as a formal power series in the u^i's.
That is: There exist scaling variables, at least as formal power series.

Correction :

Let us first assume that there is only one variable g, and hence only one beta function $\beta(g)$. The generalisation to multi-variable is simple but cumbersome.

(i) The beta function is not a function but a vector field. Hence, it transforms as a vector field under a change of coordinate. If we change $g \to u = u(g)$, then $\beta(u) = \beta(g)\frac{\partial u}{\partial g}$.
Let $\beta(g)$ to second order $\beta g = yg - \frac{C}{2}g^2 + \cdots$. Let us do this change of variable with $u = g + \frac{a}{2}g^2 + \cdots$, or equivalently $g = u - \frac{a}{2}u^2 + \cdots$. Then

$$\begin{aligned}\beta(u) = \beta(g)\frac{\partial u}{\partial g} &= (yg - \frac{C}{2}g^2 + \cdots)(1 + ag + \cdots) \\ &= y(u - \frac{a}{2}u^2) - \frac{C}{2}u^2 + ayu^2 + \cdots .\end{aligned}$$

So that $\beta(u) = yu + O(u^3)$ if we choose $ay = C$ (which is possible if $y \neq 0$, i.e. if g (or u) is not a marginal coupling).

(ii) The strategy of the proof is the same. Let us assume that $\beta g = -\frac{C}{2}g^2 + \frac{D}{6}g^3 + \cdots$ and let us implement a change of variable $g \to u = g + \frac{a}{2}g^2 + \frac{b}{6}g^3 + \cdots$. Inversely $g = u - \frac{a}{2}u^2 + \frac{3a^2-b}{6}g^3 + \cdots$. Then

$$\begin{aligned}\beta(u) = \beta(g)\frac{\partial u}{\partial g} &= (-\frac{C}{2}g^2 + \frac{D}{6}g^3 + \cdots)(1 + ag + \frac{b}{2}g^2 + \cdots) \\ &= -\frac{C}{2}g^2 + \frac{D}{6}g^3 - \frac{aC}{2}g^3 + \cdots \\ &= -\frac{C}{2}(g - \frac{a}{2}g^2 + \cdots)^2 + \frac{D}{6}g^3 + \cdots .\end{aligned}$$

Hence, to order $O(u^4)$, we have $\beta u = -\frac{C}{2}u^2 + \frac{D}{6}u^3 + \cdots$. This means that, for marginal perturbation, the second and third loop beta function coefficients are independent of the way to parmaterize the marginal couplings.

(iii) The proof is exactly as in the two previous questions: change variable keeping in mind that the beta functions are vector fields by writing u as a Taylor series $u = g + \sum_{n>0} \frac{a_n}{n!}g^n$. The only interesting point is to observe that the beta functions can be diagonalize only if the 'non-resonance' condition $p_i y_i - p_j y_j \notin \mathbb{Z}$ for any integers p_i, p_j.

1.5 Chapter 6: Free field theory

- *Exercise 6.1: Translation invariance and the stress-tensor*

The aim of this exercise is to see some aspect of the relation between translation invariance and the stress-tensor. Let us consider classical scalar field theory with Lagrangian $\mathcal{L}[\phi, \partial\phi]$ and action $S[\phi] = \int d^D x\, \mathcal{L}[\phi, \partial\phi]$. Recall that maps extremalizing this action are said to be solution of the classical equations of motion, which reads

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)(x)} \Big) = \frac{\partial \mathcal{L}}{\partial \phi(x)}.$$

These equations are the Euler-Lagrange equations.

(i) Consider an infinitesimal field transformation $\phi(x) \to \phi(x) + \epsilon(\delta\phi)(x)$. Suppose that, under such transformation the Lagrangian variation is $\delta\mathcal{L}[\phi, \partial\phi] = \epsilon\partial_\mu G^\mu$ so that the action is invariant. Show that the following Noether current

$$J^\mu = (\delta\phi)\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} - G^\mu,$$

is conserved on solutions of the equations of motion.

(ii) Let us look at translations $x \to x - \varepsilon\, a$. How does a scalar field ϕ transforms under such translation? Argue that if the Lagrangian density is a scalar, then $\delta\mathcal{L} = \varepsilon\, a^\mu \partial_\mu \mathcal{L}$. Deduce that the action is then translation invariant and that associated conserved Noether current is $J^\mu_a = T^\mu_\nu a^\nu$ with

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\, (\partial_\nu \phi) - \delta^\mu_\nu\, \mathcal{L}.$$

This tensor is called the stress-tensor. It is conserved: $\partial_\mu T^\mu_\nu = 0$.

(iii) Find the expression of the stress-tensor T^μ_ν for a scalar field theory with action $S[\phi] = \int d^D x\, \big(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big)$.

Correction :

(i) We have the hypothesis, the chain rule, and the equation of motion:

$$\delta\mathcal{L} = \partial_\mu G^\mu\,, \tag{19}$$

$$\delta\mathcal{L} = \delta\phi \frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu(\delta\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,, \tag{20}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{21}$$

Now compute $\partial_\mu J^\mu$ for the proposed form of the current, using again the chain rule on the first term:

$$\partial_\mu J^\mu = \partial_\mu(\delta\phi) \cdot \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \delta\phi \cdot \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) - \partial_\mu G^\mu$$

Use the equation of motion on the second term:

$$\partial_\mu J^\mu = \partial_\mu(\delta\phi) \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \delta\phi \cdot \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu G^\mu$$

Use the above chain rule on the first two terms:

$$\partial_\mu J^\mu = \delta \mathcal{L} - \partial_\mu G^\mu$$

Finally, using the hypothesis, we have $\partial_\mu J^\mu = 0$.

(ii) Under $x \to x - \epsilon a$ the scalar field must transform in such a way, $\phi(x) \to \phi'(x)$, that we have the invariance $\phi'(x') = \phi(x)$. This means that

$$\phi(x) \to \phi'(x) = \phi(x + \epsilon a) = \phi(x) + \epsilon a^\mu \partial_\mu \phi(x)\,,$$

where we have developed to first order. If $\mathcal{L}$ is itself an invariant scalar, it therefore also transforms as

$$\delta \mathcal{L} = \epsilon a^\mu \partial_\mu \mathcal{L}\,,$$

so we are have the situation of (i) with $G^\mu = a^\mu \mathcal{L}$. By the result of (i), the conserved current is thus

$$J^\mu = a^\nu (\partial_\nu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - a^\mu \mathcal{L}\,.$$

We write this as $J^\mu = T^\mu_\nu a^\nu$, so that

$$T^\mu_\nu = (\partial_\nu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \delta^\mu_\nu \mathcal{L}\,.$$

The conservation law $\partial_\mu J^\mu = 0$ then reads $\partial_\mu T^\mu_\nu = 0$.

(iii) We apply this to a scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + V(\phi)\,.$$

Applying the result of (ii) we get

$$\begin{aligned} T^\mu_\nu &= (\partial^\mu \phi)(\partial_\nu \phi) - \delta^\mu_\nu \mathcal{L} \\ &= \left[(\partial^\mu \phi)(\partial_\nu \phi) - \frac{1}{2}\delta^\mu_\nu (\partial\phi)^2 \right] - \delta^\mu_\nu V(\phi)\,. \end{aligned}$$

Remark. Consider the trace of the stress tensor for a free massless field, $V(\phi) = 0$. We have

$$\Theta \equiv T^\mu_\mu = (\partial\phi)^2 - \frac{D}{2}(\partial\phi)^2 = \left(\frac{2-D}{2}\right)(\partial\phi)^2\,.$$

So in $D = 2$ the stress tensor is also traceless, $\Theta = 0$. This carries over to the quantum case and is an important property in CFT.

- *Exercise 6.2: Lattice scalar field and lattice Green function*

Recall that lattice scalar free theory is defined by the action

$$S[\phi] = \frac{a^{D-2}}{2} \sum_x \phi_x \left[(-\Delta^{\text{dis}} + a^2 m^2) \right] \phi_x,$$

where ϕ_x are the value of the field at point x on the lattice and Δ_{dis} discrete Laplacian on that lattice. We here consider only D-dimensional square lattice of mesh size a, i.e. $a\mathbb{Z}^D$. Let us also recall that the Fourier transforms in $a\mathbb{Z}^D$ are defined by

$$\hat{\phi}_k = a^D \sum_{\mathbf{n}} e^{-ix\cdot k}\,\phi_x, \quad \phi_x = \int_{\text{BZ}} \frac{d^D k}{(2\pi)^D}\, e^{ix\cdot k}\,\hat{\phi}_k$$

where the integration is over the Brillouin zone, which is the hyper-cube $\text{BZ} \equiv [-\frac{\pi}{a}, \frac{\pi}{a}]^D$.

(i) Verify that the Laplacian acts diagonally in the Fourier basis, with

$$(-\Delta^{\text{dis}} + a^2m^2)_k = 2\sum_{\alpha} \big(\eta - \cos(a\,k_\alpha)\big),$$

with $\eta = 1 + \frac{a^2m^2}{2D}$ and k_α the component of the momentum k in the direction α.

(ii) Verify that in the Fourier basis the free field action reads

$$S[\phi] = \frac{1}{2}\int_{\text{BZ}} \frac{d^D k}{(2\pi/a)^D}\,\hat{\phi}_{-k}(-\Delta^{\text{dis}} + m^2)_k\,\hat{\phi}_k.$$

(iii) Deduce that in Fourier space, a scalar free field is thus equivalent to a collection of i.i.d. Gaussian variables, indexed by the momentum k, with mean and covariance

$$\langle\hat{\phi}_k\rangle = 0, \quad \langle\hat{\phi}_k\hat{\phi}_p\rangle = \frac{1}{(-a^2\Delta^{\text{dis}} + m^2)_k}\,(2\pi)^D\,\delta(k+p).$$

Correction :

Remark: Many factors of a have been corrected.

(i) Notice that the factors of a are such that the action is dimensionless. The discrete Laplacian is

$$(\Delta^{\text{dis}} f)(x) = \sum_{\alpha=1}^{D} [f(x + ae_\alpha) - 2f(x) + f(x - ae_\alpha)]\ .$$

We have

$$(-\Delta^{\text{dis}} + a^2m^2)_k = \sum_\alpha (2 - 2\cos(ak_\alpha)) + a^2m^2 = 2\sum_\alpha (\eta - \cos(ak_\alpha))$$

with $\eta = 1 + \frac{a^2m^2}{2D}$.

(ii) Insert the Fourier transform for the two factors of ϕ_x in the action:

$$S[\phi] = \frac{a^{D-2}}{2}\sum_x \int \frac{d^D p}{(2\pi)^D}\int \frac{d^D k}{(2\pi)^D} e^{i(k+p)x}\hat{\phi}_p(-\Delta^{\text{dis}} + a^2m^2)_k\hat{\phi}_k\,.$$

The sum over x gives

$$\sum_x e^{i(k+p)x} = \left(\frac{2\pi}{a}\right)^D \delta(k+p)\,,$$

so that

$$S[\phi] = \frac{a^{-2}}{2}\int \frac{d^Dk}{(2\pi)^D}\hat{\phi}_{-k}(-\Delta^{\rm dis}+a^2m^2)_k\hat{\phi}_k$$
$$= \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\hat{\phi}_{-k}(-a^{-2}\Delta^{\rm dis}+m^2)_k\hat{\phi}_k\,.$$

(iii) This is directly read off the action, by the usual result on Gaussian kernels.

- *Exercise 6.4: Fractal dimension of free paths*

The fractal dimension $D_{\rm frac}$ of a set embedded in a metric space may be defined through the minimal number $\mathcal{N}_\epsilon$ of boxes of radius ϵ need to cover it by $D_{\rm frac} = \lim_{\epsilon\to 0}\log\mathcal{N}_\epsilon/\log(1/\epsilon)$.

(i) Prove that the fractal dimension of free paths is $D_{\rm frac} = 2$ using the fact that the composite operator ϕ^2, with ϕ a (massless) Gaussian free field, is the operator conditioning on two paths emerging from its insertion point.

Correction :

We give two arguments, the first is quicker than the second but much less general.

(i-a) Free random paths are statistically equivalent (in a sense to make more precise) to Brownian motions. The later may be defined as a scaling limit of random walks. The scaling relation links the linear size of the walker steps ϵ to the number of steps N_ϵ via $N_\epsilon\,\epsilon^2 = t$ with t fixed. A Brownian motion drawn during a time t as a typical linear extension of size $L\sim\sqrt{t}$ so that fixing t amounts to fix the macroscopic size of the Brownian curves. Thus the number of boxes of size ϵ needed to cover a Brownian curve of size L is

$$N_\epsilon \sim (L/\epsilon)^2.$$

Hence the fractal dimension of a Brownian curve is $D_{\rm frac}=2$.

(i-b) This first step is to find an estimation of the probability that a (random) fractal set visit a small ball of radius ϵ. Let us consider such random fractal set γ embedded in the D dimensional Euclidean space. Let $D_{\rm frac}$ be its fractal dimension. Let L be the macroscopic linear spatial extension of this set, that is the set fits in a macroscopic ball of size L^D. The number of small ball a radius ϵ needed to cover this macroscopic ball scales $(L/\epsilon)^D$. The number of small ball a radius ϵ needed to cover the fractal scales $(L/\epsilon)^{D_{\rm frac}}$, by definition of the fractal dimension. Thus, the probability that the fractal visit a given small ball of radius ϵ can be estimated by the ratio

$$\mathbb{P}[\gamma\,\text{visit a ball of radius}\,\epsilon] \simeq \frac{(L/\epsilon)^{D_{\rm frac}}}{(L/\epsilon)^D} \simeq (\epsilon/L)^{D-D_{\rm frac}}.$$

Now, let ϕ_2 be the operator conditioning on two free random paths to emerge from its insertion point, or alternatively, the operator conditioning a free random path to visit a neighbourhood of its insertion point. From its microscopic definition, we may write

$$\phi_2 = \lim_{\epsilon\to 0}\epsilon^{-\Delta_2}\mathbb{I}_{\gamma\text{visit a ball of radius}\,\epsilon},$$

with $\mathbb{I}$ the indicator function conditioning the the written condition and Δ_2 the scaling dimension of the operator. Thus by identifying the two above formula, using that $\mathbb{P}[\gamma$ visit a ball of radius $=$ $\mathbb{E}[\gamma$visit a ball of radius $\epsilon]$, we get that

$$\Delta_2 = D - D_{\text{frac}} \quad \text{or} \quad D_{\text{frac}} = D - \Delta_2.$$

For Gaussian free random paths and its lattice description, the operator conditioning the path to visit to visit a neighbourhood of its insertion point is $:\phi^2:$, with ϕ a (massless) Gaussian free field. That is: $\phi_2 =: \phi^2:$. For the free Gaussian theory in dimension D, the scaling dimension of $:\phi^2:$ is $\Delta_2 = 2\frac{D-2}{2} = D-2$. Hence, the fractal dimension of Gaussian free random paths is

$$D_{\text{frac}} = D - \Delta_2 = D - (D-2) = 2.$$

Notice that, as a consequence, two independent free Gaussian random paths almost never cross in dimension $D > 4$.

- *Exercise 6.6: Two ways to compute the free energy*

The aim of this exercise is to compute the free energy, or the partition function, of a massless free boson in space dimension $d = 1$ at temperature $T = 1/\beta$. Let $D = d + 1$. Recall that the partition function is defined as $Z = \text{Tr}(e^{-\beta H})$ where the trace is over the quantum Hilbert space with H the hamiltonian. Let us suppose that the quantum theory is define dover an interval $\mathbb{I}$ of length L. We shall be interested in the large L limit.

(i) Argue (see Chapter 3) that the partition function is given by the Euclidean path integral on the cylinder $\mathbb{I} \times \mathbb{S}_1$ with a radius β:

$$Z = \int_{\phi(\mathbf{x},\beta)=\phi(\mathbf{x},0)} D\phi\, e^{-S[\phi]}.$$

We shall compute the partition function by quantizing the theory along two different channels (see Figure):
(a) either taking the direction $\mathbb{S}_1$ as time, this Euclidean time is then period with period β;
(b) or taking the direction $\mathbb{I}$ as time, this time then runs from 0 to L with $L \to \infty$.
Global rotation invariance implies that this to way of computing gives identical result. Let us check. On the way this will give us a nice relation about the Riemann ζ-function.

(ii) Explain why the first computation gives $Z = e^{-\beta L \mathcal{F}(\beta)}$, where $\mathcal{F}$ the free energy.

(iii) Explain why the second computation gives $Z = e^{-LE_0(\beta,A)}$ with $E_0(\beta) = \beta\, \mathcal{E}_0(\beta)$ where E_0 is the vacuum energy and $\mathcal{E}_0$ is the vacuum energy density (this is the Casimir effect).

(iv) Show that the free energy density of a massless boson in one dimension is:

$$\mathcal{F} = \frac{1}{\beta} \int \frac{dk}{2\pi} \log(1 - e^{-\beta|k|}) = \frac{1}{\beta^2} \int_0^\infty \frac{dx}{\pi} \log(1 - e^{-x}).$$

(v) Compute the integral to write this free energy density as

$$\mathcal{F} = -\frac{1}{\pi\beta^2}\zeta(2).$$

We have introduce the so-called *zeta*-regularisation. Let $\zeta(s) := \sum_{n>0}\frac{1}{n^s}$. This function was introduced by Euler. This series is convergent for $\Re s > 2$. It is defined by analytic continuation for other value of s via an integral representation.

(vi) Show that the vacuum energy density is $\mathcal{E}_0(\beta) = \frac{1}{\beta}\sum_n \frac{1}{2}\left|\frac{2n\pi}{\beta}\right|$.

(vii) This is divergent. Argue that a regularization based on analytic continuation gives

$$\mathcal{E}_0(\beta) = \frac{2\pi}{\beta^2}\zeta(-1).$$

(viii) Conclusion: A remarkable fact is that $\zeta(2) = \frac{\pi^2}{6}$ and that the analytic continuation of ζ gives $\zeta(-1) = -\frac{1}{12}$. Thus

$$\mathcal{F}(\beta) = \mathcal{E}_0(\beta) = -\frac{\pi}{6\beta^2}.$$

Actually, we could reverse the logic: physics tells us that $\zeta(-1)$ has to be equal to $-\frac{1}{12}$ because $\mathcal{E}_0$ has to be equal to $\mathcal{F}$.

Correction :

[To Do...]

• *Exercise 6.7: Radial quantization (at least in 2D).*

[... To be completed...]

Correction :

[To Do...]

• *Exercise 6.8: Spanning trees of a graph*

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The edges $e \in E$ are equipped with an arbitrary but fixed orientation. An example is shown in Fig. 2.

The *discrete Laplacian* of G is a matrix M of size $|V| \times |V|$ with elements m_{ij}. For $i \neq j$, $m_{ij} = -k$ if there are $k = 0, 1, 2, \ldots$ edges between the vertices i and j; and for $i = j$, m_{ii} is the number of edges incident on the vertex i.

The *incidence matrix* of G is a matrix A_0 of size $|V| \times |E|$ with elements a_{ij}. These are $a_{ij} = 1$ if the edge j goes out of the vertex i; $a_{ij} = -1$ if the edge j goes into the vertex i; and $a_{ij} = 0$ otherwise (i.e., if the edge j is not incident on the vertex i).

(i) Write A_0 and M for the example in Fig. 2.

(ii) Show that, in the general case, $M = A_0 \cdot A_0^{\mathrm{T}}$.

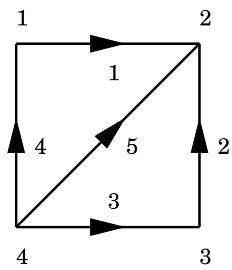

Figure 2: A graph G with 4 vertices and 5 oriented edges.

(iii) Show that the rank of A_0 is $|V| - \mathcal{C}$, where $\mathcal{C}$ denotes the number of connected components of G [Kirchhoff 1847].

We henceforth suppose that G is a connected graph, $\mathcal{C} = 1$. The *reduced incidence matrix* A is obtained from A_0 by erasing its last row.

Define a *spanning tree* of $G = (V, E)$ to be a sub-graph $G' = (V, E')$ with $E' \subseteq E$, so that G' is connected and has no cycles (i.e., the edges E' generate no closed loop).

(iv) Show that if B is a square sub-matrix of A, either B is singular, or $\det(B) = \pm 1$ [Poincaré 1901].

(v) Show that if the size of B is maximal (i.e., B is a $(|V|-1) \times (|V|-1)$ matrix), then B is non-singular if and only if the edges corresponding to its columns generate a spanning tree of G [Chuard 1922].

We recall the *Binet-Cauchy theorem*:
Let R be a $p \times q$ matrix and S a $q \times p$ matrix, with $p \leq q$. Let R' and S' be $p \times p$ sub-matrices of R and S respectively. Then,

$$\det(R \cdot S) = \sum \det(R') \cdot \det(S') \,,$$

where the sum is over all possible ways of forming sub-matrices R' and S'.

(vi) Prove the *matrix-tree theorem*: If A is the reduced incidence matrix of a graph G, then $\det(A \cdot A^{\mathrm{T}})$ equals the number of spanning trees of G.

(vii) Check this result for the example of Fig. 2.

One introduces a pair of fermionic fields (Grassmann variables) $\eta_1(i)$, $\eta_2(i)$ per vertex of G. By definition, and two of these variables anticommute ($\eta\tilde{\eta} + \tilde{\eta}\eta = 0$) and one integrates over them using the definitions $\int \mathrm{d}\eta \, 1 = 0$ and $\int \mathrm{d}\eta \, \eta = 1$. To lighten the notation, we shall denote, for $k = 1, 2$, $\mathrm{d}\eta_k \equiv \prod_{i=1}^{|V|} \mathrm{d}\eta_k(i)$ and $\eta_k \equiv [\eta_k(1), \eta_k(2), \ldots, \eta_k(|V|)]$.

(viii) Show that

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \, \mathrm{e}^{\eta_1 \cdot M \cdot \eta_2} = \det(M) = 0.$$

Deduce that

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, \eta_1(|V|)\eta_2(|V|)\mathrm{e}^{\eta_1 \cdot M \cdot \eta_2} \tag{22}$$

is the number of spanning trees of G.

Correction :

(i) For the example in the figure we have

$$A_0 = \begin{bmatrix} +1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & +1 & +1 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

(ii) Let us compute the element

$$\tilde{m}_{ij} = (A_0 \cdot A_0^{\mathrm{T}})_{ij} = \sum_k (A_0)_{ik} (A_0^{\mathrm{T}})_{kj} \,,$$

where i, j is a pair of vertices. The factors are only non-zero if the edge k is incident on both vertices i and j. If $i = j$, there are as many non-zero terms in the sum as there are edges indicent on i, and each term contributes $(\pm 1)^2 = 1$. So indeed $\tilde{m}_{ij} = m_{ij}$ in this case. If $i \neq j$, both factors are non-zero only if the edge k connects vertices i and j. Moreover the two factors have opposite signs, since k goes out of i and into j, or into i and out of j. So again $\tilde{m}_{ij} = m_{ij}$.

(iii) If G is not a connected graph, A_0 is block-diagonal with one block per connected component of G. In each block, the sum of the rows is a null vector (since any edge k belonging to that connected component contributes both $+1$ and -1 to the k'th element of the vector). It is easy to see that no other linear combination of the rows vanishes. Therefore $\mathrm{rank}(A_0) = |V| - \mathcal{C}$.

(iv) Each column of A contains exactly two non-zero entries, so each column of B can contain at most two non-zero entries. Suppose that B is non-singular. If there were some column of B with all entries zero, then clearly $\det(B) = 0$, a contradiction. Therefore each column of B must contain one or two non-zero entries. But if they all contained two non-zero entries, the sum of all rows of B would be the zero vector, so that $\det(B) = 0$, again a contradiction. It follows that B must have a column with precisely one non-zero element (viz. ± 1). By expanding the determinant along this column, the result $\det(B) = \pm 1$ follows by induction.

(v) Let now $H = (V, E_B)$ be the sub-graph of G whose $|V| - 1$ edges correspond to the columns of B. By construction, B is then the reduced incidence matrix of H. It is obvious that $\det(B) \neq 0$ iff $\mathrm{rank}(B) = |V| - 1$. But by [Kirchhoff 1847] this happens iff H is connected. Finally, it is easy to see that a connected graph with $|V|$ vertices and $|V| - 1$ edges is necessarily a spanning tree.

(vi) By the Binet-Cauchy theorem,

$$\det(A \cdot A^{\mathrm{T}}) = \sum \det(B) \cdot \det(B^{\mathrm{T}}) = \sum [\det(B)]^2 \,.$$

By [Poincaré 1901], this sum is simply the number of non-singular $(|V| - 1) \times (|V| - 1)$ sub-matrices of A. And by [Chuard 1922], this gives the number of spanning trees of G.

(vii) In the example, let M_0 denote the discrete Laplacian with the last row and column deleted. We compute $\det(M) = 8$. And indeed there are eight spanning trees on G (four without edge 5, and four more containing edge 5).

(viii) We first have

$$\mathrm{e}^{\eta_1 \cdot M \cdot \eta_2} = \mathrm{e}^{\sum_{i,j} \eta_1(i) m_{ij} \eta_2(j)} = \prod_{i,j} \mathrm{e}^{\eta_1(i) m_{ij} \eta_2(j)} = \prod_{i,j} \left[1 + \eta_1(i) m_{ij} \eta_2(j)\right],$$

where the last equality follows from the fact that $(\eta)^2 = 0$. We then develop the product, keeping only terms that contain each of the factors $\eta_1(i)$ and $\eta_2(j)$ exactly one; only such terms will give a non-zero contribution after integration. In each of these terms, we will have to permute the fermions to obtain the form $\eta_2(|V|)\eta_1(|V|)\cdots\eta_2(1)\eta_1(1)$ which is necessary for performing each integral in $\mathrm{d}\eta_1 \mathrm{d}\eta_2$ in the correct order. This rearrangement leads to a sign which is the signature of the corresponding permutation. It follows that the end result is $\det(M)$. Moreover, we have already seen that M has (exactly) one zero eigenvalue, since the sum of its rows is zero. Therefore $\det(M) = 0$. Finally, to evaluate the second integral (22), we observe that the insertion of $\eta_1(|V|)\eta_2(|V|)$ is tantamount to eliminating the last row of M. The integral thus equals $\det(A \cdot A^{\mathrm{T}})$, which is the number of spanning trees of G by the matrix-tree theorem.

1.6 Chapter 7: Interacting field theory: basics

- *Exercise 7.1: The effective potential and magnetization distribution functions*

The aim of this exercise to probability distribution function of the total magnetization is governed by the effective potential — and this gives a simple interpretation of the effective potential.

Let $M_\phi := \int d^Dx\,\phi(x)$ be the total magnetization. It is suppose to be typically extensive so let m_ϕ be the spatial mean magnetization, $m_\phi = \mathrm{Vol.}^{-1}\,M_\phi$.

(i) Find the expression of the generating function of the total magnetization, $\mathbb{E}[e^{zM_\phi}]$, in terms of the generating function $W[\cdot]$ of connected correlation functions.
Recall that if the source $J(x)$ is uniform, i.e. $J(x) = j$ independent of x, then $W[J]$ is extensive in the volume: $W[J(x) = j] = \mathrm{Vol.}\,w(j)$.

(ii) Let $P(m)dm$ be the probability density for the random variable m_ϕ. Show that at large volume, we have

$$P(m) \simeq e^{-\mathrm{Vol.}\,V_{\mathrm{eff}}(m)},$$

with $V_{\mathrm{eff}}(m)$ the effective potential, defined as the Legendre transformed of $w(j)$.

This has important consequence, in particular the most probable mean magnetization is at the minimum of the effective potential, and phase transition occurs when this minimum changes value.

Correction :

(i) By construction, since $M_\phi := \int d^Dx\,\phi(x)$ the path integral representation of the expectation $\mathbb{E}[e^{zM_\phi}]$ is

$$\mathbb{E}[e^{zM_\phi}] = \int [D\phi]\, e^{-S[\phi]}\, e^{zM_\phi} = \int [D\phi]\, e^{-S[\phi]+z\int d^Dx\,\phi(x)}.$$

It corresponds to coupled the theory to a uniform constant source. This yields $\mathbb{E}[e^{zM_\phi}] = e^{W[j(x)=z]}$. We expect $W[j(x) = z]$ to be extensive in the volume: $W[J(x) = z] = \mathrm{Vol.}\,w(z)$.

(ii) By definition of the probability density for the random variable m_ϕ we have:

$$\mathbb{E}[e^{zM_\phi}] = \mathbb{E}[e^{z\,\mathrm{Vol.}\,m}] = \int dm P(m)\, e^{z\,\mathrm{Vol.}\,m}.$$

Let us check that $P(m) \simeq e^{-\mathrm{Vol.}\,V_{\mathrm{eff}}(m)}$ reproduces the expected result. Indeed

$$\int dm\, e^{-\mathrm{Vol.}\,V_{\mathrm{eff}}(m)}\, e^{z\,\mathrm{Vol.}\,m} \simeq \int dm\, e^{-\mathrm{Vol.}\big(V_{\mathrm{eff}}(m)-z\,m\big)} \simeq e^{\mathrm{Vol.}\,w(z)}$$

with $w(z)$ the Legendre transform of $V_{\mathrm{eff}}(m)$. In the last line, we evaluated the integral using a saddle point approximation. The extremum of $(V_{\mathrm{eff}}(m) - z\,m)$ over m yields the Legendre transform $w(z)$.
This result has a clear interpretation: The effective potential specifies the probability distribution of the magnetization. This distribution is exponentially picked around the minimum of the effective potential (which then defines the mean magnetization) because of the extensive nature of this potential.

• *Exercise 7.2: Two-point correlation and vertex functions*

Prove that the two-point connected correlation function and the two-point vertex function are inverse one from the other, that is:

$$\hat{\Gamma}^{(2)}(k)\,\hat{G}_c^{(2)}(k) = 1,$$

as mentioned in the text.

Correction :

See the lecture notes.
Indeed recall that $\varphi(x_1) = \frac{\delta W[j]}{\delta j(x_1)}$. Differentiating this relation with respect to $\varphi(x_2)$ and then taking the limit $j = 0$ yields:

$$\begin{aligned}
\delta(x_1 - x_2) &= \frac{\delta}{\delta\varphi(x_2)}\frac{\delta W[j]}{\delta j(x_1)}\Big|_{j=0} = \int d^Dy\, \frac{\delta j(y)}{\delta\varphi(x_2)}\,\frac{\delta^2 W[j]}{\delta j(x_1)\delta j(y)}\Big|_{j=0} \\
&= \int d^Dy\, \langle\phi(x_1)\phi(y)\rangle^c\, \frac{\delta j(y)}{\delta\varphi(x_2)}\Big|_{j=0} \\
&= \int d^Dy\, \langle\phi(x_1)\phi(y)\rangle^c\, \Gamma^{(2)}(y, x_2) = \int d^Dy\, G_c^{(2)}(x_1, y)\,\Gamma^{(2)}(y, x_2)
\end{aligned}$$

In the first line we used the chain rule to compute the derivative. In the second and third line we used the fact that $W[j]$ and $\Gamma[\varphi]$ are the generating function of the connected and vertex functions, respectively. In Fourier space, this gives the above relation $\hat{\Gamma}^{(2)}(k)\,\hat{G}_c^{(2)}(k) = 1$.
Thus knowing $\hat{\Gamma}^{(2)}(k)$ is enough to reconstruct the connected two-point function.

• *Exercise 7.2bis: Ward identities for the stress-tensor*

The aim of this exercise is to derive the Ward identities associated to translation symmetry. This will allows us to make contact with the stress tensor.

We consider a scalar field ϕ in D-dimensional Euclidean flat space with action

$$S[\phi] = \int d^Dx\,\big[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big].$$

Translations act on the field as $\phi(x) \to \phi(x - a)$ for any vector a. The infinitesimal transformation is $\phi(x) \to \phi(x) - \epsilon a^\mu(\partial_\mu\phi)(x)$.

(i) Let us consider an infinitesimal transformation $\phi(x) \to \phi(x) - \epsilon^\mu(x)(\partial_\mu\phi)(x)$ with the space dependent vector fields $\epsilon(x)$.
Prove that the variation of the action is (assuming that the boundary terms do not contribute)

$$\delta S[\phi] = -\int d^Dx\,(\partial^\mu\epsilon^\sigma)(x)\,T_{\mu\sigma}(x) = \int d^Dx\,\epsilon^\sigma(x)\,(\partial^\mu T_{\mu\sigma})(x),$$

with $T_{\mu\sigma}(x)$ the so-called stress-tensor ($g_{\mu\sigma}$ is the Euclidean flat metric):

$$T_{\mu\sigma}(x) = \partial_\mu\phi\,\partial_\sigma\phi - g_{\mu\sigma}\big[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\big].$$

(ii) Prove that the stress tensor is conserved, that is: $\partial_\mu T_{\mu\nu}(x) = 0$ inside any correlation functions away from operator insertions.

(iii) Prove the following Ward identities (here we use the notation $\partial_j^\nu = \partial/\partial_{y_j^\nu}$):

$$\langle (\partial_\mu T_\mu^\nu)(x)\, \phi(y_1)\cdots\phi(y_p)\rangle = \sum_j \delta(x-y_j)\, \partial_j^\nu \,\langle \phi(y_1)\cdots\phi(y_p)\rangle,$$

in presence of scalar field insertion of the form $\phi(y_1)\cdots\phi(y_p)$.

(iv) Do the same construction but for rotation symmetry.

Correction :

[To Do...]

- *Exercise 7.3: Generating functions $Z[J]$, $W[J]$ and $\Gamma[\varphi]$ for a ϕ^3-theory in $D=0$.*

We consider a very simple theory in dimension $D=0$ with action

$$S[\phi] = \frac{1}{2}\phi^2 + \frac{g}{3!}\,\phi^3. \tag{23}$$

The partition function, with an external source J, is defined by

$$Z[J] = \int d\phi\, \exp\left[-\frac{1}{\hbar}\big(S[\phi] - J\phi\big)\right]. \tag{24}$$

The parameter $\hbar$ is an expansion parameter from the classical solution obtained in the limit $\hbar \to 0$. This theory has a meaning only in perturbation theory (because the potential ϕ^3 is unbounded from below). We are going to study it perturbatively. By convention we assume $g > 0$.

(i) We aim at calculating $Z[J]$ at one loop. We set $\phi = \phi_c(J) + \sqrt{\hbar}\,\chi$ where $\phi_c(J)$ minimizes the action $S[\phi; J] = S[\phi] - J\phi$ in presence of an external source. Compute $\phi_c(J)$ and the corresponding action $S[\phi_c; J]$.

(ii) Compute $Z[J]$ at leading order up to $O(\hbar)$.

(iii) Compute $W[J] = \hbar\,\log Z[J]$ up to $O(\hbar^{3/2})$ and expand in power of J up to order J^4 included. We set:

$$W[J] = \sum_{n=0}^{\infty} \frac{1}{n!}\, W^{(n)}\, J^n.$$

Determine $W^{(n)}$ for $n = 0, 1, 2, 3, 4$.

(iv) Compare the previous results with a direct computation via (connected) Feynman diagrams up to order g^4.

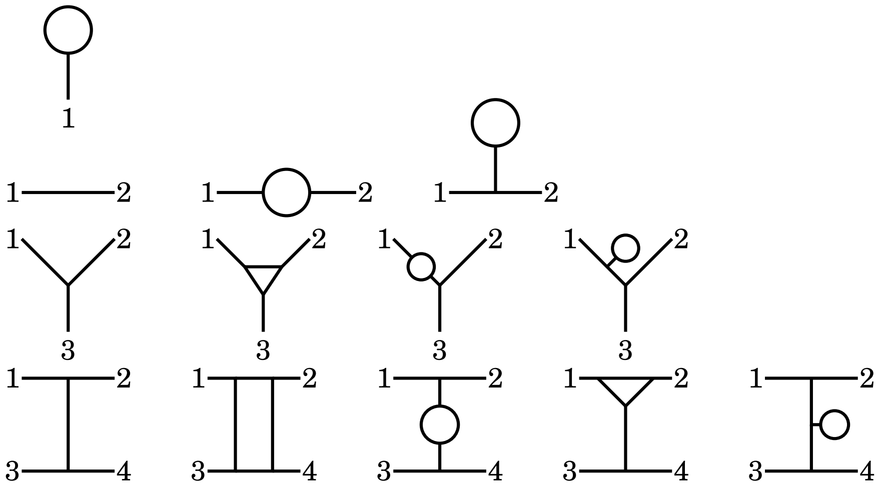

Figure 3: Feynman diagrams with 1, 2, 3 et 4 external lines.

(v) We now define the effective action $\Gamma[\varphi]$ via the Legendre transform:

$$\Gamma[\varphi] = J\varphi - W[J], \quad \text{with} \quad \varphi = \frac{\partial W[J]}{\partial J}.$$

Compute φ up to order J^3 included and neglecting terms of order $\hbar^2$ (i.e. up to two loop diagrams). Invert this relation to get J as a series in ρ with

$$\rho = \varphi + \frac{1}{2} g\hbar,$$

up to order ρ^3 included. To do this series expansion, assume that both J and ρ are small.

(vi) Show that the definition of Γ implies $\frac{\partial \Gamma}{\partial \varphi} = \frac{\partial \Gamma}{\partial \rho} = J$. Compute $\Gamma[\rho]$, up to terms of order ρ^5 or $\hbar^{3/2}$, by integrating $J[\rho]$ with to respect to ρ.
Let

$$\Gamma[\rho] = \sum_{n=1}^{\infty} \frac{1}{n!} \, \Gamma^{(n)} \, \rho^n.$$

Determine $\Gamma^{(n)}$ for $n = 1, 2, 3, 4$.
Show that these results coincide with the one particle irreducible (1-PI) diagrams.

Correction :

(i) The action with source is $S[\phi; J] = \frac{1}{2}\phi^2 + \frac{g}{3!}\,\phi^3 - J\phi$. It has an extremum at

$$\phi_c + \frac{g}{2}\phi_c^2 - J = 0,$$

which gives

$$\phi_c^\pm = -\frac{1}{g} \pm \frac{1}{g}\sqrt{1+2gJ}. \tag{25}$$

At small g we have $\phi_c^+ = J + \mathcal{O}(g)$ and $\phi_c^- = -\frac{2}{g} - J + \mathcal{O}(g)$.
In order to compute the classical action, it is simpler to use the equation $\phi_c + \frac{g}{2}\phi_c^2 = J$ to reduce it to an expression linear in ϕ_c and then to insert the explicit formula $\phi_c^\pm = -\frac{1}{g} \pm \frac{1}{g}\sqrt{1+2gJ}$. The result is:

$$\begin{aligned} S[\phi_c^\pm; J] &= \frac{J}{3g} - \frac{\phi_c^\pm}{3g} - \frac{2}{3}J\phi_c^\pm \\ &= \frac{1}{3g^2}\left[1 + 3gJ \mp (1+2gJ)^{3/2}\right]. \end{aligned} \tag{26}$$

We suppose $|2gJ < 1|$ so that $\sqrt{1+2gJ} > 0$. Then, ϕ_c^+ correspond to a minimum and ϕ_c^- to a maximum, and $S[\phi_c^+; J] < S[\phi_c^-; J]$. Thus we keep ϕ_c^+ and set $S_c = S[\phi_c^+; J]$.

(ii) We set $\phi = \phi_c^+ + \sqrt{\hbar}\chi$. Because ϕ_c^+ extremizes the action we have

$$S[\phi; J] = S_c[J] + \frac{\hbar}{2}\chi^2\, S''[\phi_c^+; J] + O(\hbar^{3/2}).$$

Since $S''[\phi; J] = 1 + g\phi$ and $1 + g\phi_c^+ = \sqrt{1+2gJ}$ we get

$$S[\phi; J] = \frac{1}{3g^2}\left[1 + 3gJ - (1+2gJ)^{3/2}\right] + \frac{\hbar}{2}(1+2gJ)^{1/2}\chi^2 + \mathcal{O}(\hbar^{3/2}).$$

Note that the first term is regular as $g \to 0$.
To compute the partition function to this order we simply have to evaluate the Gaussian integral. The result is (using $d\phi = \sqrt{\hbar}d\chi$)

$$Z[J] = \exp\left[-\frac{1}{3g^2\hbar}\left(1 + 3gJ - (1+2gJ)^{3/2}\right)\right]\sqrt{\frac{2\pi\hbar}{(1+2gJ)^{1/2}}} + \mathcal{O}(\hbar). \tag{27}$$

The factor $\sqrt{\hbar}$ comes from the Jacobian $d\phi/d\chi = \sqrt{\hbar}$.

(iii) From the previous formula, we have

$$\begin{aligned} W[J] &\equiv \hbar \log Z[J] \qquad (28) \\ &= -\frac{1}{3g^2}\left[1 + 3gJ - (1+2gJ)^{3/2}\right] - \frac{\hbar}{4}\log(1+2gJ) + \frac{\hbar}{2}\log(2\pi\hbar) + \mathcal{O}(\hbar^{3/2}). \end{aligned}$$

Recall that the first term is $-S[\phi_c^+; J] = -\frac{1}{3g^2}\left[1 + 3gJ - (1+2gJ)^{3/2}\right]$.
Developing up to order 4 in J, we get $W^{(0)} = \frac{\hbar}{2}\log(2\pi\hbar)$ and

$$\begin{array}{ll} W^{(1)} = -\frac{1}{2}g\hbar & W^{(2)} = 1 + g^2\hbar \\ W^{(3)} = -g - 4g^3\hbar & W^{(4)} = 3g^2 + 24g^4\hbar \end{array} \tag{29}$$

Recall that we defined $W^{(n)}$ via $W[J] = \sum_{n=0}^{\infty} \frac{1}{n!}\, W^{(n)}\, J^n$.

(iv) We compute the connected Feynman diagrams which are drawn in Figure 3 up to order 4. The Feynman rules are:
— Propagator: $\hbar$ (external lines included);
— Vertex: $(-g/3!\hbar)$;
— An extra weight $\hbar^{1-n}$ for n external lines (because of the normalisation of $W[J]$ and the source);
— Combinatorial factors ($1/k!$ at order k times the number of ways of drawing the diagram).
We make explicit the different contributions, including the combinatorial factors.

- n=1: There is one vertex, one external line and one loop. The symmetry factor is 3. Hence $W^{(1)} = \hbar^0 \times (-\frac{g}{3!\hbar}) \times \hbar^2 \times (3)$. Hence:

$$W^{(1)} = -\frac{1}{2} g\hbar.$$

- n=2: The first diagram has two external lines, one propagator and no symmetry factor: its contribution is $\hbar^{-1} \times \hbar = 1$. The second diagram has two external lines, two vertices (order two diagram), four propagators (two forming a loop), and combinatorial factor $(6 \cdot 3 \cdot 2)$: its contribution is $\hbar^{-1} \times \frac{1}{2!}(-\frac{g}{3!\hbar})^2 \times \hbar^4 \times (3!^2) = \frac{1}{2} g^2 \hbar$. The third diagram with two external lines at order two has four propagators and combinatorial factor $(6 \cdot 2 \cdot 3)$: its contribution is also $\frac{1}{2} g^2 \hbar$. Hence

$$W^{(2)} = 1 + 2 \times \frac{1}{2} g^2 \hbar = 1 + g^2 \hbar.$$

- n=3: The first diagram is at first order with three external lines, three propagators, one vertex and a symmetry factor $(3 \cdot 2)$: its contribution is $\hbar^{-2} \times (-\frac{g}{3!\hbar}) \times \hbar^3 \times (3!) = -g$. The second is at order three with a symmetry factor $(9 \cdot 6 \cdot 3 \cdot 4 \cdot 2)$: its contribution is $\frac{1}{3!} \times \hbar^{-2} \times (-\frac{g}{3!\hbar})^3 \times \hbar^6 \times (3!^4) = -g^3 \hbar$. The two last diagrams are at order three with a symmetry factor $(9 \cdot 6 \cdot 2 \cdot 2 \cdot 3)$: their contributions are $\frac{1}{3!} \times \hbar^{-2} \times (-\frac{g}{3!\hbar})^3 \times \hbar^6 \times (3!^3 \cdot 3) = -\frac{1}{2} g^3 \hbar$. Each of these two diagrams appear in three copies with cyclic permutations of the indices $(1, 2, 3)$. Hence the total contribution is:

$$W^{(3)} = -g - g^3 \hbar - 2 \times 3 \times \frac{1}{2} g^3 \hbar = -g - 4 g^3 \hbar.$$

- n=4: We leave to the reader the pleasure to verify that

$$W^{(4)} = 3g^2 + 3g^4 \hbar \big(1 + \frac{5}{2} + 2 + \frac{5}{2}\big) = 3g^2 + 24 g^4 \hbar.$$

Remark: Alternativement, ceci peut être comparé avec un calcul faisant intervenir des diagrammes de Feynman. Les régles de Feynman sont les suivantes:
— $W^{(n)}$ donne les diagrammes connexes à n pattes externes *fixées.*
— Il y a un facteur de $\hbar$ par boucle, et $(-g)$ par vertex.
— Chaque diagramme est pondéré par l'inverse de son facteur de symétrie (les pattes externes étant fixées).
Les diagrammes avec jusqu'à 4 pattes externes sont montrées sur la Figure 3. Calculons leurs contributions:

- $n = 1$: Il y a un facteur de symétrie de 2, une boucle, et un vertex. Donc,

$$W^{(1)} = -\frac{1}{2} g\hbar.$$

- $n = 2$: Les deux derniers diagrammes ont un facteur de symétrie de 2. Donc,

$$W^{(2)} = 1 + 2 \times \frac{1}{2} g^2 \hbar = 1 + g^2 \hbar.$$

- $n = 3$: Les deux derniers diagrammes apparaissent en trois versions (avec $1, 2, 3 \to 2, 3, 1$ et $1, 2, 3 \to 3, 1, 2$) et ont un facteur de symétrie de 2. Donc,

$$W^{(3)} = -g - g^3 \hbar \big(1 + 2 \times \frac{3}{2}\big) = -g - 4 g^3 \hbar.$$

- $n=4$: Tous les diagrammes apparaissent en trois versions (avec $1,2,3,4 \to 1,3,2,4$ et $1,2,3,4 \to 1,4,2,3$. En plus, le 4ème diagramme doit être multiplié par deux (le triangle qui décore le vertex peut être en haut où en bas), et dans le 3ème et 5ème diagramme le tadpole peut apparaitre en cinq positions différentes. Donc,

$$W^{(4)} = 3g^2 + 3g^4\hbar\big(1 + \frac{5}{2} + 2 + \frac{5}{2}\big) = 3g^2 + 24g^4\hbar.$$

(v) Recall that $S[\phi;J] = S[\phi] - J\phi$. From eq.(28) and the definition of φ, we get

$$\begin{aligned}\varphi &= \frac{\partial}{\partial J}\left\{-S[\phi_c^+;J] - \frac{\hbar}{4}\log(1+2gJ) + \mathcal{O}(\hbar^{3/2})\right\}\\ &= -\frac{1}{g} + \frac{(1+2gJ)^{1/2}}{g} - \frac{\hbar}{2}\frac{g}{1+2gJ} + \mathcal{O}(\hbar^{3/2}).\end{aligned}$$

To go from the first to the second line we used $\partial_J S[\phi_c^+;J] = (\partial_J\phi_c^+)S'[\phi_c^+;J] - \phi_c^+$ which reduces to $\partial_J S[\phi_c^+;J] = -\phi_c^+$ because ϕ_c^+ is an extremum of the action. This is a cubic equation for $\sqrt{1+2gJ}$.
We can develop in power of J to get:

$$\varphi = J - \frac{g}{2}J^2 + \frac{g^2}{2}J^3 + \hbar\left[-\frac{g}{2} + g^2J - 2g^3J^2 + 4g^4J^3\right] + \mathcal{O}(J^4,\hbar^{3/2}).$$

The definition $\rho \equiv \varphi + \frac{1}{2}g\hbar$ is motivated by the following simplification of this expansion:

$$\rho = (1+g^2\hbar)J - \frac{g}{2}(1+4g^2\hbar)J^2 + \frac{g^2}{2}(1+8g^2\hbar)J^3 + \mathcal{O}(J^4,\hbar^{3/2}).$$

The definition $\rho \equiv \varphi + \frac{1}{2}g\hbar$ can also be read as $\rho = \varphi - W^{(1)}$ so that it is such as to absorb the term linear in J in $W[J]$ (which is usually not present).
To inverse the series and express ρ as a function of J, we set $\rho = \sum_k b_k J^k$ and compute the coefficient b_k recursively. After a (tedious) computation, we get

$$J[\rho] = (1-g^2\hbar)\rho + \frac{g}{2}(1+g^2\hbar)\rho^2 - \frac{g^4\hbar}{2}\rho^3 + \mathcal{O}(\rho^4,\hbar^{3/2}).$$

(vi) Defining $\Gamma[\varphi] = \varphi J - W[J]$, we have (by standard formula of Legendre transform)

$$J = \frac{\partial\Gamma}{\partial\varphi} = \frac{\partial\Gamma}{\partial\rho}.$$

To get $\Gamma[\rho]$ we can integrate the formula $J[\rho]$ w.r.t. ρ:

$$\Gamma[\rho] = \Gamma[\rho=0] + \frac{1}{2}(1-g^2\hbar)\rho^2 + \frac{g}{6}(1+g^2\hbar)\rho^3 - \frac{g^4\hbar}{8}\rho^4 + \mathcal{O}(\rho^5,\hbar^{3/2}). \tag{30}$$

From the above formula we read :

$$\begin{aligned}&\Gamma^{(1)} = 0 && \Gamma^{(2)} = 1 - g^2\hbar\\ &\Gamma^{(3)} = g + g^3\hbar && \Gamma^{(4)} = -3g^4\hbar\end{aligned} \tag{31}$$

From the general theory, $\Gamma^{(n)}$ should reproduce the 1-PI diagrams. More precisely, we should have $\Gamma^{(1)} = 0$, $\Gamma^{(2)} = 1/W^{(2)}$ and $\Gamma^{(n)}$ for $n \geq 3$ gives *minus* the 1-PI diagrams contributing to $W^{(n)}$.
This works perfectly in the present setting (as it should). Notice that $-\Gamma^{(3)} = -g - g^3\hbar$ and $-\Gamma^{(4)} = 3g^4\hbar$ agree with the 1-PI diagrams (i.e diagrams number 1 et 2 for $W^{(3)}$, and number 2 pour $W^{(4)}$). Of course, computing only the 1-PI diagrams reduce enormously the task in any concrete situation.

• *Exercise 7.4: Effective action and one-particle irreducible diagrams.*

The aim of this exercise is to prove the equality between the effective action and the generating function of 1-PI diagrams. To simplify matter, we consider a 'field' made of N ($N \gg 1$) components ϕ^j, $j = 1, \cdots, N$. We view ϕ^j as random variables.

Let us define a 'partition function' $Z_\epsilon[J]$ by

$$Z_\epsilon[J] = \int D\phi\, e^{-\epsilon^{-1}\left[\Gamma[\phi]-(J,\phi)\right]}, \quad \text{with } D\phi = \prod_j \frac{d\phi^j}{\sqrt{2\pi\epsilon}}.$$

with J a source $(J, \phi) = J_j \phi^j$, and $\Gamma[\phi]$ an action which we define via its (formal) series expansion (summation over repeated indices is implicit):

$$\Gamma[\phi] = \frac{1}{2}\Gamma^{(2)}_{jk}\phi^j\phi^k - \sum_{n\geq 3}\frac{1}{n!}\Gamma^{(n)}_{j_1\cdots j_n}\phi^{j_1}\cdots\phi^{j_n}.$$

We shall compute this partition function in two different ways: via a saddle point approximation or via a perturbation expansion.

(i) Justify that this integral can be evaluating the integral via a saddle-point when $\epsilon \to 0$.
Prove that

$$\log Z_\epsilon = \frac{1}{\epsilon}W[J]\,\big(1 + O(\epsilon)\big),$$

where $W[J]$ is the Legendre transform of the action Γ: $W[J] = (J, \phi_*) - \Gamma(\phi_*)$ with ϕ_* determined via $\frac{\partial\Gamma}{\partial\phi^j}(\phi_*) = J_j$.
Hint : Do the computation formally which amounts to assume that the integral converges and that there is only one saddle point.

Let us now compute $Z_\epsilon[J]$ in perturbation theory. Let us decompose the action as the sum of its Gaussian part plus the rest that we view as the interaction part: $\Gamma[\phi] = \frac{1}{2}\Gamma^{(2)}_{jk}\phi^j\phi^k - \hat{\Gamma}[\phi]$.

(ii) Write

$$Z_\epsilon[J] = \int D\phi\, e^{-\frac{1}{2\epsilon}\Gamma^{(2)}_{jk}\phi^j\phi^k}\, e^{\epsilon^{-1}\hat{\Gamma}[\phi]}\, e^{\epsilon^{-1}(J,\phi)}.$$

We view J/ϵ as source, and we aim at computing the connected correlation function using Feynman diagrams perturbative expansion.
Show that the propagator is $\epsilon\, G^{jk}$ with $G = (\Gamma^{(2)})^{-1}$ and the vertices are $\epsilon^{-1}\,\Gamma^{(n)}_{j_1\cdots j_n}$ with $n \geq 3$.

(iii) Compute the two-, three- and four-point connected correlations $G_{(n)}$, $n = 1, 2, 3$, at the level tree, defined by

$$G^{j_1\cdots j_n}_{(n)} = \frac{\partial^n}{\partial J_{j_1}\cdots\partial J_{j_n}}\log Z_\epsilon[J]\Big|_{\text{tree}}.$$

Show that they are of order ϵ^{-1}. Draw their diagrammatic representations (in terms of propagators and vertices) and compare those with the representations of the connected correlation functions in terms of 1-PI diagrams.

(iv) Prove that, when $\epsilon \to 0$, the leading contribution comes only from the planar tree diagrams and that all these diagrams scale like $1/\epsilon$. That is:

$$\log Z_\epsilon[J] = \frac{1}{\epsilon}\Big(\text{planar tree diagrams} + O(\epsilon)\Big).$$

Hint : Recall that, for a connected graph drawn on a surface of genus g (i.e. with g handles, $g > 0$), one has $V - E + L + 1 = 2 - g$ with V its number of vertices, E its number of edges and L its numbers of loops (this is called the Euler characteristics). Then, argue that each Feynman graph contributing to the N point connected functions is weighted by (symbolically) $(\epsilon\, G)^E\, (-\epsilon^{-1}\, \Gamma^{(n)})^{V_{\rm int}}\, (\epsilon^{-1}\, J)^N$ with $V_{\rm int} + N$ total number of vertices.

(v) By inverting the Legendre transform, deduce the claim that the effective action is the generating function of 1-PI diagrams.

Correction :

(i) In the limit $\epsilon \to 0$ the integral can be evaluate via a saddle point method. The saddle point is at the extremum of the action $S[\phi] = \Gamma[\phi] - (J, \phi)$. So it is at ϕ_* solution of $\partial S[\phi]/\partial \phi^i = 0$, i.e.

$$J_i = \frac{\partial \Gamma[\phi]}{\partial \phi^i}\Big|_{\phi_*}.$$

This equation determines ϕ_* as a function of J.
To leading order in ϵ, the integral is given by the value of the integrant at the saddle point. Hence

$$\log Z_\epsilon[J] = \epsilon^{-1} W[J] + O(\epsilon^0), \quad W[J] = (J, \phi_*) - \Gamma[\phi_*].$$

In $W[J]$, we recognize the Legendre transformed of $\Gamma[\phi]$.

(ii) We can expand the action in power of ϕ and extract the quadratic term and write

$$\epsilon\, S[\phi] = \Gamma[\phi] - (J, \phi) = \Gamma^{(2)}[\phi] - \hat{\Gamma}[\phi] - (J, \phi).$$

This the Gaussian part of the action is $e^{-\frac{1}{\epsilon}\Gamma^{(2)}[\phi]}$. We write $\Gamma^{(2)} = \frac{1}{2}\Gamma^{(2)}_{ij}\phi^i\phi^j$. The propagator associated to this Gaussian action is $\epsilon G^{(2)}$ with $G^{(2)jk}$ the inverse of $\Gamma^{(2)}_{jk}$ We can decompose the integrand as

$$e^{-S[\phi]} = e^{-\frac{1}{2\epsilon}\Gamma^{(2)}[\phi]}\, e^{+\frac{1}{\epsilon}\hat{\Gamma}[\phi] + \frac{1}{\epsilon}(J,\phi)}.$$

We treat perturbatively the term $e^{+\frac{1}{\epsilon}\hat{\Gamma}[\phi] + \frac{1}{\epsilon}(J,\phi)}$ in this action. When computing perturbatively the integral, we then get Feynamnn rules with:
— Propagators: $[\epsilon\, G^{(2)}]$.
— Vertices of k lines (with $k = 3, 4, \cdots$): $[\epsilon^{-1}\, \Gamma^{(k)}]$.
— External lines with $[\epsilon^{-1}\, J]$ per line.

(iii) Let us compute the connected correlation functions at the tree level (See figures in the lecture note). We write the formula symbolically without indices in order to them make them more readable.
— For the 2-point connected function:

$$\langle \phi\phi \rangle^c = \epsilon\, G^{(2)}$$

and

$$\log Z_\epsilon[J]^{(2)}_{\rm tree} = [\epsilon^{-1}\, J][\epsilon^{-1}\, J][\epsilon\, G^{(2)}] = \epsilon^{-1}\,[JJG^{(2)}].$$

— For the 3-point connected function, only one diagram, with a 3-line vertex, contributes:

$$\langle\phi\phi\phi\rangle^c = [\epsilon\, G^{(2)}][\epsilon\, G^{(2)}][\epsilon\, G^{(2)}][\epsilon^{-1}\,\Gamma^{(3)}],$$

that is $\langle\phi\phi\phi\rangle^c = \epsilon^{-2}\, G^{(3)}$ with $G^{(3)} = [G^{(2)}G^{(2)}G^{(2)}\Gamma^{(3)}]$. Hence

$$\log Z_\epsilon[J]^{(3)}_{\rm tree} = [\epsilon^{-1}\, J][\epsilon^{-1}\, J][\epsilon^{-1}\, J][\epsilon^{-2}\, G^{(3)}] = \epsilon^{-1}\,[JJJG^{(3)}].$$

— For the 4-point connected function, two diagrams contribute: one with only one 4-line vertex and one with two 3-lines vertices link with a propagators (and their permutations). The contribution of the first one is

$$[\epsilon\, G^{(2)}]^4\,[\epsilon^{-1}\,\Gamma^{(4)}] = \epsilon^3\,[G^{(2)}]^4\,\Gamma^{(4)}.$$

The contribution of diagrams of the second type is

$$[\epsilon\, G^{(2)}]^2[\epsilon^{-1}\,\Gamma^{(3)}][\epsilon\, G^{(2)}][\epsilon^{-1}\,\Gamma^{(3)}][\epsilon\, G^{(2)}]^2 = \epsilon^3\,[G^{(2)}]^2\,\Gamma^{(3)}\,G^{(2)}\,\Gamma^{(3)}\,[G^{(2)}]^2.$$

They are all of order ϵ^3. To get their contribution to the tree partition function $\log Z_\epsilon[J]^{4)}_{tree;}$ we have to multiply them by $[[\epsilon^{-1}\, J]^4$, so their contribute is of order ϵ^{-1} to the partition function.
We see that all tree diagrams contribute to order ϵ^{-1} to the tree partition function.

(iv) Let us look at the loop expansion of the partition function $\log Z_\epsilon[J]$. Only connected diagrams matter. These diagramm are form with the propagator, the vertcies and the external lines as discussed above. Consider a diagramm with N external lines (attached to the source), V vertices (which are of different degrees), L loops, and E edges (which are the propagators). To each of this diagram is attached an abstract diagram whose number of edges is $E_{tot} + E$ (one for each propagator), number of vertices is $V_{tot} = V + N$ (because the source are vertices with only one external line) and number of faces is $F_{tot} = L + 1$ (the term 1 comes from the external face in contact with all sources). Euler characteristic formula then tell us that such diagram can be drawn a surface of genus (alias the number of handle of the surface) $g \geq 0$ with

$$2 - g = (V + N) - E + (L + 1).$$

Each diagram of this type is weighted by a factor $\epsilon^{-V}\,\epsilon^{E}$, because each propagator gives a factor ϵ and each vertex a factor ϵ^{-1}. To get it contribution to $\log Z_\epsilon[J]$ we have to multiply by $[\epsilon^{-1}J]$ for each external line, hence by a factor ϵ^{-N}. Hence each diagram of this type contribute to $\log Z_\epsilon[J]$ with a weight proportional to

$$\epsilon^{-V}\,\epsilon^{E}\,\epsilon^{-N} = \epsilon^{-V+E-N} = \epsilon^{-1}\,\epsilon^{L+g}.$$

Since L and g are positive (non-negative), the leading contribution is for $L = 0$ and $g = 0$, that is planar tree diagrams.

(v) Thus, $\log Z_\epsilon[J]$ scale as ϵ^{-1} to leading order in ϵ. On one hand, this leading contribution is given the Legendre transform of the effective action $S[\phi]$. On the other hand, it is given by the tree planar connected diagrams made $\Gamma[\phi]$ (with the quadratic part defining the propagator). If we take the coefficient of $\Gamma[\phi]$ to the 1-PI diagram of field theory, the tree planar connected diagrams made $\Gamma[\phi]$ are by definition of the connected correlation function of that field theory.
Hence, we prove that the Legendre transform of the generating function of the 1-PI diagrams is the generating function $W[J]$ of the connected correlation function. Since, the Legendre transform of $W[J]$ is the effective action, we prove that the generating function of the 1-PI diagrams is the effective action.

• *Exercise 7.5: Computation of the one-loop effective potential*

Prove the formula for the one-loop effective potential of the ϕ^4-theory given in the text. Namely

$$V^{\text{eff}}_{1-\text{loop}}(\varphi) = \frac{1}{2!} A_\Lambda \, \varphi^2 + \frac{1}{4!} B_\Lambda \, \varphi^4 + \frac{\hbar}{(8\pi)^2} \big(V''(\varphi)\big)^2 \log[\frac{V''(\varphi)}{\mu^2}],$$

with

$$\begin{aligned} A_\Lambda &= m_0^2 + \frac{\hbar g_0}{2}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m_0^2}{(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2})\Big) + O((\hbar g_0)^2), \\ B_\Lambda &= g_0 - \hbar g_0^2 \frac{3}{2(4\pi)^2}\log(\frac{\Lambda^2}{\mu^2}) + O(g_0(\hbar g_0)^2) \end{aligned}$$

with μ^2 an arbitrary scale that we introduced by dimensional analysis.

Analyse this potential and conclude.

Correction :

Cf lecture notes.

• *Exercise 7.6: Computation of one-loop Feynman diagrams*

[...To be completed...]

Correction :

Cf lecture notes.

• *Exercise 7.7: The $O(N)$ vector model with $N \to \infty$ in $D = 3$*

We study a model of N-component spins $\vec{\Phi}$ governed by the Hamiltonian

$$H[\vec{\Phi}] = \frac{1}{2}\int d\vec{x} \left\{ \sum_{\alpha=1}^{N} \left(\frac{\partial \Phi_\alpha}{\partial \vec{x}}\right)^2 + r_0 \sum_{\alpha=1}^{N} (\Phi_\alpha)^2 + \frac{u}{12N}\left(\sum_{\alpha=1}^{N}(\Phi_\alpha)^2\right)^2 \right\}.$$

The dimension of the embedding space is fixed as $D = 3$. In this exercise we shall compute the critical exponents for large N, and more precisely, first for infinite N and then the corrections to order $1/N$.

(i) Write the propagator and interaction vertex in Fourier space. Which diagrams contribute to the correlation function

$$G(k) = \langle \tilde{\Phi}_\alpha(-\vec{k}) \tilde{\Phi}_\alpha(\vec{k}) \rangle \ ?$$

Show that in the limit $N \to \infty$, the only surviving diagrams are of the “cactus” type (see Figure 4), where the solid line represents the bare (free) propagator.

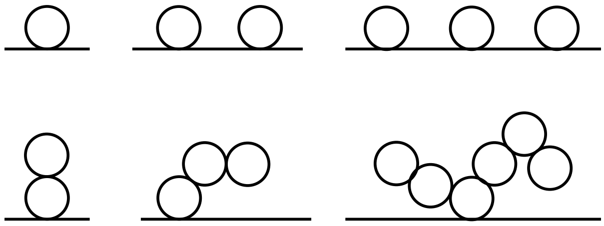

Figure 4: Some examples of "cactus" diagrams.

(ii) Deduce the implicit equation

$$\frac{1}{G_\Phi^\infty(k)} = k^2 + r_0 + \frac{u}{6}\int_{q<\Lambda} \frac{d^3q}{(2\pi)^3}\, G_\Phi^\infty(q)$$

satisfied by the dressed propagator $G_\Phi^\infty(k)$, where Λ denotes an ultraviolet cut-off. (*Hint*: Formally sum up subclasses of diagrams in a geometric series.)

(iii) Interpret the identity (1.6). Use it to compute first the critical temperature and next the exponents η^∞ and ν^∞ in the limit $\Lambda \to \infty$. (*Hint*: The critical temperature is such that the renormalised mass vanishes.)

(iv) We wish to recover this result by the saddle point method. By introducing a new scalar field $\sigma(\vec{x})$, show that one may rewrite the partition function of the above model in the form

$$Z = \int D\vec{\Phi}(\vec{x})\; D\sigma(\vec{x})\; \exp\left(-H[\vec{\Phi},\sigma]\right) ,$$

where

$$H[\vec{\Phi},\sigma] = \frac{1}{2}\int d\vec{x}\, \left\{ \sum_{\alpha=1}^{N}\left(\frac{\partial \Phi_\alpha}{\partial \vec{x}}\right)^2 + \left(r_0 + i\sqrt{\frac{u}{3N}}\,\sigma\right)\sum_{\alpha=1}^{N}(\Phi_\alpha)^2 + \sigma^2 \right\} .$$

(v) Integrate over the fields $\vec{\Phi}$. Which effective action for the field σ does one arrive at?

(vi) In the limit $N \to \infty$ one can obtain Z by computing the saddle point of the preceeding action. One supposes that this saddle point is uniform (that is, independent of x). Show that we hence recover the implicit equation (1.6).

(vii) Verify that the classical solution obtained is indeed a local minimum of the action.

(viii) We now use the action found in (v) as the starting point for computing the corrections of order $1/N$ to the critical exponents. Write down the bare propagators of the fields σ and $\vec{\Phi}$, as well as the interaction vertex.

In the following questions we focus on obtaining the propagator of σ in the $N \to \infty$ limit. This step is necessary in order to go to the next order in the computation of the Φ-propagator.

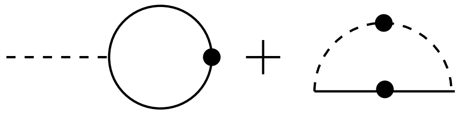

Figure 5: Development of the self-energy.

(ix) Show that in the limit $N \to \infty$ the dressed propagator $G_\sigma^\infty(k)$ of σ satisfies the implicit equation illustrated in Figure **??**. (Here the solid line represents the propagator of Φ and the dashed line that of σ. The presence of a point on a propagator means that it is dressed.) Could this equation have been anticipated from the answer to question (iv)?

(x) Compute the integral

$$I = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(\mathbf{1}-\mathbf{q})^2} \cdot \frac{1}{\mathbf{q}^2} = \frac{1}{8}\,,$$

where $\mathbf{1}$ represents a unit vector. (*Hint*: use polar coordinates.)

(xi) Deduce that at the critical temperature we have

$$G_\sigma^\infty(k) \simeq \frac{48}{u}\, k \qquad (k \to 0)\,.$$

The final stage of the exercise is now to obtain the propagator of Φ to order $1/N$.

(xii) Show that the self-energy $\Sigma_\Phi^N(k)$ of the dressed propagator $G_\Phi^N(k)$ of Φ is given by Figure 5, where the first term represents the finite contribution for $N \to \infty$ that has been studied in question (iv), while the second term is the sought-for contribution at order $1/N$.

(xiii) Infer that

$$\Sigma_\Phi^N(k) - \Sigma_\Phi^N(0) \simeq \frac{8}{3\pi^2 N}\, k^2 \ln k \qquad (k \to 0)\,.$$

(xiv) Deduce the value of the exponent η to order $1/N$.

Correction :

(i) We work in three spatial dimensions. Let us set

$$\tilde{\phi}_\alpha(\mathbf{k}) = \int \mathrm{d}^3 x \phi_\alpha(\mathbf{x}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}},$$

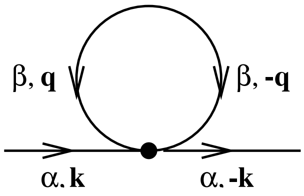

Figure 6: A diagram contributing to $G(\mathbf{k})$.

and transform Fourier transform the Hamiltonian. Doing the integral d^3x produces a momentum conserving delta function:

$$\begin{aligned} H[\vec{\phi}] &= \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{\alpha=1}^{N}\tilde{\phi}_\alpha(\mathbf{k})(k^2+r_0)\tilde{\phi}_\alpha(-\mathbf{k}) \\ &+ \frac{u}{24N}\sum_{\alpha,\beta=1}^{N}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\frac{\mathrm{d}^3k_3}{(2\pi)^3}\tilde{\phi}_\alpha(\mathbf{k}_1)\tilde{\phi}_\alpha(\mathbf{k}_2)\tilde{\phi}_\beta(\mathbf{k}_3)\tilde{\phi}_\beta(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \end{aligned} \tag{32}$$

From this expression we immediately read off the bare propagator

$$\left\langle \tilde{\phi}_\alpha(\mathbf{k})\tilde{\phi}_\beta(\mathbf{k}')\right\rangle = \frac{1}{k^2+r_0}\,\delta(\mathbf{k}+\mathbf{k}')\delta_{\alpha,\beta}$$

and the interaction vertex

$$-\frac{u}{24N}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}.$$

By convention, the four legs carry vector indices α_i and wave vectors $\mathbf{k}_i$ with $i=1,2,3,4$. By convention, the latter are counted positively (resp. negatively) at a vertex when their orientation is incoming (resp. outgoing). The summations $\sum_{\alpha,\beta=1}^{N}$ in (32) teach us that $\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ equals 1 if the indices are equal in pairs (when $\alpha\neq\beta$) and also 1 if all four indices are equals (when $\alpha=\beta$ in the sums). Otherwise $\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is zero. This can be written explicitly (albeit in a cumbersome manner):

$$\chi^{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4}+\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}+\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}-2\delta_{\alpha_1,\alpha_2}\delta_{\alpha_1,\alpha_3}\delta_{\alpha_1,\alpha_4}.$$

Let us know examine a diagram that contributes to $G(\mathbf{k})$; see Figure 6. To leading order in N we can suppose that $\alpha\neq\beta$. We then need to contract

$$\langle \phi_\mathrm{l}\cdot\phi_\alpha\phi_\alpha\phi_\beta\phi_\beta\cdot\phi_\mathrm{r}\rangle,$$

where ϕ_g and ϕ_d are the external legs on the left and the right of the diagram. One should first choose whether to contract the external legs with the two α or with the two β indices (yielding a factor of 2) and next which one of the indices connects to ϕ_l (giving another factor of 2). The sum over β produces a factor of N. The diagram therefore equals

$$-\frac{u}{24N}\times 4N\times\int \mathrm{d}^3q\,\frac{1}{q^2+r_0}\left(\frac{1}{k^2+r_0}\right)^2+\mathcal{O}(1/N)$$

which is finite in the $N\to\infty$ limit. More generally, we see that to stay of order 1, one is obliged to compensate all the factors of $1/N$ coming from the vertices by the same number

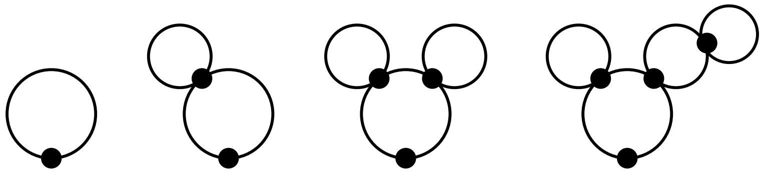

Figure 7: Diagrams contributing to the selfenergy in the limit $N \to \infty$.

of factors of N comming from the summation over the loops. The contributing diagrams therefore have an equal number of vertices and *independent* loops. Put otherwise, each diagram which is not a tadpole of a tadpole vanishes in the limit, and all finite diagrams have the "cactus" from shown in Figure 4.
Example: Consider the 2nd diagram in Figure 4, but modified so that the two loops touch one another in a point. There are then *three* independent loops for the wave vectors, but the conservation of the $O(N)$ indices at the vertices leaves only two independent loops for the vector indices. This modified diagram is thus of order $1/N$.

(ii) Let us now isolate the one-particle irreducible (1PI) contributions to $G(\mathbf{k})$. The corresponding diagrams are those contributing to the selfenergy $\Sigma(\mathbf{k})$ in the limit $N \to \infty$; they are shown in Figure 7. Denoting by $G_0 = 1/(k^2 + r_0)$ the bare propagator, we have the identity

$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \cdots = G_0 \left(\frac{1}{1 - \Sigma G_0} \right),$$

or

$$\frac{1}{G} = \frac{1}{G_0} - \Sigma.$$

Notice that since the entire "current" flows between the external legs, we have necessarily $\Sigma(\mathbf{k}) = \Sigma(\mathbf{0})$.
To produce the diagrams contributing to Σ, it suffices to take those of G, bend down the two external legs and connect them in a point. Since this operation adds one more vertex, the contribution must be multiplied by $-\frac{u}{24N} \times 4N$ (where the factor $4N$ occurs for the same combinatorial reason as explained above). Therefore,

$$\Sigma(\mathbf{0}) = -\frac{u}{6} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G(\mathbf{q}),$$

and we have the desired identity:

$$\frac{1}{G(\mathbf{k})} = k^2 + r_0 + \frac{u}{6} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G(\mathbf{q}).$$

(iii) One can use this identity to study the renormalisation of the mass (squared) away from its bare value r. To this end, write the dressed propagator $G(\mathbf{k}) = \frac{1}{k^2 + r}$ with

$$r = r_0 + \frac{u}{6} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{q^2 + r}. \tag{33}$$

Now to go spherical coordinates and impose an ultraviolet cut-off Λ so that

$$r_0 - r = -\frac{u}{6} \int_0^\Lambda \frac{\mathrm{d}q}{(2\pi)^3} \frac{4\pi q^2}{q^2 + r}. \tag{34}$$

The critical temperature is related with the value of the bare mass for which the renormalised mass vanishes: $r(r_0) = 0$. We have then

$$r_\mathrm{c} = -\frac{u}{6}\int_0^\Lambda \mathrm{d}q \frac{1}{2\pi^2} = -\frac{u\Lambda}{12\pi^2}$$

which depends explicitly on the UV cut-off. It is not a problem that r_c diverges for $\Lambda \to \infty$. In fact, we need the finite cut-off in order to control precisely what it means to be in the vicinity of the critical point. Let us subtract and add r_c:

$$r = r_0 - r_\mathrm{c} + \frac{u}{12\pi^2}\int_0^\Lambda \mathrm{d}q \left(\frac{q^2}{q^2+r} - 1\right). \tag{35}$$

A change of variables, $\tilde{q} = q/\sqrt{r}$, then produces

$$\mathrm{d}q\left(\frac{q^2}{q^2+r} - 1\right) = -\sqrt{r}\frac{\mathrm{d}\tilde{q}}{\tilde{q}^2+1},$$

and we can now take the $\Lambda \to \infty$ limit in the integral:

$$\int_0^\Lambda \mathrm{d}q\left(\frac{q^2}{q^2+r} - 1\right) = -\sqrt{r}\,[\arctan\tilde{q}]_0^{\Lambda/\sqrt{r}} \to -\sqrt{r}\frac{\pi}{2}.$$

Obviously this computation is only correct to order $\sqrt{r}$, so to remain consistent we can set to zero the square of the small quantity, i.e., set $r = 0$ dans (35):

$$0 \simeq (r_0 - r_\mathrm{c}) - \frac{u}{24\pi}\sqrt{r} + \mathcal{O}(r).$$

Finally, we define the reduced temperature as $\tau = r_0 - r_\mathrm{c}$. Since r plays the role of the mass squared, we can set $r \sim 1/\xi^2$. Thus,

$$\tau \sim \frac{u}{24\pi}\sqrt{r} \sim \frac{u}{24\pi}\frac{1}{\xi},$$

whence $\xi \sim \tau^{-1}$, and we have thus identified the critical exponent

$$\nu^\infty = 1.$$

We recall that the mean-field result, valid for $D \to \infty$, is $\nu_\mathrm{MF} = \frac{1}{2}$. So we have already found something non-trivial.
At r_c the renormalised mass vanishes, so we have $G_\infty(\mathbf{q}) = 1/q^2$, to be compared with the general form of the propagator $G(\mathbf{q}) \sim 1/q^{2-\eta}$. We have thus

$$\eta^\infty = 0.$$

This looks a bit trivial, and in particular on may well wonder how it would be possible to obtain a non-zero value for η; the answer to this question will appear at the end of the exercise.

(iv) We can get rid of the quartic term at the price of introducing a new scalar field $\sigma(\mathbf{x})$:

$$\exp\left[-\int \mathrm{d}^3x \frac{u}{24N}\left(\vec{\phi}^2\right)^2\right] = c\int \mathcal{D}\sigma \exp\left[-\int \mathrm{d}^3x\left\{\frac{\sigma^2}{2} - i\sqrt{\frac{u}{12N}}\sigma\vec{\phi}^2\right\}\right],$$

as can be easily seen by rewriting

$$\frac{\sigma^2}{2} - i\sqrt{\frac{u}{12N}}\sigma\vec{\phi}^2 = \frac{1}{2}\left(\sigma - i\sqrt{\frac{u}{12N}}\vec{\phi}^2\right)^2 + \frac{u}{24N}\left(\vec{\phi}^2\right)^2.$$

(The value of the constant c is not important.) The partition function then becomes

$$Z = c\int \mathcal{D}\sigma \exp\left[-\int \mathrm{d}^3x \frac{\sigma^2}{2}\right]\int \mathcal{D}\vec{\phi}\exp\left[-\frac{1}{2}\int \mathrm{d}^3x\left\{\left(\nabla\vec{\phi}\right)^2 + \left(r_0 + i\sqrt{\frac{u}{3N}}\sigma\right)\vec{\phi}^2\right\}\right],$$

(v) We can now perform the integral over $\mathcal{D}\vec{\phi}$ with result

$$\left(\det\left[\delta(\mathbf{x}-\mathbf{y})\left(-\nabla^2_{\mathbf{x}} + r_0 + i\sqrt{\frac{u}{3N}}\sigma(\mathbf{x})\right)\right]\right)^{-N/2}.$$

Change the scale: $\tilde{\sigma}(\mathbf{x}) = N^{-1/2}\sigma(\mathbf{x})$, to find

$$Z = \int \mathcal{D}\tilde{\sigma}(\mathbf{x})\mathrm{e}^{-\frac{N}{2}S[\tilde{\sigma}]}$$

where the action reads

$$S[\tilde{\sigma}] = \int \mathrm{d}^3x\tilde{\sigma}(\mathbf{x})^2 + \log\det\left[\delta(\mathbf{x}-\mathbf{y})\left(-\nabla^2_{\mathbf{x}} + r_0 + i\sqrt{\frac{u}{3}}\sigma(\mathbf{x})\right)\right]. \qquad (36)$$

The fact that S is multiplied by a global factor of N shows that the saddle point method is *exact* in the limit $N \to \infty$.

(vi) Let us look for a saddle point of the form

$$\tilde{\sigma}(x) = \tilde{\sigma}_{\mathrm{c}} + \epsilon(x),$$

where $\epsilon \ll \tilde{\sigma}$ represents the small fluctuations around the classical solution. The classical action for a system of finite volume V is then

$$S[\tilde{\sigma}_{\mathrm{c}}] = V\cdot\tilde{\sigma}_{\mathrm{c}}^2 + \log\det\left[\delta(\mathbf{x}-\mathbf{y})\left(-\nabla^2_{\mathbf{x}} + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\mathrm{c}}\right)\right].$$

The last term can be rewritten as

$$\mathrm{Tr}\log\left[\delta(\mathbf{k}-\tilde{\mathbf{k}})\left(\mathbf{k}^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\mathrm{c}}\right)\right] = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\log\left(\mathbf{k}^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\mathrm{c}}\right)\langle\mathbf{k}|\mathbf{k}\rangle$$

where we have, in Fourier space,

$$\langle\mathbf{k}|\mathbf{k}\rangle = \delta(\mathbf{0}) = \int \mathrm{d}\mathbf{x}\mathrm{e}^{i\mathbf{x}\cdot\mathbf{0}} = V.$$

The result for the classical is therefore

$$S[\tilde{\sigma}_{\mathrm{c}}] = V\left[\tilde{\sigma}_{\mathrm{c}}^2 + \int\frac{\mathrm{d}^3k}{(2\pi)^3}\log\left(\mathbf{k}^2 + r_0 + i\sqrt{\frac{u}{3}}\tilde{\sigma}_{\mathrm{c}}\right)\right].$$

The corresponding equation of motion reads $\partial S/\partial\tilde{\sigma}=0$, or

$$2\tilde{\sigma}+\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2+r_0+i\sqrt{\frac{u}{3}}\tilde{\sigma}}\cdot i\sqrt{\frac{u}{3}}=0.$$

To recover the form previously encountered, we must set $r=r_0+i\sqrt{\frac{u}{3}}\tilde{\sigma}$. Multiplying by $\frac{i}{2}\sqrt{\frac{u}{3}}$ we then obtain

$$i\sqrt{\frac{u}{3}}\tilde{\sigma}+\frac{1}{2}\left(i\sqrt{\frac{u}{3}}\right)^2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2+r}=0,$$

or

$$(r-r_0)-\frac{u}{6}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2+r}=0,$$

which coincides with relation (33).

(vii) This is a rather technical question, which however shows a few neat tricks. Let us omit the tildes on $\tilde{\sigma}$ for notational convenience. We compute the Hessian ("matrix" of second derivatives) of the action (36):

$$\frac{\delta^2S[\sigma]}{\delta\sigma(x)\delta\sigma(y)}=2\delta(x-y)+\frac{\delta^2}{\delta\sigma(x)\delta\sigma(y)}\log\det M(x,y),$$

where we have defined the "matrix" (with continuous indices)

$$M(x,y)=\delta(x-y)\left(-\nabla_x^2+r_0+i\sqrt{\frac{u}{3}}\sigma(x)\right).$$

To compute the second term, we first remark that

$$\begin{aligned}\frac{\delta\log\det M}{\delta M(z_1,z_2)} &= \frac{1}{\det M}\cdot\frac{\delta\det M}{\delta M(z_1,z_2)}\\ &= \frac{1}{\det M}\times[\text{co-facteur de }M](z_1,z_2)\\ &= M^{-1}(z_1,z_2),\end{aligned}$$

and that

$$\begin{aligned}\frac{\delta M(z_1,z_2)}{\delta\sigma(x)} &= i\sqrt{\frac{u}{3}}\delta(z_1-z_2)\frac{\delta\sigma(z_1)}{\delta\sigma(x)}\\ &= i\sqrt{\frac{u}{3}}\delta(z_1-z_2)\delta(z_1-x)\\ &= i\sqrt{\frac{u}{3}}\delta(z_1-x)\delta(z_2-x).\end{aligned}$$

By generalising the chain rule of differentiation to continuous indices, we thus have

$$\frac{\delta\log\det M}{\delta\sigma(x)} = \int\mathrm{d}z_1\mathrm{d}z_2\frac{\delta\log\det M}{\delta M(z_1,z_2)}\cdot\frac{\delta M(z_1,z_2)}{\delta\sigma(x)} \quad (37)$$

$$= \int\mathrm{d}z_1\mathrm{d}z_2M^{-1}(z_1,z_2)i\sqrt{\frac{u}{3}}\delta(z_1-x)\delta(z_2-x) \quad (38)$$

$$= i\sqrt{\frac{u}{3}}M^{-1}(x,x).$$

To derive one more term, we need the lemma

$$\frac{\delta M^{-1}(x_1,x_2)}{\delta M(z_1,z_2)} = -M^{-1}(x_1,z_1)M^{-1}(z_2,x_2).) \qquad (39)$$

This can be proven by deriving the trivial identity

$$\int \mathrm{d}x_3 M^{-1}(x_1,x_3)M(x_3,x_4) = \delta(x_1,x_4)$$

with respect to $M(z_1,z_2)$. We find

$$0 = \int \mathrm{d}x_3 \left(\frac{\partial M^{-1}(x_1,x_3)}{\partial M(z_1,z_2)} M(x_3,x_4) + M^{-1}(x_1,x_3)\delta(x_3,z_1)\delta(x_4,z_2) \right).$$

One next multiplies by $\int \mathrm{d}x_4 M^{-1}(x_4,x_2)$:

$$\int \mathrm{d}x_3 \mathrm{d}x_4 \frac{\partial M^{-1}(x_1,x_3)}{\partial M(z_1,z_2)} M(x_3,x_4)M^{-1}(x_4,x_2) = \\ -\int \mathrm{d}x_3\mathrm{d}x_4 M^{-1}(x_1,x_3)M^{-1}(x_4,x_2)\delta(x_3,z_1)\delta(x_4,z_2).$$

The left-hand side becomes

$$\int \mathrm{d}x_3 \frac{\partial M^{-1}(x_1,x_3)}{\partial M(z_1,z_2)}\delta(x_3,x_2) = \frac{\partial M^{-1}(x_1,x_2)}{\partial M(z_1,z_2)},$$

whereas the right-hand side can be written $-M^{-1}(x_1,z_1)M^{-1}(z_2,x_2)$. This concludes the proof of the lemma (39).
Using this, we have finally

$$\begin{aligned}
\frac{\delta^2 \log\det M}{\delta\sigma(x)\delta\sigma(y)} &= i\sqrt{\frac{u}{3}} \int \mathrm{d}z_1\mathrm{d}z_2 \frac{\delta M^{-1}(x,x)}{\delta M(z_1,z_2)} \cdot \frac{\delta M(z_1,z_2)}{\delta\sigma(y)} \\
&= i\sqrt{\frac{u}{3}} \int \mathrm{d}z_1\mathrm{d}z_2 \left[-M^{-1}(x,z_1)M^{-1}(x,z_2)\right] \cdot \left[i\sqrt{\frac{u}{3}}\delta(z_1-y)\delta(z_2-y)\right] \\
&= \frac{u}{3}\left[M^{-1}(x,y)\right]^2.
\end{aligned}$$

At the saddle point, this equals

$$\frac{u}{3}\left(\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{e}^{ik(x-y)}}{k^2+r_0+i\sqrt{\frac{u}{3}}\sigma_{\mathrm{c}}} \right)^2 = \\ \frac{u}{3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{e}^{ik(x-y)}}{\left(q^2+r_0+i\sqrt{\frac{u}{3}}\sigma_{\mathrm{c}}\right)\left((\mathbf{k}-\mathbf{q})^2+r_0+i\sqrt{\frac{u}{3}}\right)},$$

where we have named the integration variables $\mathbf{q}$ and $\mathbf{k}-\mathbf{q}$, respectively.
The conclude, the Hessian can be diagonalised by Fourier transform, and its eigenvalues are

$$\lambda_k = 2 + \frac{u}{3}\int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{\left(q^2+r_0+i\sqrt{\frac{u}{3}}\sigma_{\mathrm{c}}\right)\left((\mathbf{k}-\mathbf{q})^2+r_0+i\sqrt{\frac{u}{3}}\right)}.$$

One can easily convince oneself that λ_k is well-defined for all $r_0>0$, and that the real part is always positive. It follows that the fluctuations around the saddle point are bounded, as required.

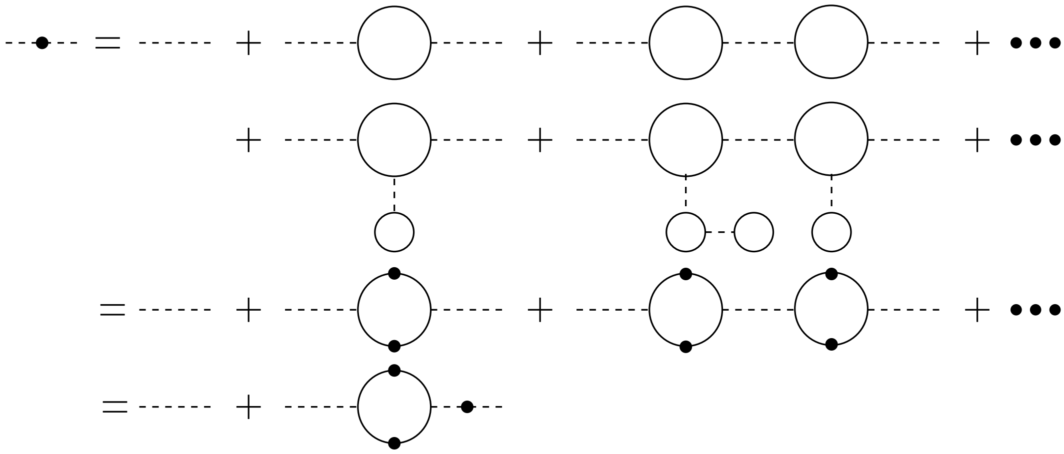

Figure 8: Derivation of the implicit relation satisfied by the propagators.

(viii) We use the action found in (v) to read off the diagrammatic rules. The bare propagator of the scalar field σ is simply

$$\langle\tilde{\sigma}(\mathbf{k})\tilde{\sigma}(-\mathbf{k}\rangle = 1,$$

and that of the vector field ϕ is

$$\langle\tilde{\phi}_\alpha(\mathbf{k})\tilde{\phi}_\beta(-\mathbf{k})\rangle = \frac{\delta_{\alpha,\beta}}{k^2 + r_0}$$

where we note the usual conservation of $\mathrm{O}(N)$ indices.
Finally, there is a trivalent interaction vertex $\sigma\phi_\alpha\phi_\beta$ with value

$$-i\sqrt{\frac{u}{12N}}\delta_{\alpha,\beta}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)(2\pi)^3.$$

We stress that the decomposition of a tetravalent interaction into a pair of trivalent interactions, by means of an auxiliary field (here σ), is a very common and useful trick in field theory.

(ix) The equation satisfied by the propagators in the limit $N \to \infty$ is shown graphically in Figure 8. Each trivalent vertex contributes a factor of $1/\sqrt{N}$, and each independent loop gives a factor of N. In the limit $N \to \infty$ we thus need exactly twice more vertices than loops to get a non-zero result, so only tadpoles of tadpoles survive (first line of Fig. 8). The insertion of ϕ-loops into other ϕ lines amounts to renormalising the propagator of ϕ (second line). And finally, we dress the propagator of σ in order to count all the dressed tadpoles in one fell swoop (third line).
In algebraic terms, Fig. 8 reads

$$G_\sigma^\infty(\mathbf{k}) = 1 + 1\cdot\int\frac{\mathrm{d}^3q}{(2\pi)^3}G_\phi^\infty(-\mathbf{q})G_\phi^\infty(-\mathbf{k}+\mathbf{q})G_\sigma^\infty(\mathbf{k})\cdot\left(-i\sqrt{\frac{u}{12N}}\right)^2\cdot 2N,$$

where the factor of $2N$ comes from the two ways of contracting the ϕ in $\langle\sigma\phi\phi|\phi\phi\sigma\rangle$, and from the sum over the vector index in the loop. We can then isolate $G_\sigma^\infty(\mathbf{k})$:

$$G_\sigma^\infty(\mathbf{k}) = \frac{1}{1+\frac{u}{6}\int\frac{\mathrm{d}^3q}{(2\pi)^3}G_\phi^\infty(\mathbf{k}-\mathbf{q})G_\phi^\infty(\mathbf{q})}. \tag{40}$$

At the critical point, $G_\phi^\infty(\mathbf{q}) = 1/\mathbf{q}^2$ and one can compute the integral. Let us first set

$$\begin{aligned}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} G_\phi^\infty(\mathbf{k}-\mathbf{q}) G_\phi^\infty(\mathbf{q}) &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{q})^2} \cdot \frac{1}{\mathbf{q}^2} \\ &= \frac{1}{|\mathbf{k}|} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(\mathbf{1}-\mathbf{q})^2} \cdot \frac{1}{\mathbf{q}^2} \equiv \frac{1}{|\mathbf{k}|} \cdot I,\end{aligned}$$

where $\mathbf{1}$ is an arbitrary unit vector (we integrate over the orientations of $\mathbf{q}$), and the factor $\frac{1}{|\mathbf{k}|}$ can be found by dimensional analysis.

(x) We now compute the integral I. Let θ be the angle between $\mathbf{1}$ and $\mathbf{q}$. Going to polar coordinates,

$$I = \int_0^\infty \frac{\mathrm{d}q}{(2\pi)^3} \frac{q^2}{q^2} \int_0^\pi \mathrm{d}\theta 2\pi \sin\theta \frac{1}{1+q^2-2q\cos\theta},$$

and with the change of variables $x = -\cos\theta$:

$$\begin{aligned} I &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}q \int_{-1}^1 \mathrm{d}x \frac{1}{1+q^2-2qx} \\ &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}q \frac{1}{2q} \left\{ \log(1+q^2+2q) - \log(1+q^2-2q) \right\} \\ &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}q \frac{1}{q} \log\left|\frac{1+q}{1-q}\right|. \end{aligned}$$

One easily sees that $\int_1^\infty = \int_0^1$ by making the change of variables $q = 1/u$. Therefore,

$$I = \frac{1}{2\pi^2} \int_0^1 \mathrm{d}q \frac{1}{q} \log\left(\frac{1+q}{1-q}\right),$$

and we can expand

$$\log\left(\frac{1+q}{1-q}\right) = \sum_{n=0}^\infty \left[\left(1 + \frac{q^n}{n}\right) - \left(1 + \frac{(-q)^n}{n}\right) \right] = 2 \sum_{p=0}^\infty \frac{1}{2p+1} q^{2p+1}.$$

Integrating term by term leads to

$$I = \frac{1}{2\pi^2} \cdot 2 \sum_{p=0}^\infty \frac{1}{(2p+1)^2},$$

and since $\sum_{n=1,3,5,\ldots} = \sum_{n=0}^\infty - \sum_{n=0,2,4,\ldots}$,

$$\sum_{p=0}^\infty \frac{1}{(2p+1)^2} = \sum_{n=0}^\infty \left(\frac{1}{n^2} - \frac{1}{(2n)^2}\right) = \left(1 - \frac{1}{4}\right) \sum_{n=0}^\infty \frac{1}{n^2} = \frac{3}{4} \cdot \frac{\pi^2}{6}.$$

Finally,

$$I = \frac{1}{2\pi^2} \cdot 2 \cdot \frac{\pi^2}{8} = \frac{1}{8}. \tag{41}$$

(xi) Combining these pieces, it follows that (40) becomes, in the limit $k \to 0$,

$$G_\sigma^\infty(\mathbf{k}) \simeq \frac{48}{u} |\mathbf{k}|. \tag{42}$$

(xii) Let us return to (1.6):

$$\frac{1}{G_\phi^N(\mathbf{k})} = k^2 + r_0 - \Sigma_\phi^N(\mathbf{k}). \tag{43}$$

The first term in Fig. 5 is the contribution to the self-energy which remains finite when $N \to \infty$. This contribution has already been accounted for—more precisely in Fig. 4—but it reappears here in the formalism using trivalent vertices. Indeed, if we provide external legs of the ϕ type, the first term yields precisely the diagrams in Fig. 7). The second term, however, is of order $1/N$, since there is no ϕ-loop to provide the compensating factor of N. On can verify that no other diagrams contribute at this order.

(xiii) We therefore set out to compute the new diagram. There is a combinatorial factor due to the contraction $\langle \phi_\mathrm{l} \cdot \phi^1\phi^1\sigma | \sigma\phi^2\phi^2 \cdot \phi_\mathrm{r}\rangle$:

- A factor $\frac{1}{2}$ coming from the expansion of the exponential (there are two vertices).
- A factor 2 due to the choice of contracting ϕ_l with either a ϕ^1 or a ϕ^2.
- A factor 2^2, since we must decide which one of the ϕ^1 (or ϕ^2) we wish to contract with an external leg.

Donc, au total un facteur de 4.
The diagram, expressed as a function of the wave number $\mathbf{k}$ flowing between external legs, then leads to the integral

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} G_\sigma^\infty(\mathbf{q}) G_\phi^\infty(\mathbf{k}-\mathbf{q}) \cdot 4 \left(i\sqrt{\frac{u}{12N}} \right)^2 ,$$

which is seen to have an explicit factor of $1/N$. It is therefore consistent to evaluate the propagators for $N \to \infty$. (This is a usual phenomenon in perturbation theory, seen also in ϕ^4 theory for example: some quantities are lagging one order behind others, in order to define a consistent perturbative framework.)
We are not interested in this integral *per se*, but only in its dependence on $\mathbf{k}$, so it is permissible to subtract the same equation with $\mathbf{k} = \mathbf{0}$:

$$\Delta\Sigma_\phi^N(\mathbf{k}) = 4 \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G_\sigma^\infty(\mathbf{q}) \left[G_\phi^\infty(\mathbf{k}-\mathbf{q}) - G_\phi^\infty(-\mathbf{q}) \right] \left(i\sqrt{\frac{u}{12N}} \right)^2 ,$$

where $[\cdots]$ equals $\frac{1}{(\mathbf{k}-\mathbf{q})^2} - \frac{1}{\mathbf{q}^2}$ at the critical point.
The behaviour of $\Delta\Sigma$ for small $|\mathbf{k}|$ (i.e., at large distances in real space) is determined by the small values of $\mathbf{q}$ in the integral over $\mathbf{q}$. When using (42), we thus suppose that $G_\sigma^\infty(\mathbf{q}) \simeq \frac{48}{u}|\mathbf{q}|$ when $q < \Lambda$, where Λ denotes an appropriate UV cut-off. In the limit $\mathbf{k} \to 0$, we therefore have

$$\begin{aligned}
\Delta\Sigma_\phi^N(\mathbf{k}) &\simeq 4 \int_{q<\Lambda} \frac{\mathrm{d}^3 q}{(2\pi)^3} |\mathbf{q}| \left[(\mathbf{k}-\mathbf{q})^{-2} - \mathbf{q}^{-2} \right] \frac{48}{u} \cdot \frac{-u}{12N} \\
&= -\frac{2}{\pi^3 N} \int_{q<\Lambda} \frac{\mathrm{d}^3 q}{(2\pi)^3} |\mathbf{q}| \left[(\mathbf{k}-\mathbf{q})^{-2} - \mathbf{q}^{-2} \right]
\end{aligned}$$

We still need to compute the integral. Unlike the previous case, we do not need an explicit evaluation, but only the leading-order asymptotics for small $\mathbf{k}$. We go to polar coordinates:

$$I(k) = 2\pi \int_0^\Lambda \mathrm{d}q q^3 \int_0^\pi \mathrm{d}\theta \sin\theta \left[\frac{1}{k^2 + q^2 - 2kq\cos\theta} - \frac{1}{q^2} \right] .$$

We first perform the integral $\int \mathrm{d}\theta$ by substituting $x = -\cos\theta$:

$$\int_{-1}^{1} \mathrm{d}x \left[\frac{1}{k^2 + q^2 + 2kqx} - \frac{1}{q^2} \right] = \frac{1}{kq} \log \left| \frac{k+q}{k-q} \right| - \frac{2}{q^2},$$

and next set $y = q/k$ to arrive at

$$I(k) = 2\pi k^2 \int_0^{\Lambda/k} \mathrm{d}y\, y^2 \left[\log \left| \frac{1+y}{1-y} \right| - \frac{2}{y} \right].$$

For $k \ll 1$, we have $y \gg 1$, and the first non-zero order in $[\cdots]$ yields $\simeq \frac{2}{3y^3}$. Notice that there is no infrared divergence, as can be seen *before* doing the series expansion. Thus,

$$\begin{aligned} I(k) &\simeq 2\pi k^2 \cdot \frac{2}{3} \log \left| \frac{\Lambda}{k} \right| \\ &\simeq -\frac{4\pi}{3} k^2 \log k + \cdots . \end{aligned}$$

There is a small subtlety here: At the critical point, we are interested in large distances, hence small k (IR limit). However, the change of variables means that we are interested in large Λ/k, so we should study the UV limit of the one-loop integral. The final result is therefore:

$$\Delta\Sigma_\phi^N(k) \simeq \frac{8}{3\pi^2 N} k^2 \log k.$$

(xiv) At the critical point

$$G(k) \sim \frac{1}{k^{2-\eta}} \simeq \frac{1}{k^2(1 - \eta \log k + \cdots)},$$

and from (43) we see that

$$\Sigma(k) = \eta k^2 \log k + \cdots .$$

The above result therefore implies that

$$\eta = \frac{8}{3\pi^2 N} + \mathcal{O}\left(\frac{1}{N^2} \right).$$

The subject is treated in chapter 26 of the book by Zinn-Justin, but omitting all the technical details of the computation (see also the book by Parisi). It is mentioned there that one can push the computation to higher order:

$$\begin{aligned} \eta &= \frac{\eta_1}{N} + \frac{\eta_2}{N^2} + \frac{\eta_3}{N^3} + \ldots \\ \eta_1 &= \frac{8}{3\pi^2} \simeq 0.27019 \\ \eta_2 &= -\frac{8}{3}\eta_1^2 \simeq -0.19467 \\ \eta_3 &= \eta_1^3 \left[-\frac{797}{84} - \frac{61}{24}\pi^2 + \frac{27}{8}\psi''\left(\frac{1}{2}\right) + \frac{9}{2}\pi^2 \ln 2 \right] \simeq -1.19502. \end{aligned}$$

Here $\psi(x) = \frac{\mathrm{d} \log \Gamma(x)}{\mathrm{d}x}$ denotes the digamma function.
To study the 3D Ising model one could naively set $N = 1$ in these results. A high-temperature series expansion gives $\eta = 0.055 \pm 0.014$. This seems to compare rather favourably with the change from the first-order result $\eta_1 = 0.27$ to the second-order one $\eta_1 + \eta_2 = 0.076$. Naive hopes are quickly quenched when observing that at third order $\eta_1 + \eta_2 + \eta_3 = -1.12$. The resolution of this situation is that the perturbative series is only asymptotic and should in fact be handled with resommation techniques (more details in the second semester optional course given by Kay Wiese).

1.7 Chapter 8: Conformal field theory: basics

- *Exercise 8.1: Conformal mappings in 2D.*

(i) Verify that the map $z \to w = \frac{z-i}{z+i}$ is holomorphic map from the upper half plane $\mathbb{H} = \{z \in \mathbb{C},\ \mathrm{Im} z > 0\}$ to the unit disc $\mathbb{D} = \{w \in \mathbb{C},\ |w| < 1\}$ centred at the origin 0.

(ii) Similarly verify that the map $w \to z = e^{w/\beta}$ is a holomorphic map from the cylinder with radius β to the complex z-plane with the origin and the point at infinity removed.

Correction :

(i) Clearly, $w(z) = \frac{z-i}{z+i}$ is holomorphic in the upper half plane. For $z \in \mathbb{R}$ on the real line, w is on the unit circle centred at the origin, $|w| = 1$. For $z = i$, $w(z) = 0$. Then, by continuity, $z \to w(z)$ maps the upper half plane to the unit disc centred at the origin. A fact which can also be checked directly.

(ii) The cylinder of radius β can be parametrized by point $w = \beta(r + i\theta)$ with $r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$ modulo 2π. Then the point $z(w) = e^r\, e^{i\theta}$ covers the complex plane with the point at the origin and at infinity removed (they correspond to $r = \pm\infty$).

- *Exercise 8.2: The group of conformal transformations.*

The aim of this exercise is to fill the missing steps in determining all infinitesimal conformal transformations in the flat Euclidean space $\mathbb{R}^D$.

Let us recall a few basic facts from the lectures. A diffeomorphism $x \to y$ is called conformal if it changes the metric by a space-dependent factor:

$$\hat{g}_{\mu\nu}(x) = \left(\frac{\partial y^\sigma}{\partial x^\nu}\right)\left(\frac{\partial y^\rho}{\partial x^\nu}\right) g_{\sigma\rho}(y(x)) := \mathrm{e}^{2\phi(x)} g_{\mu\nu}(x)\,. \tag{44}$$

Here ϕ is called the conformal factor. Now apply this to an infinitesimal transformation $x^\mu \to x^\mu + \epsilon\xi^\mu(x) + \cdots$, where $\xi^\mu(x)$ is the vector field generating conformal transformations. By developing the left- and right-hand sides of (44) to first order in the small parameter ϵ and comparing we obtain

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2(\delta\phi)\delta_{\mu\nu}\,. \tag{45}$$

Another useful relation is obtained by multiplying this by $\delta^{\nu\mu}$ on both sides. We then get $\partial^\nu \xi_\nu + \partial^\mu \xi_\mu = 2(\delta\phi)\delta^\nu_\nu = 2D(\delta\phi)$, or in other words

$$D(\delta\phi) = (\partial_\mu \xi^\mu)\,. \tag{46}$$

We denote $\partial \cdot \xi := \partial_\mu \xi^\mu$ and the Euclidean Laplacian $\Delta := \partial^\mu \partial_\mu$, with the summation convention throughout.

(i) Take derivatives of the previous equation to deduce that $D\,\Delta\xi_\nu = (2-D)\partial_\nu(\partial\cdot\xi)$, with Δ the Euclidean Laplacian.

(ii) Take further derivatives, either w.r.t ∂_ν or w.r.t. ∂_μ, to get two new equations: $(D-1)\Delta(\partial\cdot\xi)=0$, and $2(2-D)\,\partial_\mu\partial_\nu(\partial\cdot\xi) = D\,\Delta(\partial_\mu\xi_\nu+\partial_\nu\xi_\mu)$.

(iii) Deduce that $(2-D)\partial_\mu\partial_\nu(\partial\cdot\xi)=0$, and hence that, in dimension $D>2$, the conformal factor $\delta\varphi(x)$ is linear in x.

Let us write $\delta\varphi(x) = k + b_\nu x^\nu$ with k and b_ν integration constantes. We thus have

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 2(k+b_\sigma x^\sigma)\,\delta_{\mu\nu}.$$

A way to determine ξ consists in getting information on the difference $\partial_\mu\xi_\nu - \partial_\nu\xi_\mu$.

(iv) By taking derivates of the previous equation w.r.t ∂_σ and permuting the indices, deduce that $\partial_\nu(\partial_\sigma\xi_\mu - \partial_\mu\xi_\sigma) = 2(b_\sigma\delta_{\mu\nu} - b_\mu\delta_{\nu\sigma})$, and hence, by integration, that

$$\partial_\sigma\xi_\mu - \partial_\mu\xi_\sigma = 2(b_\sigma x_\mu - b_\mu x_\sigma) + 2\theta_{\mu\sigma},$$

where $\theta_{\sigma\mu} = -\theta_{\mu\sigma}$ are new integration constants.

(v) Integrate the last equations to prove that

$$\xi_\nu(x) = a_\nu + kx_\nu + \theta_{\nu\sigma}x^\sigma + [(b\cdot x)x_\nu - \frac{1}{2}(x\cdot x)b_\nu],$$

where a_ν are new, but last, integration constants.

(vi) Find the explicit formula for all finite –not infinitesimal– conformal transformations in dimension D.
Hint: It is advantageous to consider (a) the flow generated by the above vector fields $\xi(x)$, i.e. to consider the one parameter family of transformations $x\to y_t$ such that $\partial_t y_t = \xi(y_t)$ with initial condition $y_{t=0}=x$, and (b) to change coordinate to $Y_t := \frac{y_t}{(y_t\cdot y_t)}$.

(vii) *Optional:* Verify that the Lie algebra of the group of conformal transformation in dimension D is isomorphic to $so(D+1,1)$.

Correction :

(i) Apply first $\partial^\mu(\cdots)$ to (45) to obtain

$$\Delta\xi_\nu + \partial_\nu(\partial\cdot\xi) = 2\partial_\nu(\delta\phi) = \frac{2}{D}\partial_\nu(\partial\cdot\xi)\,.$$

Multiplying by D and rearranging this we get

$$D\Delta\xi_\nu = (2-D)\partial_\nu(\partial\cdot\xi)\,. \tag{47}$$

(ii) Now take $\partial^\nu(\cdots)$ of (47) to obtain $D\Delta(\partial\cdot\xi) = (2-D)\Delta(\partial\cdot\xi)$, so that $(D-1)\Delta(\partial\cdot\xi) = 0$. This implies that

$$\Delta(\partial\cdot\xi) = 0 \tag{48}$$

for any $D \neq 1$. Alternatively we take $\partial_\mu(\cdots)$ of (47) to obtain

$$D\Delta(\partial_\mu\xi_\nu) = (2-D)\partial_\mu\partial_\nu(\partial\cdot\xi)\,.$$

Symmetrising this with respect to $\mu \leftrightarrow \nu$ we get

$$D\Delta(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) = 2(2-D)\partial_\mu\partial_\nu(\partial\cdot\xi)\,. \tag{49}$$

(iii) Combining (48) with (46) we get $\Delta(\delta\phi) = 0$ and hence by (45) $\Delta(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) = 0$. Using also (49) we arrive at $(2-D)\partial_\mu\partial_\nu(\partial\cdot\xi) = 0$. This implies that

$$\partial_\mu\partial_\nu(\partial\cdot\xi) = 0$$

for any $D \neq 2$.
This is an important conclusion: when written out in terms of the coordinates, the conformal factor is linear. Recalling (46) we can thus write

$$D^{-1}(\partial\cdot\xi) = (\delta\phi) = k + (b\cdot x)\,,$$

where $(b\cdot x) := b_\nu x^\nu$, for some constants k and b_ν. Multiply by $2\delta_{\mu\nu}$ and use (45) to get

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 2(k + (b\cdot x))\delta_{\mu\nu}\,. \tag{50}$$

We see in particular from (45) that the infinitesimal dilatation factor is

$$(\delta\phi) = k + (b\cdot x)\,.$$

(iv) We thus have (50) for the sum of $\partial_\mu\xi_\nu$ and $\partial_\nu\xi_\mu$. We now seem some information on the corresponding difference, so that we can eventually isolate ξ. To that end, take $\partial_\sigma(\cdots)$ of (50) to obtain

$$\partial_\mu\partial_\sigma\xi_\nu + \partial_\nu\partial_\sigma\xi_\mu = 2b_\sigma\delta_{\mu\nu}\,.$$

Subtract from this the same equation in which we have relabeled $\mu \leftrightarrow \sigma$. Then the first term on the left-hand side drops out and we get

$$\partial_\nu(\partial_\sigma\xi_\mu - \partial_\mu\xi_\sigma) = 2(b_\sigma\delta_{\mu\nu} - b_\mu\delta_{\sigma\nu})\,.$$

Integrating this equation we obtain

$$\partial_\sigma\xi_\mu - \partial_\mu\xi_\sigma = 2(b_\sigma x_\mu - b_\mu x_\sigma) + 2\theta_{\mu\sigma}\,, \tag{51}$$

where $2\theta_{\mu\sigma}$ are some integration constants. Because of the anti-symmetry of the left-hand side we must have $\theta_{\mu\sigma} = -\theta_{\sigma\mu}$.

(v) Add now the sum (50) and the difference (51), and divided by two, to obtain

$$\partial_\mu\xi_\nu = (k + (b\cdot x))\delta_{\mu\nu} + \theta_{\nu\mu} + (b_\mu x_\nu - b_\nu x_\mu)\,.$$

Integrating this we find

$$\xi_\nu = a_\nu + kx_\nu + \theta_{\nu\mu}x^\mu + \left[(b\cdot x)x_\nu - \frac{1}{2}(x\cdot x)b_\nu\right]\,, \tag{52}$$

where a_ν are some other integration constants.
The four terms in ξ_ν have a clear interpretation: they generate respectively translations, dilatations, rotations, and special conformal transformations.
Recall from the lectures that the conserved current is $J_\mu = T_{\mu\nu}\xi^\nu$ with $T_{\mu\nu}$ being the stress tensor. We have the conservation law $\partial^\mu J_\mu = 0$, which implies that $\partial^\mu T_{\mu\nu} = 0$ (the stress tensor is conserved) and $T^\mu_\mu = 0$ (it is traceless).

(vi) After a change of notation ($b \to 2b$) the infinitesimal conformal transformation now takes the form

$$\xi^\mu = a^\mu + kx^\mu + \theta^{\mu\nu}x_\nu + [2(b\cdot x)x^\mu - (x\cdot x)b^\mu] \,. \qquad (53)$$

The parts which are constant or linear in x_ν are responsible for translations, rotations and dilatations. We hence focus on the quadratic part, which corresponds to the infinitesimal *special conformal transformation* (SCT):

$$\xi^\mu_b = 2(b\cdot x)x^\mu - (x\cdot x)b^\mu \,.$$

We now wish to find the corresponding global SCT. To this end we should integrate the flow equation

$$\partial_\epsilon y^\mu_\epsilon = \xi^\mu_b(y_\epsilon)\,, \qquad \text{with } y^\mu_{\epsilon=0} = x^\mu \qquad (54)$$

and set $\epsilon = 1$ in the end. Explicitly, we must integrate

$$\partial_\epsilon y^\mu_\epsilon = 2(b\cdot y_\epsilon)y^\mu_\epsilon - (y_\epsilon\cdot y_\epsilon)b^\mu \,. \qquad (55)$$

The trick is to consider the inversion, defined by

$$Y^\mu = \frac{y^\mu_\epsilon}{(y_\epsilon\cdot y_\epsilon)} \,. \qquad (56)$$

Direct differentiation gives (denote $y = y_\epsilon$ for simplicity)

$$\partial Y^\mu = \frac{\partial y^\mu}{(y\cdot y)} - 2\frac{(y\cdot\partial y)}{(y\cdot y)^2}y^\mu \,. \qquad (57)$$

The second term follows from (55): $(y\cdot\partial y) = 2(b\cdot y)(y\cdot y) - (b\cdot y)(y\cdot y) = (b\cdot y)(y\cdot y)$. Inserting this and (55) into (57) gives

$$\partial Y^\mu = \frac{1}{(y\cdot y)}\left[2(b\cdot y)y^\mu - (y\cdot y)b^\mu - 2(b\cdot y)y^\mu\right] = -b^\mu \,.$$

In this form, the flow equation can be immediately integrated to finite ϵ. The result is

$$\frac{y^\mu_\epsilon}{(y_\epsilon\cdot y_\epsilon)} = \frac{x^\mu}{(x\cdot x)} - \epsilon b^\mu \,,$$

and setting $\epsilon = 1$ we obtain

$$\frac{y^\mu}{(y\cdot y)} = \frac{x^\mu - (x\cdot x)b^\mu}{(x\cdot x)} \,. \qquad (58)$$

To obtain the $(y\cdot y)$ factor, we consider the square of this relation,

$$\frac{1}{(y\cdot y)} = \frac{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}{(x\cdot x)} \,,$$

and inserting this into (58) we get the final form of the global SCT:

$$y^\mu = \frac{x^\mu - (x\cdot x)b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}\,. \tag{59}$$

Remark. Let us consider the composition of an inversion, a translation by b^μ, and another inversion:

$$\frac{\frac{x^\mu}{(x\cdot x)} - b^\mu}{\left(\frac{x^\mu}{(x\cdot x)} - b^\mu\right)^2} = \frac{x^\mu - (x\cdot x)b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}\,.$$

We see that this is equal to the SCT. Therefore it is useful to think of the SCT as an object that adds to the usual transformations (translation, rotation, dilatation) also the inversion.

(vi) For the isomorphisms to $so(D+1,1)$: check Lie algebra relations, see Denis' handwritten notes.

• *Exercise 8.3: The two- and three-point conformal correlation functions.*

The aim of this exercise is to fill the missing steps in determining the two and three point function of conformal fields in conformal field theory. Let $G^{(2)}(x_1,x_2) = \langle \Phi_1(x_1)\Phi_2(x_2)\rangle$ be the two point function of to scalar conformal fields of scaling dimension h_1 and h_2 respectively.

(i) Prove that translation and rotation invariance implies that $G^{(2)}$ is a function of the distance $r = |x_1 - x_2|$ only.

(ii) Prove that dilatation invariance of the 2-point function demands that

$$[h_1 + x_1\cdot\partial_1 + h_2 + x_2\cdot\partial_2]G_2(x_1,x_2) = 0.$$

Deduce that $G_2(x_1,x_2) = \text{const.}\, r^{-(h_1+h_2)}$.

(iii) Prove that invariance under special conformal transformations (also called inversions) implies that

$$\sum_{j=1,2}\,[h_j(b\cdot x_j) + [(b\cdot x_j)x_j^\nu - \frac{1}{2}(x_j\cdot x_j)b^\nu]\partial_{x_j^\nu}]G_2(x_1,x_2) = 0.$$

Deduce that $G^{(2)}(x_1,x_2)$ vanishes unless $h_1 = h_2$.

Let us now look at the three point functions of scalar conformal fields. Let $G^{(3)}(x_1,x_2,x_3) = \langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle$, be their correlation functions.

(iv) Prove that invariance under infinitesimal conformal transformations demands that

$$\sum_{j=1,2,3}[h_j\, D^{-1}(\partial.\xi)(x_j) + \xi^\mu(x_j)\partial_{x_j^\mu}]G^{(3)}(x_1,x_2,x_3) = 0,$$

for any conformal vector $\xi^\mu(x)$. See previous exercise.

(v) Integrate this set of differential equations to determine the explicit expression of $G^{(3)}(x_1, x_2, x_3)$ up to constant.

Correction :

(i) We consider the two-point function $G^{(2)}(x_1, x_2) = \langle \Phi_1(x_1)\Phi_2(x_2)\rangle$ of two scalar conformal fields, Φ_1 and Φ_2, of respective conformal weights h_1 and h_2. Invariance under translation and rotation imply that

$$G^{(2)}(x_1, x_2) = G^{(2)}(x_1 - x_2, 0) = G^{(2)}(|x_1 - x_2|, 0)\,,$$

so the dependence is on $r := |x_1 - x_2|$ only.

(ii) The transformation law for a (quasi-primary) field Φ under a conformal transformation $x \to y$ is

$$\Phi(x) \to \hat{\Phi}(y) = \left|\frac{\partial y}{\partial x}\right|^{-h/D} \Phi(x)\,,$$

where h is the corresponding conformal weight. For a two-point function this becomes

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \left|\frac{\partial y}{\partial x}\right|^{h_1/D}_{x=x_1} \left|\frac{\partial y}{\partial x}\right|^{h_2/D}_{x=x_2} \left\langle \hat{\Phi}_1(y_1)\hat{\Phi}_2(y_2)\right\rangle\,.$$

Now insert the infinitesimal transformation $x^\mu \to y^\mu = x^\mu + \epsilon\xi^\mu(x)$, develop to order ϵ and compare the two sides. This gives the invariance equation

$$\sum_{j=1,2}\left[\frac{h_j}{D}(\partial\cdot\xi)(x_j) + \xi^\mu(x_j)\partial_{x_j^\mu}\right] G^{(2)}(x_1, x_2) = 0\,. \tag{60}$$

This equation is quite general, and can be applied as well to higher-order correlation functions (with the sum being over all points x_j).
For dilatation we have seen that $\xi_\nu = kx_\nu$. It follows that $\partial\cdot\xi = \partial^\nu\xi_\nu = Dk$. Dividing by parameter k we then have

$$[h_1 + x_1\partial_1 + h_2 + x_2\partial_2]\, G^{(2)}(x_1, x_2) = 0\,. \tag{61}$$

Dilatation invariance implies that $G^{(2)}(r) = ar^\Delta$ for some exponent Δ and a constant a. Applying (61) to this form we get $(h_1 + h_2 + \Delta)ar^\Delta = 0$, and it follows that

$$G^{(2)}(x_1, x_2) = \frac{a}{|x_1 - x_2|^{h_1+h_2}}\,. \tag{62}$$

(iii) We now turn to the SCT. We have seen that it is generated by

$$\xi_\nu = (b\cdot x)x_\nu - \frac{1}{2}(x\cdot x)b_\nu\,.$$

We actually obtained this expression by integration of $\partial_\mu\xi_\nu = (b\cdot x)\delta_{\mu\nu} + (b_\mu x_\nu - b_\nu x_\mu)$. Multiplying the latter expression by $\delta^{\mu\nu}$ we obtain

$$(\partial\cdot\xi) = D(b\cdot x) + (b\cdot x) - (b\cdot x) = D(b\cdot x)$$

for the SCT. Inserting this in the general form (60) we have

$$\sum_{j=1,2}\left[h_j(b\cdot x_j)+\left[(b\cdot x_j)x_j^\nu-\frac{1}{2}(x_j\cdot x_j)b^\nu\right]\partial_{x_j^\nu}\right]G^{(2)}(x_1,x_2)=0\,.$$

If we inject the form (62) of the two-point function, we obtain after some computation that $G^{(2)}(x_1,x_2)$ must vanish unless $h_1=h_2$.
If we further assume that there is just a single field of a given conformal weight h, then the constant a can be set to one upon normalising that field. We have then

$$G^{(2)}(x_1,x_2)=\frac{\delta_{h_1,h_2}}{|x_1-x_2|^{h_1+h_2}}\,. \tag{63}$$

(iv) For the form of three- and four-point functions, see the treatment in the AIMES lecture notes.

• *Exercise 8.4: Diff $\mathbb{S}^1$ and its central extension.*

The aim of this exercise is to study the Lie algebra Diff $\mathbb{S}^1$ of vector fields in the circle and its central extension the Virasoro algebra. Let $z=e^{i\theta}$ coordinate on the unit circle. A diffeomorphism is on application $\theta\to f(\theta)$ from $\mathbb{S}^1$ onto $\mathbb{S}^1$. Using the coordinate z, we can write it as $z\to f(z)$ so that it is, at least locally, identified with a holomorphic map (again locally holomorphic). They act on functions $\phi(z)$ by composition: $\phi(z)\to(f\cdot\phi)(z)=\phi(f^{-1}(z))$. For an infinitesimal transformation, $f(z)=z+\epsilon\,v(z)+\cdots$ avec $\epsilon\ll1$, the transformed function is

$$(f\cdot\phi)(z)=\phi(z)+\epsilon\,\delta_v\phi(z)+\cdots\,,\quad \text{with } \delta_v\,\phi(z)=-v(z)\,\partial_z\phi(z).$$

(i) Take $v(z)=z^{n+1}$, with n integer. Verify that $\delta_v\phi(z)=\ell_n\phi(z)$ with $\ell_n\equiv-z^{n+1}\partial_z$. Show t

$$[\ell_n,\ell_m]=(n-m)\,\ell_{n+m}.$$

This Lie algebra is called the Witt algebra.

(ii) Let us consider the (central) extension of the Witt algebra, generated by the ℓ_n and the central element c, with the following commutation relations

$$[\ell_n,\ell_m]=(n-m)\ell_{n+m}+\frac{c}{12}(n^3-n)\delta_{n+m;0},\quad [c,\ell_n]=0.$$

Verify that this set of relation satisfy the Jacobi identity. This algebra is called the Virasoro algebra.

(iii) Prove that this is the unique central extension of the Witt algebra.

Correction :

To be completed...

• *Exercise 8.5: The stress-tensor OPE in 2D CFT*

Let ϕ be a massless Gaussian free field in 2D with two point function $\langle\phi(z,\bar z)\phi(w,\bar w)\rangle = -\log(|z-w|^2/R^2)$. Recall that the (chiral component of the) stress-tensor of a massless 2D Gaussian field is $T(z) = -\frac{1}{2} : (\partial_z\phi)^2(z) :$.

(i) Prove, using Wick's theorem, that it satisfies the OPE

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + [\frac{2}{(z_1-z_2)^2} + \frac{1}{(z_1-z_2)}\partial_z]T(z_2) + \text{reg.}$$

Correction :

Cf lecture notes.

- *Exercise 8.6: Transformation of the stress-tensor in 2D CFT.*

Under a conformal transformation $z \to w = w(z)$, the transformation rules for the stress tensor in two-dimensional CFT is

$$T(z) \to \hat T(w) = \left[z'(w)\right]^2 T(z(w)) + \frac{c}{12} S(z;w)\,, \tag{64}$$

where $z'(w)$ is the derivative of z with respect to w, while $S(z;w)$ denotes the Schwarzian derivative:

$$S(z;w) = \left[\frac{z'''(w)}{z'(w)}\right] - \frac{3}{2}\left[\frac{z''(w)}{z'(w)}\right]^2 . \tag{65}$$

(i) Let us consider two conformal transformations $z \to w = w(z)$ and $w \to \xi = \xi(w)$ and their composition $z \to \xi = \xi(z)$. Prove that consistency of the stress-tensor transformation rules demands that:

$$S(z;\xi) = S(w;\xi) + [\xi'(w)]^2\, S(z,w).$$

Verify this relation from the definition of $S(z;w)$.

(ii) Use this formula to compute the stress-tensor expectation for a CFT defined over a infinite cylinder of radius R. Show that

$$\langle T(z)\rangle_{\text{cylinder}} = -c\,\frac{\pi}{12\,R^2}.$$

Correction :

(i) We consider two conformal transformations, $z \to w = w(z)$ and $w \to \xi = \xi(w)$. Under the combined transformation $z \to w \to \xi$ we have $T(z) \to \hat T(w) \to \widetilde T(\xi)$. Under the second transformation, the stress tensor transforms as

$$\widetilde T(\xi) = [w'(\xi)]^2\, \hat T(w(\xi)) + \frac{c}{12} S(w;\xi)\,,$$

and inserting into this the first transformation we obtain

$$\widetilde T(\xi) = [w'(\xi)]^2 \left\{ [z'(w)]^2\, T(z) + \frac{c}{12} S(z;w) \right\} + \frac{c}{12} S(w;\xi)\,. \tag{66}$$

Another way of relating $\widetilde{T}(\xi)$ to $T(z)$ is to apply directly the combined transformation $z \to \xi$:

$$\widetilde{T}(\xi) = \left[z'(\xi)\right]^2 T(z(\xi)) + \frac{c}{12} S(z;\xi) \,. \tag{67}$$

We now wish impose the compatibility of (66) and (67). We first see that we must have $z'(\xi) = z'(w)w'(\xi)$, which can also be written $\frac{\mathrm{d}z}{\mathrm{d}\xi} = \frac{\mathrm{d}z}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}\xi}$. This is of course just the familiar rule for deriving a composite function. To avoid confusing in the sequel, let us denote the reverse conformal mappings by $f : \xi \to w$ and $g : w \to z$. In this notation we have just used $(g \circ f)' = (g' \circ f) \times f'$. With this being settled, the important constraint coming from comparing (66) and (67) reads

$$S(z;\xi) = \left[w'(\xi)\right]^2 S(z;w) + S(w;\xi) \,. \tag{68}$$

We now examine the constraint (68). In the previous notation we have

$$\begin{aligned} S(w;\xi) &= \frac{f'''}{f'} - \frac{3}{2}\left[\frac{f''}{f'}\right]^2 , \\ S(z;w) &= \frac{g''' \circ f}{g' \circ f} - \frac{3}{2}\left[\frac{g'' \circ f}{g' \circ f}\right]^2 , \end{aligned}$$

so that the right-hand side of (68) becomes

$$\frac{f'''}{f'} - \frac{3}{2}\left[\frac{f''}{f'}\right]^2 + (f')^2 \left(\frac{g''' \circ f}{g' \circ f} - \frac{3}{2}\left[\frac{g'' \circ f}{g' \circ f}\right]^2 \right) . \tag{69}$$

The left-hand side meanwhile reads

$$S(z;\xi) = \frac{(g \circ f)'''}{(g \circ f)'} - \frac{3}{2}\left[\frac{(g \circ f)''}{(g \circ f)'}\right]^2 . \tag{70}$$

To make this explicit we need the following rules (basically a binomial expansion):

$$\begin{aligned} (g \circ f)' &= (g' \circ f) \times f' \,, \\ (g \circ f)'' &= (g'' \circ f) \times (f')^2 + (g' \circ f) \times f'' \,, \\ (g \circ f)''' &= (g''' \circ f) \times (f')^3 + 3(g'' \circ f) \times f'' f' + (g' \circ f) \times f''' \,. \end{aligned}$$

Inserting this in (70) we obtain

$$\frac{(g''' \circ f)}{(g' \circ f)}(f')^2 + 3\frac{(g'' \circ f)}{(g' \circ f)} f'' + \frac{f'''}{f'} - \frac{3}{2}\left[\frac{(g'' \circ f)}{(g' \circ f)} f' + \frac{f''}{f'}\right]^2 .$$

In this expression, the second term cancels with the crossed term of $[\cdots]^2$. What remains is precisely (69).

Remark. One can actually show that the constraint (68) fixes the explicit form (65) of the Schwarzian derivative. Showing this is however a bit more tricky.

(ii) To apply these results, we consider the conformal mapping from the plane to an infinite cylinder of circumference R:

$$w(z) = \frac{R}{2\pi} \log z \,.$$

The inverse mapping is $z(w) = \exp(2\pi w/R)$ and we compute that

$$S(z;w) = -\frac{2\pi^2}{R^2} \,.$$

Now apply the transformation law (64). In the plane, all one-point functions (except that of the identity operator) are zero, so $\langle T(z)\rangle = 0$. It follows that on the cylinder

$$\langle \hat{T}(w)\rangle = -\frac{\pi^2 c}{6R^2}\,.$$

This is analogous to the Casimir effect between two uncharged metallic plates. According to quantum electrodynamics, the vanishing of the wave function on the plates induces a force between them. In the present setting, the finite size of the cylinder imposes a non-zero value of a one-point function.

One can go a step further and relate this to the finite-size scaling of the free energy per unit area $f_0(R)$ on the cylinder. This gives (the steps of the computation are spelled out starting from eq. (5.139) in the yellow CFT book)

$$f_0(R) = f_0(\infty) - \frac{\pi c}{6R^2} + o(R^{-2})\,. \tag{71}$$

This is an extremely useful way of determining c. Indeed the finite-size term can be obtained analytically in solvable models from the Euler-MacLaurin formula, or evaluated numerically by diagonalising the transfer matrix.

- *Exercise 8.7: Regularization of vertex operators*

In the text, we use the connection with lattice model to argue for the anomalous transformation of vertex operators in gaussian conformal field theory. The aim of this exercise is to derive (more rigorously) this transformation within field theory (without making connection with lattice models).

Let $\phi(z,\bar{z})$ a Gaussian free field normalized by $\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\log(|z-w|^2/R^2)$ with R the IR cut-off tending to infinity. In order to regularized the field we introduce a smeared version ϕ_ϵ of ϕ defined by integrating it around a small circle, of radius ϵ, centred at z:

$$\phi_\epsilon(z,\bar{z}) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, \phi(z_\epsilon(\theta), \bar{z}_\epsilon(\theta)),$$

with $z_\epsilon(\theta)$ be point on this circle, $0 < \theta < 2\pi$. The small radius ϵ play the role of UV cutoff.

(i) Prove that (notice that we consider the smeared at the same central position z but with two different cutoff ϵ and ϵ')

$$\langle \phi_\epsilon(z,\bar{z})\phi_{\epsilon'}(z,\bar{z})\rangle = \min(\log(R/\epsilon)^2, \log(R/\epsilon')^2).$$

In particular $\langle \phi_\epsilon(z,\bar{z})^2\rangle = \log(R/\epsilon)^2$.

(ii) Verify that $\langle e^{i\alpha\phi_\epsilon(z,\bar{z})}\rangle = (\epsilon/R)^{\alpha^2}$, for α real. Let us define the vertex operator by

$$V_\alpha(z,\bar{z}) = \lim_{\epsilon\to 0} \epsilon^{-\alpha^2}\, e^{i\alpha\phi_\epsilon(z,\bar{z})}.$$

Argue that this limit exists within any expectation values.

(iii) Let us now consider a conformal transformation $z \to w = w(z)$ or inversely $w \to z = z(w)$. Show that a small circle of radius $\hat{\epsilon}$, centred at point w, in the w-plane is deformed into a small close curve in the z-plane which approximate a circle of radius $\epsilon = |z'(w)|\,\hat{\epsilon}$, centred at $z(w)$.
Deduce that under such conformal transformation the vertex operator transforms as follows:
$$\hat{V}_\alpha(w, \bar{w}) = |z'(w)|^{\alpha^2}\, V_\alpha(z, \bar{z}).$$
That is: the anomalous scaling transformation of the vertex operator arises from the fact that the regularization scheme/geometry is not preserved by the conformal transformations.

Correction :

(i) We pick a circle with centre z and radius ϵ, parameterised by
$$z_\epsilon(\theta) = z + \epsilon \mathrm{e}^{i\theta}\,, \quad \text{with } \theta \in [0, 2\pi]\,.$$
Here $\epsilon \ll 1$ plays the role of the UV cut-off. We compute the two-point function at coinciding points, but at different UV cut-offs:
$$\begin{aligned} \langle \phi_\epsilon(z,\bar{z})\phi_{\epsilon'}(z,\bar{z})\rangle &= -\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\frac{\mathrm{d}\theta'}{2\pi} \log\left|\frac{z_\epsilon(\theta) - z_{\epsilon'}(\theta')}{R}\right|^2 \\ &= -\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \log\left|\frac{\epsilon' - \epsilon \mathrm{e}^{i\theta}}{R}\right|^2 . \end{aligned} \tag{72}$$
Let us assume (without loss of generality) that $\epsilon < \epsilon'$ and set $\nu = \epsilon/\epsilon' < 1$. Then
$$\log\left|\frac{\epsilon' - \epsilon \mathrm{e}^{i\theta}}{R}\right|^2 = 2\log(\epsilon'/R) + \log|1 - \nu \mathrm{e}^{i\theta}|^2\,.$$
Integrating the last term we obtain
$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \log\left|1 - \nu \mathrm{e}^{i\theta}\right|^2 = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\left[\log(1 - \nu \mathrm{e}^{i\theta}) + \log(1 - \nu \mathrm{e}^{-i\theta})\right]$$
Since $\nu < 1$ we can replace the integrand by its series expansion
$$-2\sum_{k=1}^{\infty} \cos(k\theta)\frac{\nu^k}{k}\,.$$
Since each term integrates to zero we have shown that
$$\begin{aligned} \langle \phi_\epsilon(z,\bar{z})\phi_{\epsilon'}(z,\bar{z})\rangle &= -2\log(\epsilon'/R) \\ &= 2\log\left[\min\left(\frac{R}{\epsilon}, \frac{R}{\epsilon'}\right)\right] = \min\left[\log\left(\frac{R}{\epsilon}\right)^2, \log\left(\frac{R}{\epsilon'}\right)^2\right] \end{aligned} \tag{73}$$
In particular, setting $\epsilon' = \epsilon$, we obtain the variance
$$\left\langle \phi_\epsilon(z,\bar{z})^2 \right\rangle = \log\left(\frac{R}{\epsilon}\right)^2 ,$$
which diverges for $\epsilon \to 0$ as it should in field theory.

(ii) According to question 4 of exercise 2.8.1 the Gaussian random variable $\phi_\epsilon(z,\bar{z})$ satisfies

$$\left\langle \mathrm{e}^{i\alpha\phi_\epsilon(z,\bar{z}}\right\rangle = \mathrm{e}^{-\frac{\alpha^2}{2}\langle\phi_\epsilon(z,\bar{z})^2\rangle} = \mathrm{e}^{-\frac{\alpha^2}{2}\log\left(\frac{R}{\epsilon}\right)^2} = \left(\frac{\epsilon}{R}\right)^{\alpha^2} .$$

In particular we see that the vertex operator defined by

$$V_\alpha(z,\bar{z}) = \lim_{\epsilon\to 0}\left[\epsilon^{-\alpha^2}\mathrm{e}^{i\alpha\phi_\epsilon(z,\bar{z})}\right]$$

has a well-defined limit inside the expectation value $\langle\cdots\rangle$.

(iii) Now perform a conformal transformation $z \to w = w(z)$. The small parameterised circle transforms like

$$z_\epsilon(\theta) \to \hat{w}_\epsilon = w\left(z+\epsilon\mathrm{e}^{i\theta}\right) = w(z) + w'(z)\epsilon\mathrm{e}^{i\theta} + O(\epsilon^2)\,.$$

To first order in ϵ this is again a circle, of radius $\hat{\epsilon} = w'(z)\epsilon$. (Note that for the projective transformations, circles map to circles globally.) Inverting this we get $\epsilon = \hat{\epsilon} z'(w)$'. Because of the $\epsilon^{-\alpha^2}$ factor put into the definition of $V_\alpha(z,\bar{z})$ the latter thus transforms like

$$\hat{V}_\alpha(w,\bar{w}) = |z'(w)|^{\alpha^2}\, V_\alpha(z,\bar{z})$$

For a primary field the scale factor should be $|dz/dw|^h|d\bar{z}/d\bar{w}|^{\bar{h}}$. Thus we have here the conformal weights

$$h = \bar{h} = \frac{\alpha^2}{2}\,.$$

Remark. Vertex operators play an important role in the so-called Coulomb gas construction in CFT. In this case the Gaussian fields are deformed by adding an extra term to the action (compactified bosonic fields) that couples the field to the so-called background electric field. In particular this deforms the central charge away from $c = 1$.

1.8 Chapter 9: Scaling limits and the field theory renormalisation group

- *Exercise 9.1: Explicit RG flows*

The aim of this exercise is to study simple, but important, examples beta functions and solutions of the Callan-Symanzik equation.

(i) Consider a field theory with only one relevant coupling constant g and suppose that its beta function is $\beta(g) = yg$.
Show that the RG flow, solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ is $g(\lambda) = g_1\, \lambda^y$.
Show that the RG mass scale, solution of $\beta(g)\partial_g m(g) = m(g)$ is $m(g) = m_*\, g^{1/y}$.
Consider the two point function $G(r;g)$ of a scaling field Φ of scaling dimension Δ, i.e. $G(r,g) = \langle\Phi(r)\Phi(0)\rangle_g$. Prove (using the Callan-Symanzik equation) that

$$G(r;g) = r^{-2\Delta}\, F(m(g)r),$$

with $m(g)$ the RG mass scale defined above.

(ii) Consider a field theory with only one marginal coupling constant g and suppose that its beta function is $\beta(g) = cg^2$ ($c > 0$ corresponds to marginally relevant, $c < 0$ to marginally irrelevant).
Prove that the RG flow, solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ is $g(\lambda) = g_\mu/(1-cg_\mu \log(\lambda/\mu))$.
Notice that $g_\lambda \to 0^+$, if $c < 0$, while g_λ flows up if $c > 0$, as $\lambda \to \infty$ (with $g_\mu > 0$ initially). Prove that the RG mass scale, solution of $\beta(g)\partial_g m(g) = m(g)$ is $m(g) = m_*\, e^{-1/cg}$.
Notice that this mass scale is non perturbative in the coupling constant.
Consider the two point function $G(r;g)$ of a scaling field Φ whose matrix of anomalous dimension is $\gamma(g) = \Delta + \gamma_0 g$. Prove (using the Callan-Symanzik equation) that $G(r/\lambda; g(\lambda)) = Z(\lambda)^2 G(r,g)$ with

$$Z(\lambda) = \text{const.}\, \lambda^\Delta\, [g(\lambda)]^{\gamma_0/c}.$$

Deduce from this that, in the case marginally irrelevant perturbation (i.e. $c < 0$) and asymptotically for r large,

$$G(r;g_a) \simeq \text{const.}\, r^{-2\Delta}\, [\log(r/a)]^{-2\gamma_0/c}.$$

This codes for logarithmic corrections to scaling.

Correction :

(i) The solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ with $\beta(g) = yg$ is $g(\lambda) = g_1\, \lambda^y$. The fact that $\beta(g) = yg$, with no extra terms, means that g is a scaling variable (which transform homogeneously un RG transformation), or alternatively that the RG transformation have been diagonalized by choosing the variable g.
The RG mass scale solution of $\beta(g)\partial_g m(g) = m(g)$ with with $\beta(g) = yg$ is $m(g) = m_*\, g^{1/y}$ with m_* a integration constant (equals to $m(g=1) = m_*$).
The Callan-Symanzik equation for the two-point function $G(r;g)$ is (with the $\gamma(g) = \Delta$)

$$\big(r\partial_r + 2\Delta - \beta(g)\partial_g\big)\, G(r;g) = 0,$$

with the anomalous dimension chosen to be $\gamma(g) = \Delta$ (to make this choice means that we have defined the field Φ as scaling fiels with well defined scaling dimension, ie.e we have again diagonalize the RG transformation). This type of equation are solved by the so-called 'methods of characteristics' (which means the solution are transported according to the flow specified by the beta function). Let us insert $G(r;g) = r^{-2\Delta}\, F(m(g)r)$ into this this equation. Then, for $G(r;g)$ to be a solution we need to have

$$\beta(g)\partial_g m(g) = m(g).$$

That is $m(g)$ has to be the RG mass scale (which for $\beta(g) = yg$ is $m(g) = m_*\, g^{1/y}$).

(ii) For $\beta(g) = cg^2$, the RG flow equation $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ reads $\frac{dg}{g^2} = c\frac{d\lambda}{\lambda}$ whose solutions is

$$\frac{1}{g(\mu)} - \frac{1}{g(\lambda)} = c\log(\lambda/\mu),$$

or equivalently $g(\lambda) = g(\mu)/(1 - cg(\mu)\log(\lambda/\mu))$. The analysis of the behaviour of $g(\lambda)$, depending on the sign of c, follows from this equation.
The equation for the RG mass scale $\beta(g)\partial_g m(g) = m(g)$ now reads $\frac{dm}{m} = c\frac{dg}{g^2}$, whose solution is $m(g) = m_*\, e^{-1/cg}$.
The beta function $\beta(g) = cg^2$ has no term of order g means that the variable g (and the corresponding perturbation) is marginal, that is it has no bare/classical dimension. But the fact that it has a term of order g^2 means that it is not exactly marginal (it is marginally relevant if $c > 0$ and marginally irrelevant if $c < 0$). This means that the fluctuation have introduced a scale, which can be parametrized by the RG mass scale $m(g)$, and in the literature this phenomena is often referred to the 'dynamical generation of a mass scale').
For $\beta(g) = cg^2$ and fields with anomalous scaling dimension $\gamma(g) = \Delta + \gamma_0 g$ the Callan-Symanzik equation for the two-point function $G(r;g)$ reads

$$\left(r\partial_r + 2(\Delta + \gamma_0 g) - cg^2\partial_g\right) G(r;g) = 0.$$

By construction (but this can be checked directly from the equation) this equation codes for the fact that $Z^{-2}(\lambda)G(r/\lambda; g(\lambda))$ is independent of λ provided that $g(\lambda)$ is the RG running coupling constant, solution of $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda))$ and that the (sometimes called 'wave function renormalization) function $Z(\lambda)$ satisfies

$$\lambda\partial_\lambda \log Z(\lambda) = \gamma(g(\lambda)) = \Delta + \gamma_0 g(\lambda).$$

Since $\lambda\partial_\lambda g(\lambda) = \beta(g(\lambda)) = cg^2(\lambda)$, this can be written as $\lambda\partial_\lambda[\lambda^{-\Delta}\log Z(\lambda)] = \gamma_0 g(\lambda) = \frac{\gamma_0}{c}\lambda\partial_\lambda \log g(\lambda)$. Hence

$$Z(\lambda) = \text{const.}\, \lambda^{\Delta}\, [g(\lambda)]^{\gamma_0/c}.$$

For $c < 0$ (i.e for a marginally irrelevant perturbation), the RG running coupling constant $G(\lambda)$ goes to 0^+ as $\lambda \to \infty$. Thus, let us take $\lambda = r/a$ with r the large IR scale and a the UV cutoff. Using the RG equation $Z(\lambda)^{-2}\, G(r/\lambda; g(\lambda)) = G(r,g)$ we have, for large r (at a fixed),

$$G(r, g_a) \simeq Z(r/a)^{-2}\, G(a; 0^+).$$

Alternatively, using the expression for $Z(\lambda)$, we get

$$G(r; g_a) \simeq \text{const.}\, r^{-2\Delta}\, [\log(r/a)]^{-2\gamma_0/c}.$$

This codes for logarithmic corrections to scaling due to marginally irrelevant operators.

- *Exercise 9.2: Anomalous dimensions and beta functions*

(i) Prove the relation $\gamma^\sigma_\alpha(g) = D\delta^\sigma_\alpha - \partial_\alpha \beta^\sigma(g)$ between the matrix of anomalous dimensions and the beta functions.

(ii) Give two proofs of the formula $\gamma^\sigma_\alpha(g) = \Delta_\alpha \delta^\sigma_\alpha + S_D \sum_i g^i C^\sigma_{i\alpha}$ for the matrix of anomalous dimensions to first order in perturbation theory (Here g^i are the perturbative coupling constant and S_D the volume of the D-dimensional unit sphere): one proof comes from using the previous result, the second proof comes from analysing the perturbative expansion of the correlation functions.

Correction :

See the lecture notes.

- *Exercise 9.3: Renormalisation of ϕ^3 in $D = 6$: One-particle irreducible functions*

In this exercise and the following, we consider the ϕ^3 action of the scalar field ϕ defined by

$$S[\phi] = \int d^d x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{6} g m^{\epsilon/2} \phi^3 \right],$$

where $\epsilon = 6 - d$.

In this first part, we shall compute the one-particle irreducible functions $\Gamma^{(n)}$ for $n = 1, 2, 3$.

(i) What is the dimension of ϕ and of the coupling constant g? Determine the superficial degree of (ultra-violet) divergence of $\Gamma^{(n)}$ to L loops. For which values of d the theory is renormalisable, super-renormalisible, non-renormalisable?

(ii) We first work in $d = 6$ dimensions. Which Feynman diagrams are superficially divergent? Is their number finite or infinite? Same question for one-particle irreducible diagrams.

(iii) Compute $\Gamma^{(1)}$, $\Gamma^{(2)}(p, -p)$ and $\Gamma^{(3)}(p_1, p_2, -p_1 - p_2)$ to one-loop order. To this end, use dimensional regularisation and the formulae

$$\frac{1}{a_1\, a_2} = \int_0^1 dx \frac{1}{[a_1 x + a_2(1-x)]^2}$$

$$\frac{1}{a_1\, a_2\, a_3} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[a_1 x + a_2 y + a_3(1-x-y)]^3},$$

as well as

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + 2\vec{q}.\vec{k} + p^2)^n} = \frac{\Gamma(n - \frac{d}{2})}{(4\pi)^{d/2} \Gamma(n)} \, (p^2 - k^2)^{\frac{d}{2} - n}.$$

(iv) Give expressions for $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$, neglecting terms of order ϵ. To this end, use the following propreties of the Euler Γ function:

$$\begin{aligned} \Gamma(x+1) &= x\,\Gamma(x) \\ \Gamma(x) &= \frac{1}{x} + \psi(1) + O(x) \qquad (x \to 0) \\ \psi(x) &= \frac{d}{dx}\log\Gamma(x). \end{aligned}$$

Express the results in terms of the two functions

$$\begin{aligned} f_1(u) &= \int_0^1 dx[1+ux(1-x)]\log[1+ux(1-x)] \\ f_2(u,v,w) &= \int_0^1 dx \int_0^{1-x} dy \log[1+ux(1-x)+vy(1-y)+2wxy]. \end{aligned}$$

(v) Show that the divergence of $\Gamma^{(3)}$ to one-loop order can be formally eliminated by redefining the coupling constant as follows:

$$g = \tilde{g}\left(1 - \frac{\tilde{g}^2}{(4\pi)^3\epsilon}\right).$$

Verify that by replacing $1/\epsilon$ by $\log(\Lambda/m)$ in the above formula, one recovers the divergent part corresponding to a regularisation of the theory by an ultra-violet cut-off Λ.

Correction :

(i) In Fourier (momentum) space, there is a dimension -1 for each factor of x, and dimension $+1$ for each derivation. The dimensional analysis of the three terms in the action thus produces:

$$\begin{aligned} 2+2[\phi]-d &= 0 \\ 2[m]+2[\phi]-d &= 0 \\ [g]+3[\phi]-d+\frac{\epsilon}{2}[m] &= 0. \end{aligned}$$

This can be solved to give

$$[\phi] = \frac{d-2}{2}, \qquad [m]=1, \qquad [g]=0.$$

The true coupling constant $G = gm^{\epsilon/2}$ is of dimension $\frac{\epsilon}{2}$. The sign of $[G]$ teaches us that the theory is

- Super-renormalisable for $d<6$ (with $[G]$ negative)
- Non-renormalisable for $d>6$ (with $[G]$ positive)
- Renormalisable for $d=6$ (as we shall see below).

Consider now a diagram $\Gamma^{(n)}$ with n external legs, L loops and I internal lines. In general, for a ϕ^q theory (here with $q = 3$) one has

$$qV = n + 2I, \tag{74}$$

since each line is incident on two vertices (this is called the "hand-shake lemma"). The Euler relation, $L = I - V + 1$, permits us to eliminate V, obtaining

$$I = n + 3L - 3.$$

A diagram with I internal lines ($\sim 1/q^2$) and L loops ($\sim \mathrm{d}^d q$) has thus a superficial UV divergence of degree

$$\delta = dL - 2I = (d-6)L + 6 - 2n.$$

The degree of superficial IR divergence (in the limit of zero mass, $m \to 0$) is similarly $-\delta$. One distinguishes moreover three cases:

- For $d > 6$, each $\Gamma^{(n)}$ is UV divergent, provided that L is sufficiently large. On the other hand, there is no IR divergence, implying that there are no corrections to mean field theory (the short-distance details play no role).
- For $d < 6$ the degree of superficial UV divergence diminishes with L. For $d = 4, 5$, only $\Gamma^{(1)}$ et $\Gamma^{(2)}$ diverge. And for $d = 2, 3$, only $\Gamma^{(1)}$ diverges. On the other hand, there are severe IR divergences and perturbation theory is not applicable.
- Exactly at $d = 6$, notice that $\delta = 6 - 2n$ is independent of L, so only $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$ diverge. Thus, we need to handle three divergences, having at our disposal three renormalisable parameters (m, g et ϕ). This sounds possible—and it is (see the following exercise).

Here are some general remarks:

- There can be divergences due to sub-diagrams. For instance, even when having $\delta < 0$, $\Gamma^{(3)}$ can diverge if it contains a sub-diagram with two legs.
- Each $\Gamma^{(n)}$ expands as an infinite sum of diagrams. There is thus, in general, an infinite number of divergent diagrams, but at a fixed order L this number becomes finite.
- It is convenient to limit the discussion to 1PI diagrams, and invoke general results to relate general diagrams to those.

(iii) Figure 9 shows the divergent diagrams contributing to $\Gamma^{(1)}$ (up to 3 loops), to $\Gamma^{(2)}$ (up to 2 loops), and to $\Gamma^{(3)}$ (up to 2 loops).

(iv) Below we omit the (trivial) contributions to zero loop order. Hence, in Figure 9, each one of $\Gamma^{(1)}$, $\Gamma^{(2)}$, and $\Gamma^{(3)}$ corresponds to one single 1PI diagram (the one with one loop), here in their amputated version (with the external legs chopped off). By conservation of $\mathbf{k}$, these diagrams will be evaluated at zero total momentum. Finally, there is a sign difference between $\Gamma^{(n)}$ and the 1PI digrams, coming from the relation $G^{-1} = \Gamma^{(2)} = G_0^{-1} - \Sigma$.
Let us begin by the first diagram in the first line of Fig. 9. There is a symmetry factor of 2. Hence:

$$-\Gamma^{(1)}_{1\text{ boucle}} = \frac{(-g)}{2} m^{3-d/2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2 + m^2}.$$

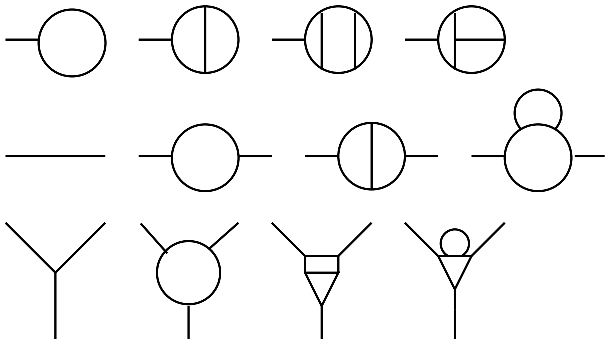

Figure 9: Divergent 1PI diagrams for ϕ^3 theory in dimension $d = 6$.

The integral can be computed by applying a trick:

$$\begin{aligned} I &\equiv \int \frac{\mathrm{d}^d q}{q^2+m^2} = \int_0^\infty \mathrm{d}\alpha \int \mathrm{d}^d q\, \mathrm{e}^{-\alpha(q^2+m^2)} \\ &= \pi^{d/2} \int_0^\infty \mathrm{d}\alpha \alpha^{-d/2} \mathrm{e}^{-\alpha m^2} \\ &= \pi^{d/2} m^{d-2} \int_0^\infty \mathrm{d}u\, u^{-d/2} \mathrm{e}^{-u} = \pi^{d/2} m^{d-2} \Gamma\left(1-\frac{d}{2}\right), \end{aligned}$$

and we find

$$\Gamma^{(1)}_{1\text{ boucle}} = \frac{g}{2} \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{d/2}} m^{d/2+1}.$$

In $d = 6$, one encounters $\Gamma(-2)$, which is divergent!
Next, the second diagram of the second line in Fig. 9 yields $-\Gamma^{(2)}_{1\text{ boucle}}(p,-p)$:

$$\begin{aligned} &\frac{(-g)^2}{2} m^{6-d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{[q^2+m^2]\,[(p-q)^2+m^2]} \\ = \;&\frac{g^2}{2} m^{6-d} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d} \left\{[(p-q)^2+m^2]x + [q^2+m^2](1-x)\right\}^{-2} \end{aligned}$$

by the trick (**??**). After some simplification and the use of (**??**) we get

$$\begin{aligned} &= \frac{g^2}{2} m^{6-d} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d} \left\{q^2 - 2xp\cdot q + p^2 x + m^2\right\}^{-2} \\ &= \frac{m^{6-d} g^2 \Gamma\left(2-\frac{d}{2}\right)}{2(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \left[p^2 x(1-x) + m^2\right]^{d/2-2}. \end{aligned}$$

So we got instead $\Gamma(-1)$. There is just one step left to get the last divergence!

Finally, for $-\Gamma^{(3)}_{1\text{ boucle}}(p_1, p_2, -p_1 - p_2)$ one finds

$$
\begin{aligned}
& (-g)^3 m^{9-d/2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{[q^2+m^2]\,[(q+p_1)^2+m^2]\,[(q-p_2)^2+m^2]} \\
= \; & -g^3 m^{9-d/2} 2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^d q}{(2\pi)^d} \left\{[(q+p_1)^2+m^2]x + [(q-p_2)^2+m^2]y + [q^2+m^2](1-x-y)\right\}^{-3} \\
= \; & -2g^3 m^{9-d/2} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^d q}{(2\pi)^d} \left\{q^2 + 2q\cdot(xp_1 - yp_2) + p_1^2 x + p_2^2 y + m^2\right\}^{-3} \\
= \; & -g^3 m^{9-d/2} \frac{\Gamma\left(3-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \left[p_1^2 x(1-x) + p_2^2 y(1-y) + 2p_1\cdot p_2 xy + m^2\right]^{d/2-3}.
\end{aligned}
$$

(v) Concerning the Euler ψ function, see the handbook by Grandshteyn, section 8.36. In particular, $\psi(1) = -C$, where

$$C = \lim_{n\to\infty} \left[\sum_{k=1}^{n-1} \frac{1}{k} - \ln n\right] = 0.577215\dots$$

is Euler's constant.
The help given on the exercise sheet permits us to develop the gamma functions:

$$
\begin{aligned}
\Gamma\left(3-\frac{d}{2}\right) &= \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} + \psi(1) + \mathcal{O}(\epsilon) \\
\Gamma\left(2-\frac{d}{2}\right) &= \frac{\Gamma\left(3-\frac{d}{2}\right)}{\frac{\epsilon}{2}-1} = -\frac{2}{\epsilon} - [\psi(1)+1] + \mathcal{O}(\epsilon) \\
\Gamma\left(1-\frac{d}{2}\right) &= \frac{\Gamma\left(2-\frac{d}{2}\right)}{\frac{\epsilon}{2}-2} = \frac{1}{\epsilon} + \frac{1}{2}\left[\psi(1) + \frac{3}{2}\right] + \mathcal{O}(\epsilon).
\end{aligned}
$$

To develop factors such as $(4\pi)^{\epsilon/2}$ one uses

$$a^{\kappa+\epsilon} = a^{\kappa}(1 + \epsilon \log a + \mathcal{O}(\epsilon^2)).$$

The final results are then

$$
\begin{aligned}
\Gamma^{(1)}_{1\text{ boucle}} &= \frac{g m^{4-\epsilon/2}}{2(2\pi)^3}\left[\frac{1}{\epsilon} + \frac{1}{2}\log 4\pi + \frac{\psi(1)}{2} + \frac{3}{4} + \mathcal{O}(\epsilon)\right] \\
\Gamma^{(2)}_{1\text{ boucle}}(p,-p) &= \frac{g^2 m^2}{(4\pi)^3}\left[\frac{1}{\epsilon}\left(1 + \frac{p^2}{6m^2}\right) + \frac{1}{2}\left(1 + \frac{p^2}{6m^2}\right)[\psi(1) + 1 + \log 4\pi]\right. \\
&\quad \left. - \frac{1}{2} f_1\left(\frac{p^2}{m^2}\right) + \mathcal{O}(\epsilon)\right] \\
\Gamma^{(3)}_{1\text{ boucle}}(p_1, p_2, -p_1-p_2) &= \frac{g^3 m^{\epsilon/2}}{(4\pi)^3}\left[\frac{1}{\epsilon} + \frac{1}{2}[\psi(1) + \log 4\pi]\right. \\
&\quad \left. - f_2\left(\frac{p_1^2}{m^2}, \frac{p_2^2}{m^2}, \frac{p_1\cdot p_2}{m^2}\right) + \mathcal{O}(\epsilon)\right]
\end{aligned}
$$

(vi) Taking only the most divergent term to one-loop order:

$$\Gamma^{(3)}(g)\Big|_{p_i=0} = g m^{\epsilon/2} + \frac{g^3}{(4\pi)^3}\frac{m^{\epsilon/2}}{\epsilon}.$$

Hence, $\Gamma^{(3)}(\tilde{g})$ is non-diverging if

$$g = \tilde{g}\left(1 - \frac{\tilde{g}^2}{(4\pi)^3\epsilon}\right).$$

We shall study further this kind of renormalisation of the coupling constant in the following exercise.
On the other hand, the diagram that we have computed for $\Gamma^{(3)}$ has a degree of superficial divergence equal to zero, and with a UV cut-off Λ, one would find the behaviour

$$\int^{\Lambda/m} \frac{\mathrm{d}q}{q} \sim \log\left(\frac{\Lambda}{m}\right),$$

so that $\log(\Lambda/m)$ plays the role of $1/\epsilon$ in $\Gamma^{(3)}$.

- *Exercise 9.4: Current-current perturbations and applications.*

  […To be completed…]

- *Exercise 9.5: Disordered random bound 2D Ising model.*

  […To be completed…]

1.9 Chapter 10: Miscellaneous applications

• *Exercise 10.1: The XY model*

The XY model is a statistical spin model with spin variables $\vec{S}_i$, on each site i of the lattice Λ, which are two component unit vectors, $\vec{S}_i^2 = 1$. The energy of a configuration $[\vec{S}]$ is defined as $E[\vec{S}] = -\sum_{[ij]} \vec{S}_i \cdot \vec{S}_j$ where the sum runs over neighboor points on Λ. Parametrising the unit spin vectors $\vec{S}_i$ by an angle Θ_i defined modulo 2π, we write the configuration energy as

$$E[\vec{S}] = -\sum_{[ij]} \cos(\Theta_i - \Theta_j).$$

The partition function is $Z = \int [\prod_i \frac{d\Theta_i}{2\pi}] \exp\left(\beta \sum_{[i,j]} \cos(\Theta_i - \Theta_j)\right)$ with $\beta = 1/k_B T$ the inverse temperature.

Here is the solution of the problem on the XY model given in Section 9.1.

IA- The XY model on a lattice: High temperature expansion

The aim of this section is to study the high temperature ($\beta \ll 1$) behavior of the XY model. It is based on rewriting the Boltzmann sums in terms of dual flow variables.

IA-1 Explain why we can expand $e^{\beta \cos\Theta}$ in series as $e^{\beta\cos\Theta} = I(\beta)\left(1+\sum_{n\neq 0} t_n(\beta) e^{in\Theta}\right)$, where $I(\beta)$ and $t_n(\beta)$ are some real β-dependent coefficients. We set $t_0(\beta) = 1$.

IA-2 By inserting this series in the defining expression of the partition function and by introducing integer variables $u_{[ij]}$ on each edge $[ij]$ of the lattice Λ, show that the partition function can be written as $Z = I(\beta)^{N_e} \cdot \hat{Z}$ with N_e the number of edges and

$$\hat{Z} = \sum_{[u],\ [\partial u = 0]} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the partition sum is over all configurations $[u]$ of integer edge variables $u_{[ij]}$ such that, for any vertex $i \in \Lambda$, the sum of these variables arriving at i vanishes, i.e. $\sum_j u_{[ij]} = 0$.
Remark: The variables u are attached to the edge of the lattice and may be thought of as ‘flow variables’. The condition that their sum vanishes at any given vertex is a divergence free condition. The divergence at a vertex i of a configuration $[u]$ is defined as $(\partial u)_i := \sum_j u_{[ij]}$.

IA-3 Let i_1 and i_2 be two points of Λ and $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2}\rangle$ be the two-point spin correlation function.
Explain why $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2}\rangle = \mathrm{Re}\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})}\rangle$.
Show that,

$$\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})}\rangle = \frac{1}{\hat{Z}} \cdot \sum_{\substack{[u] \\ [\partial u = \delta_{\cdot;i_1} - \delta_{\cdot;i_2}]}} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the sum is over all integer flow configurations such that their divergence is equal to $+1$ at point i_1, to -1 at point i_2, and vanishes at any other vertex.

IA-4 Show that $t_n(\beta) = t_{-n}(\beta) \simeq \frac{\beta^n}{2^n n!}$ as $\beta \to 0$.
Argue, using this asymptotic expression for the $t_n(\beta)$'s, that the leading contribution to

the spin correlation functions at high temperature comes from flow configurations with $u = 0$ or $u = \pm 1$ on each edge of the lattice.

IA-5 Deduce that, at high temperature, the correlation function $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ decreases exponentially with the distance between the two points i_1 and i_2.
Show that the correlation length behaves as $\xi \simeq a/\log(2/\beta)$ at high temperature.

Correction :

IA-1: $e^{\beta \cos \Theta}$ is a periodic function of Θ (with period 2π) so that it can be represented as a Fourier series. By reality, we have $t_n(\beta) = t_{-n}(\beta)$.
IA-2 We insert the representation $e^{\beta \cos \Theta} = I(\beta) \sum_n t_n(\beta) e^{in\Theta}$ in the partition function to write

$$\begin{aligned} \hat{Z} &= \int [\prod_i \frac{d\Theta_i}{2\pi}] \prod_{[ij]} \Big(\sum_{u_{[ij]}} t_{u_{[ij]}}(\beta)\, e^{iu_{[ij]}\Theta} \Big), \\ &= \int [\prod_i \frac{d\Theta_i}{2\pi}] \sum_{[u]} \Big(\prod_{[ij]} t_{u_{[ij]}}(\beta)\, e^{iu_{[ij]}\Theta} \Big). \end{aligned}$$

Integration of the Θ_i's yields the constraint $(\partial u)_i := \sum_j u_{[ij]} = 0$.
IA-3 By reality $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle = \mathrm{Re}\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle$. By definition

$$\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle = \frac{1}{\hat{Z}} \times \int [\prod_i \frac{d\Theta_i}{2\pi}] \sum_{[u]} \Big(\prod_{[ij]} t_{u_{[ij]}}(\beta)\, e^{iu_{[ij]}\Theta} \Big) \cdot e^{-i(\Theta_{i_1} - \Theta_{i_2})}.$$

Integration over the Θ_i's gives $\sum_j u_{[ij]} = 0$ for all $i \neq i_1, i_2$, but $\sum_j u_{[i_1 j]} = 1$ and $\sum_j u_{[i_2 j]} = -1$.
IA-4 By definition $t_n(\beta) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} e^{\beta \cos\theta}$. For small β we can expand $e^{\beta\cos\theta}$ in Taylor series. The first term of this series which contributes non-trivially to the integral is $\frac{1}{n!}(\beta\cos\theta)^n$. Its integral yields $t_n(\beta) \simeq \beta^n/2^n n!$ as $\beta \to 0$.
Since $t_u \simeq \mathrm{const.}\, (\beta/2)^{|u|}$ the leading contribution to the correlation function is for $u = 0$ or $|u| = 1$ (and the weight of configuration is independent of sign of u).
IA-5 Selecting the configuration with $u = 0$ or $u = \pm 1$ on each lattice edge compatible with the divergence constraints $\sum_j u_{[ij]} = 0$ for all $i \neq i_1, i_2$, but $\sum_j u_{[i_1 j]} = 1$ and $\sum_j u_{[i_2 j]} = -1$ selects a path from i_1 to i_2. The weight of such path γ is proportional to $(\beta/2)^{d_{12}(\gamma)}$ with $d_{12}(\gamma)$ its length (measured as the number of steps of the path). The leading contribution comes from the shortest path, and the correlation function decreases exponentially with the distance d_{12} between the two points as $(\beta/2)^{d_{12}}$. The correlation length is thus $a/\log(2/\beta)$, asymptotically at high temperature. The high temperature phase is disordered.

IB- Low temperature expansion

The aim of this section is to study the low temperature ($\beta \gg 1$) behavior of the XY model. It consists in expanding the interaction energy $\cos(\Theta_i - \Theta_j)$ to lowest order in the angle variables so that we write the configuration energy as (up to an irrelevant additive constant)

$$E[\vec{S}] = \mathrm{const.} + \frac{1}{2} \sum_{[i,j]} (\Theta_i - \Theta_j)^2 + \cdots .$$

This approximation neglects the 2π-periodicity of the angle variables.

IB-1 Argue that the higher order terms in this expansion, say the terms proportional to $\sum_{[i,j]} (\Theta_i - \Theta_j)^4$, are expected to be irrelevant and can be neglected.

IB-2 Write the expression of the partition function Z of the model within this approximation.
Explain why, in this approximation, the theory may be viewed as a Gaussian theory.

IB-3 Let $G_\beta(x)$ be the two-point function of this Gaussian theory. Show that $G_\beta(x) = \beta^{-1}\, G(x)$ with

$$G(x) = \int_{-\pi/a}^{+\pi/a} \frac{d^2p}{(2\pi/a)^2} \frac{e^{ip\cdot x}}{4-2(\cos ap_1 + \cos ap_2)},$$

with p_1, p_2 the two components of the momentum p and a the lattice mesh.

IB-4 Let i_1 and i_2 be two points on Λ and x_1 and x_2 be their respective Euclidean positions. Let $C_\alpha(x_1,x_2) = \langle e^{i\alpha(\Theta_{i_1}-\Theta_{i_2})}\rangle$ with α integer. Show that

$$C_\alpha(x_1,x_2) = e^{-\frac{\alpha^2}{\beta}\big(G(0)-G(x_1-x_2)\big)}.$$

IB-5 Explain why $G(x)$ is actually IR divergent[1] and what is the origin of this divergence, but that $G(0)-G(x)$ is finite for all x. Show that

$$G(0)-G(x) = \frac{1}{2\pi}\log(|x|/a) + \text{const.} + O(1/|x|).$$

IB-6 Deduce that the correlation functions C_α decrease algebraically at large distance according to

$$C_\alpha(x_1,x_2) \simeq \text{const.}\,(a/|x_1-x_2|)^{\alpha^2/2\pi\beta}.$$

Compare with the high temperature expansion.

Correction :

IB-1 This expansion is a gradient expansion, the leading term is $(\nabla\Theta)^2$. The other terms $(\nabla\Theta)^p$, with higher powers of the gradient, are irrelevant (when estimated using the leading Gaussian contribution $\int(\nabla\Theta)^2$).

IB-2 In this approximation, the partition reads (up to an irrelevant multiplicative constant)

$$Z = \int [\prod_i \frac{d\Theta_i}{2\pi}]\, e^{-\frac{\beta}{2}\sum_{[i,j]}(\Theta_i-\Theta_j)^2}.$$

This is a Gaussian theory.

IB-3 The two point function is given by the inverse of the quadratic form defining the action. Hence it is $\beta^{-1}G(x)$ with G the Green function of the lattice Laplacian on $(a\mathbb{Z}^2)$. Thus $\hat{G}(p)$, the Fourier transform of $G(x)$ is solution of

$$\big(4-2(\cos ap_1+\cos ap_2)\big)\,\hat{G}(p) = 1.$$

This yields the formula for $G(x)$ given in the text[2].

IB-4 The formula for C_α follows from the fact that the theory is Gaussian (with a even translation invariant two point function).

IB-5 The function $G(x)$ is IR divergent because the (discrete) Laplacian has the constant function

[1] So that, when defining $G(x)$, we implicitly assumed the existence of an IR cut-off, say $|p| > 2\pi/L$ with L the linear size of the box on which the model is considered.

[2] There is actually an IR divergence is this formula – due to constant zero mode of the Laplacian – so that we implicitly assume an IR regularization.

as zero mode (the constant function is in the kernel of the Laplacian) and the inverse Laplacian does not exist. For this inverse to exist one has to impose boundary conditions (say periodicity, or Dirichlet, etc.) which eliminate the constant zero mode. The IR divergence of $G(x)$ is of the form $\frac{1}{2\pi}\log L$ with L the linear size of system box (this can be seen by looking at the small momenta contribution to the integral: $\frac{1}{(2\pi)^2}\int_{|p|>2\pi/L}\frac{d^2p}{(p)^2}$.). This divergence cancels in the difference $G(0)-G(x)$ (because this is independent of the constant zero mode). Alternatively, we can write

$$G(0)-G(x)=\int_{-\pi/a}^{+\pi/a}\frac{d^2p}{(2\pi/a)^2}\frac{1-\cos(p\cdot x)}{4-2(\cos ap_1+\cos ap_2)}.$$

which is explicitly convergent.

Since $G(0)-G(x)$ only depends on $|x|/a$, the long distance behavior is identical to the continuous limit ($a\to 0$). In this limit $G(0)-G(x)$ is (minus) a Green function of the 2D Euclidean Laplacian. Hence it is equal to $\frac{1}{2\pi}\log|x|$ up to an additive constant. Dimensional analysis then fixes the constant as in the text.

IB-6 Direct application of the above formula.

II- The role of vortices in the XY field theory

The previous computations show that the model is disordered at high temperature but critical at low temperature with temperature dependent exponents. The aim of this section is to explain the role of topological configurations, called vortices, in this transition.

We shall now study the model in continuous space, the Euclidean plane $\mathbb{R}^2$, but with an explicit short distance cut-off a. We shall consider the XY system in a disc of radius L.

In the continuous formulation, the spin configurations are then maps Θ from $\mathbb{R}^2$ to $[0,2\pi]$ modulo 2π. The above Gaussian energy is mapped into the action

$$S_0[\Theta]=\frac{\kappa}{2}\int d^2x(\nabla\Theta)^2,$$

with a coefficient κ proportional to β.

II-1 Argue that the coefficient κ cannot be absorbed into a rescaling of the field variable Θ?

II-2 A vortex, centred at the origin, is a configuration such that $\Theta^\pm_{\rm v}(z)=\pm\mathrm{Arg}(z)$, with z the complex coordinate on $\mathbb{R}^2$, or in polar coordinates[3], $\Theta^\pm_{\rm v}(r,\phi)=\pm\phi$.
Show that $\Theta^\pm_{\rm v}$ is an extremum of S_0 in the sense that $\nabla^2\Theta^\pm_{\rm v}=0$ away from the origin.
Show that $\oint_{C_0}d\Theta^\pm_{\rm v}=\pm 2\pi$ for C_0 a small contour around the origin.

II-3 Let a_0 be a small short distance cut-off and let $\mathbb{D}(a_0)$ be the complex plane with small discs of radius a_0 around the vortex positions cut out. Prove that, evaluated on $\Theta^\pm_{\rm v}$, the action S_0 integrated over $\mathbb{D}(a_0)$ (with an IR cut-off L) is

$$S^{(1)}_{\rm vortex}=\frac{\kappa}{2}\int_{\mathbb{D}(a_0)}d^2x\,(\nabla\Theta^\pm_{\rm v})^2=\pi\kappa\log\big[L/a_0\big].$$

Give an interpretation of the divergence as $a_0\to 0$.

[3] We recall the expression of the gradient in polar coordinates: $\nabla\Theta=(\partial_r\Theta,\frac{1}{r}\partial_\phi\Theta)$. The Laplacian is $\nabla^2F=\frac{1}{r}\partial_r(r\partial_r)F+\frac{1}{r^2}\partial_\phi^2F$.

II-4 What is the entropy of single vortex configurations? Show that the contribution of single vortex configurations to the free energy is

$$e^{-F^{(1)}_{\rm vortex}} \simeq \text{const.} \, \big(\frac{L}{a_0}\big)^2 \, e^{-\pi\kappa \log[L/a_0]}$$

Conclude that vortex configurations are irrelevant for $\pi\kappa > 2$ but relevant for $\pi\kappa < 2$.

Correction :

II-1 Since Θ is 2π-periodic we cannot rescale it to absorb the parameter κ in a redefinition of Θ (unless we redefine the periodicity).
II-2 In polar coordinate $\nabla^2\Theta = \frac{1}{r}\partial_r(r\partial_r)\Theta + \frac{1}{r^2}\partial_\phi^2\Theta$. Thus, $\nabla^2\Theta_v^\pm = 0$ away from the origin. The gradient is $\nabla\Theta_v^\pm = \pm(0, \frac{1}{r})$. Hence, $\oint_{C_0} d\Theta_{\rm v}^\pm = \pm\int_0^{2\pi} d\phi = \pm 2\pi$.
II-3 Using $(\nabla\Theta_v^\pm)^2 = \frac{1}{r^2}$, we get (doing the integration using polar coordinate)

$$S^{(1)}_{\rm vortex} = \frac{\kappa}{2}\int_{a_0}^{L} \frac{2\pi\, rdr}{r^2} = \pi\kappa \log\big[L/a_0\big].$$

The UV divergence (with $a_0 \to 0$) is an echoe of the fact that the naive continuous limit we are using is ill-defined near the core of the vortex at which the field Θ becomes singular.
II-4 The vortex center may be positioned at any position, with a typical size of diameter a_0. Hence the entropy of single vortex configuration is $\simeq \log(L/a_0)^2$. This yields the expression of $F^{(1)}_{\rm vortex}$ given in the text. And $e^{-F^{(1)}_{\rm vortex}}$ is significantly large for $\pi\kappa < 2$ but negligeable for $\pi\kappa > 2$.

IIIA- The XY field theory: Mapping to the sine-Gordon theory

This mapping comes about when considering a gas of pairs of vortices of opposite charges $\pm$, so that the vortex system is neutral ($\sum_a q_a = 0$). We denote x_j^+ (resp. x_j^-) the positions of the vortices of charge $+$ (resp. $-$).

The vortex gas is defined by considering all possible vortex pair configurations (with arbitrary number of pairs) and fluctuations around those configurations. We set $\Theta = \Theta_{\rm v}^{(2n)} + \theta_{\rm sw}$ and associate to each such configuration a statistical weights e^{-S} with action given by

$$S = S^{(2n)}_{\rm vortex}[x_j^+, x_j^-] + S_0[\theta_{\rm sw}],$$

with $S_0[\theta_{\rm sw}]$ the Gaussian action $\frac{\kappa}{2}\int d^2x (\nabla\theta_{\rm sw})^2$. We still assume a short-distance cut-off a.

IIIA-1 Write the expression of the action $S^{(2n)}_{\rm vortex}[x_j^+, x_j^-]$ for a collection of n pairs of vortices at positions $x_j^\pm$, $j = 1, \cdots, n$.

IIIA-2 Argue that the partition function of the gas of vortex pairs is given by the product $Z = Z_{\rm sw} \times Z_{\rm vortex}$ with $Z_{\rm sw}$ the partition function for the Gaussian free field $\theta_{\rm sw}$ and

$$Z_{\rm vortex} = \sum_{n\geq 0} \frac{\mu^{2n}}{n!\cdot n!} \times \int (\prod_{j=1}^n d^2x_j^+ \prod_{j=1}^n d^2x_j^-) \, \frac{\prod_{i<j}(|x_i^+ - x_j^+|/a)^{2\pi\kappa}(|x_i^- - x_j^-|/a)^{2\pi\kappa}}{\prod_{i,j}(|x_i^+ - x_j^-|/a)^{2\pi\kappa}},$$

with $\mu = \big(\frac{a_0}{a}\big)^{\pi\kappa} e^{-\beta\epsilon_c}$.

IIIA-3 The aim of the following questions is to express Z_{vortex} as a path integral over an auxiliary bosonic field φ. Let $\tilde{S}_\kappa[\varphi] = \frac{1}{2\kappa}\int d^2x(\nabla\varphi)^2$ be a Gaussian action. Show that, computed with this Gaussian action,

$$\langle e^{i2\pi\varphi(x)}e^{-i2\pi\varphi(y)}\rangle_{\tilde{S}_\kappa} = \frac{1}{|x-y|^{2\pi\kappa}}.$$

Hint: The Green function associated to the action $\tilde{S}_\kappa[\varphi]$ is $G(x,y) = -\frac{\kappa}{2\pi}\log\big(|x-y|/a\big)$.

IIIA-4 What is the scaling dimension (computed with the Gaussian action $\tilde{S}_\kappa[\varphi]$) of the operators $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$?
Deduce that the perturbation $\cos(2\pi\varphi)$ is relevant for $\pi\kappa < 2$ and irrelevant for $\pi\kappa > 2$.
Is the the perturbation $(\nabla\varphi)^2$ relevant or irrelevant?

IIIA-5 Show that Z_{vortex} can be written as the partition function of Gaussian bosonic field with action $S_{sG}[\varphi]$,

$$Z_{\text{vortex}} = \int [D\varphi]\, e^{-S_{sG}[\varphi]},$$

where the action S_{sG} is defined as

$$S_{sG}[\varphi] = \int d^2x\Big[\frac{1}{2\kappa}(\nabla\varphi)^2 - 2\mu\cos(2\pi\varphi)\Big].$$

This is called the sine-Gordon action.
Hint: Compute perturbatively the above partition function as a series in μ while paying attention to combinatorial factors.

Correction :

IIIA-1 Vortices with charge $+$ are at positions x_j^+, those of charge $-$ are at positions x_j^- with $j = 1,\cdots,n$. Hence, the total sum of the charges vanishes, and

$$S^{(2n)}_{\text{vortex}}[x_j^+,x_j^-] \;=\; -2\pi\kappa\sum_{i<j}\log\Big[\frac{|x_i^+-x_j^+||x_i^- - x_j^-|}{a_0\,a_0}\Big] + 2\pi\kappa\sum_{i\le j}\log\Big[\frac{|x_i^+-x_j^-|}{a_0}\Big] + 2n\beta\,\epsilon_c.$$

IIIA-2 Since the statistical weights of vortex configurations and of the spin waves θ_{sw} factorize, it is clear that the partition function factorizes as $Z = Z_{\text{rm}} \times Z_{\text{vortex}}$. Next we rewrite $S^{(2n)}_{\text{vortex}}$ in term of lattice cut-off a: this amounts to replace a_0 by a in the log's and to add the term $-2n\,\pi\kappa\,\log(\frac{a}{a_0})$. The vortex gas partition function is then

$$Z_{\text{vortex}} = \sum_{n\ge 0}\frac{1}{n!}\times\frac{1}{n!}\times\int(\prod_{j=1}^n d^2x_j^+\prod_{j=1}^n d^2x_j^-)\,e^{-S^{(2n)}_{\text{vortex}}[x_j^+,x_j^-]},$$

where the factor $\frac{1}{n!}\times\frac{1}{n!}$ comes from the indistinguishability of the vortices of given charge. This proves the claim.
IIIA-3 Since $\langle\varphi(x)\varphi(0)\rangle = -\frac{\kappa}{2\pi}\log\big(|x-y|/a\big)$ (w.r.t the Gaussian theory with action $\tilde{S}_\kappa[\varphi] = \frac{1}{2\kappa}\int d^2x(\nabla\varphi)^2$), we get the result.
IIIA-4 W.r.t $\tilde{S}_\kappa[\varphi]$, the operator $\cos(2\pi\varphi)$ has scaling dimension $\pi\kappa$. It is relevant (in dimension $D=2$) for $\pi\kappa<2$ and irrelevant for $\pi\kappa>2$.
The operator $(\nabla\varphi)^2$ has dimension 2 and it is marginal. Its RG behaviour depends on the details of the perturbation. If this is only perturbing operator, it is exactly marginal. If $(\nabla\varphi)^2$ is accompanied

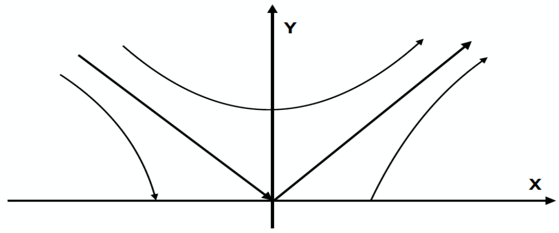

Figure 10: The XY RG flow.

with other perturbing operators, then it may be relevant or irrelevant depending on the OPE structure.
IIIA-5 We expand the sine-Gordon partition function is Taylor series in μ. By charge conservation, only the even terms in the expansion are non-vanishing. Hence,

$$Z_{\rm sG} = \sum_n \frac{(2\mu)^{2n}}{(2n)!} \langle \Big[\int d^2x \cos(2\pi\varphi) \Big]^{2n} \rangle_{\tilde S_\kappa},$$

where the expectations are computed using the Gaussian action $\tilde S_\kappa$. In these expectations, only the charge neutral combinations contribute. The terms of order $2n$ involve n operators $e^{+i2\pi\varphi}$, with charge $+$, and n operators $e^{-i2\pi\varphi}$, with charge $-$. Integrating over the positions of these charges and taking into account the combinatorial factor (corresponding to choose n amongst $2n$) we get

$$Z_{\rm sG} = \sum_n \frac{(2\mu)^{2n}}{(2n)!} \binom{2n}{n} \frac{1}{2^{2n}} \int (\prod_{j=1}^n d^2x_j^+ d^2x_j^-)\, \langle \prod_j e^{+i2\pi\varphi(x_j^+)}\, e^{-i2\pi\varphi(x_j^-)} \rangle_{\tilde S_\kappa}.$$

This proves the result.

IIIB- The XY field theory: The renormalization group analysis

IIIB-1 We now study the renormalization group flow of the action S_{sG} for κ close to the critical value $\kappa_c = 2/\pi$. We let $\kappa^{-1} = \kappa_c^{-1} - \delta\kappa$ and write

$$S_{sG}[\varphi] = \tilde S_{\kappa_c}[\varphi] - \int d^2x \Big[\frac{1}{2}(\delta\kappa)(\nabla\varphi)^2 + 2\mu\cos(2\pi\varphi)\Big]$$

Show that, to lowest order, the renormalization group equations for the coupling constants $\delta\kappa$ and μ are of the following form:

$$\begin{aligned} (\dot{\delta\kappa}) = \ell\partial_\ell\,(\delta\kappa) &= b\,\mu^2 + \cdots \\ \dot\mu = \ell\partial_\ell\,\mu &= a\,(\delta\kappa)\,\mu + \cdots \end{aligned}$$

with a and b some positive numerical constants.
Hint: It may be useful to first evaluate the OPE of the fields $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$.
IIIB-2 We redefine the coupling constants and set $X = a\,(\delta\kappa)$ and $Y = \sqrt{ab}\,\mu$ such that the RG equations now reads $\dot X = Y^2$ and $\dot Y = XY$.
Show that $Y^2 - X^2$ is an invariant of this RG flow.
Draw the RG flow lines in the upper half plane $Y > 0$ near the origin.

IIIB-3 We look at the flow with initial condition $X_I < 0$ and Y_I.
Show that if $Y_I^2 - X_I^2 < 0$ and $X_I < 0$, then the flow converges toward a point on the line $Y = 0$.
Deduce that for such initial condition the long distance theory is critical. Compare with section I-B.

IIIB-4 Show that if $Y_I^2 - X_I^2 > 0$ and $X_I < 0$, the flow drives X and Y to large values. Let $Y_0^2 = Y_I^2 - X_I^2$ with $Y_0 > 0$. Show that the solution of the RG equations are

$$\log\big(\frac{\ell}{a}\big) = \frac{1}{Y_0}\Big[\arctan\big(\frac{X(\ell)}{Y_0}\big) - \arctan\big(\frac{X_I}{Y_0}\big)\Big].$$

IIIB-5 The initial condition X_I and Y_I are smooth functions of the temperature T of the XY model. The critical temperature T_c is such that $X_I + Y_I = 0$. We take the initial condition to be near the critical line $X_I + Y_I = 0$ with $X_I < 0$. We let $X_I = -Y_I(1+\tau)$ in which $\tau \ll 1$ is interpreted at the distance from the critical temperature: $\tau \propto (T - T_c)$. For $\tau > 0$, we define the correlation length as the length ξ at which $X(\ell)$ is of order 1. Why is this a good definition?
Show that

$$\xi/a \simeq \text{const.}\ e^{\text{const.}/\sqrt{\tau}}.$$

Correction :

IIIB-1 W.r.t to $\tilde{S}_{\kappa_c}$ both operators $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$ are marginal. To lowest order the beta-functions are given by the OPE coefficients. These OPE are of the form:

$$(\nabla\varphi)^2 \times \cos(2\pi\varphi) \implies \cos(2\pi\varphi) + \text{irrelevant}$$
$$\cos(2\pi\varphi) \times \cos(2\pi\varphi) \implies (\nabla\varphi)^2 + \text{irrelevant}$$

This implies the structure of the beta-function given in the text. The coefficients are positive because these OPE coefficients are positive (one also has to take into account the signs introduced when defining the perturbed action).
IIIB-2 $Y^2 - X^2$ is proved to be a constant of the RG flow by computing its derivative.
See the picture for a representation of the flow lines.
IIIB-3 If $Y_I^2 - X_I^2 < 0$ and $X_I < 0$, the picture shows that the flow converges to the axis $Y = 0$. Alternatively, $\dot{Y} = XY$ implies that Y decreases until the flow reaches $Y = 0$ with $X = -\sqrt{X_I^2 - Y_I^2}$.
The large distance behaviour is critical because the theory at $Y = 0$ is a massless Gaussian theory which is critical. It corresponds to the low temperature phase of the XY model discussed in section IB.
IIIB-4 If $Y_I^2 - X_I^2 > 0$ and $X_I < 0$, the picture shows that the flow drives X and Y to large values (along the curve $Y^2 - X^2 = \text{const}$).
Using the fact that $Y^2 - X^2 = Y_0^2$ is an invariant of the RG flow, the latter can be written as $\dot{X} = Y_0^2 + X^2$, or alternatively $\frac{d\ell}{\ell} = \frac{dX}{Y_0^2+X^2}$. The solution given in the text is checked by computing its ℓ derivative (using that $d\arctan x = dx/(x^2+1)$).
IIIB-5 For $X_I < 0$ and $X_I = -Y_I(1+\tau)$, we get $Y_0 = Y_I\sqrt{2\tau}$, and $Y_0 \to 0$ as $\tau \to 0$. At the length scale $\ell = \xi$ the correlation length, $X(\ell)$ is of order one, as is its initial value X_I, and thus $X(\ell)/Y_0 \to \infty$ and $X_I/Y_0 \to -\infty$ as $\tau \to 0$. Hence, from the explicit solution above we get

$$\log(\xi/a) = \pi/Y_0 = (\pi/Y_I) \times (1/\sqrt{2\tau}),$$

as claimed in the text.
The correlation diverges as $\xi \simeq e^{\mathrm{const}/\sqrt{T-T_c}}$ near the transition. The transition is of infinite order. This is the Kosterlitz-Thouless (KT) transition.