TURBULENCE

The power of two dimensions

Certain aspects of two-dimensional turbulence are remarkably similar to those found in critical percolation, and show conformal invariance. But there is both less, and more, to this observation than meets the eye.

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Turbulence in fluids is one of those everyday natural phenomena for which we think we understand the underlying physics, but are nevertheless unable to make analytic predictions for the patterns we see when, for example, we stir our coffee in the morning. On page 124 of this issue[1], Denis Bernard *et al.* argue that, in two dimensions, some aspects of those patterns have much in common with those that appear in a simpler process called critical percolation. They do this by comparing the results of extensive numerical simulations of two-dimensional turbulence with some recent analytic results for percolation. Although these observations are fascinating and unexpected, they perhaps raise more questions than they answer.

Turbulence in a fluid in three dimensions occurs when it is stirred on large length scales (for example, by a coffee spoon). It can be shown that the viscosity, by which the energy is dissipated, is only effective at very short length scales. Thus there is a continual 'cascade' of energy from large to small scales. Many years ago Kolmogorov argued that[2], despite the fact that the underlying Navier–Stokes equations are deterministic, this cascade may be described statistically in terms of random velocity fluctuations, with a power-law spectrum of their Fourier components. In two dimensions, the picture is slightly different, and Kraichnan argued there should be an inverse energy cascade out to scales larger than that of the stirring[3]. In this regime, the vorticity is a random scalar quantity.

Percolation, on the other hand, is described much more simply. Imagine that we randomly colour the cells of a honeycomb lattice black or white. Each cell has a probability p of being white. If p is small, most of the cells will be black, with a few small islands, or clusters, of white. As p is increased, these white clusters grow larger until at a critical value p_c (which for this example equals 1/2, the case shown in Fig. 1) there is a non-zero probability that one of these spans the whole region, no matter how large it is. Percolation itself is important as a model for random inhomogeneous systems — for instance, if the black clusters represent untapped oil beds, it is much easier to extract the oil if they percolate.

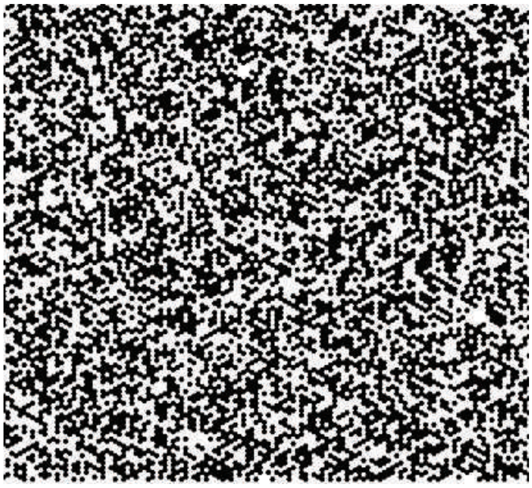

Figure 1 An example of critical percolation. The cells of a honeycomb lattice have been randomly coloured black or white with equal probability. The picture is statistically conformally invariant, and the boundaries of the clusters seem to have the same statistics at large scales as the lines of zero vorticity in two-dimensional turbulence.

The most striking feature of Fig. 1 is its scale invariance: if it is slightly out of focus, so we cannot see the details of the lattice, and we then take a part of the region and blow it up so it is the same size as the original, statistically we could not tell the difference from the original. In two dimensions it turns out that an even stronger symmetry holds: if we blow up different parts of the figure by different magnification factors (as long as angles are preserved) then statistically the picture once again looks the same. This property, called conformal invariance, is believed to be shared with other important physical systems — such as the clusters of spins that point in the same direction in certain simple ferromagnets at the Curie temperature. With this assumption, theoreticians have in the past few decades been able to derive many important properties of such systems.

Conformal invariance is much more powerful in two dimensions, when it is associated with the theory of analytic functions of a complex variable. The recent theoretical progress on percolation and related problems, which goes under the name of stochastic Loewner evolution (SLE) and was developed initially by the mathematicians Gregory Lawler, Oded Schramm and Wendelin Werner, relies very much on this property. Instead of the clusters themselves, the theory focuses on the properties of their boundaries, which are fractal curves. It makes very detailed analytic predictions for some of their average properties — how the probability that one crosses the entire region from left to right depends on the shape of the region, for example.

It is these predictions that Bernard *et al.* have tested against some features of the vorticity distribution in the inverse cascade of two-dimensional turbulence, with spectacular agreement[1]. Specifically, they consider colouring the regions of positive vorticity white, and the negative regions black. This produces pictures that, on large scales, look very much like Fig. 1, but are they really the same? By making numerical measurements on the curves that separate the black and white regions, taking statistical averages and comparing these with the predictions of SLE, these authors argue that they are.

This would seem to provide compelling evidence for the claim that at least some aspects of two-dimensional turbulence are conformally invariant. But perhaps there is less to this than meets the eye. Suppose we take a simple random function for the vorticity — not necessarily that given by solving the Navier–Stokes equations — and think of its value as giving the height at that point in a mountainous landscape. Now imagine flooding this landscape and colouring the parts above the water level white, and the rest black. If the water level is high, there will be small disconnected islands, and if it is low, there will be disconnected lakes. There is however a critical value of the sea level for which there is one large supercontinent and one large ocean. As long as the original random function has a gaussian distribution with only short-range correlations, it is believed that the large-scale properties of the coastlines correspond to percolation cluster boundaries and should be described by SLE. Perhaps, then, a simple explanation of the authors' results is that they have used a very complicated way of generating a simple random landscape, and they tell us nothing about turbulence *per se*.

But this is an oversimplification: as the authors point out, the correlations of the vorticity are not short-ranged but rather decay as the 4/3 power of the distance. The fact that lines of zero vorticity nevertheless seem to enjoy all the properties of simple percolation cluster boundaries, including conformal invariance, suggests that something much deeper is at work.

REFERENCES

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