Statistical Aspects of Quantum State Monitoring for (and by) Amateurs

Denis BERNARD ♣ and ...

♣ Laboratoire de Physique Théorique de l'Ecole Normale Supérieure de Paris, CNRS, ENS & PSL Research University, UMPC & Sorbonne Universités, France.

Abstract

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1 Introduction

2 Quantum non-demolition measurements

2.1 Indirect measurement and POVM

What kind of experiments?

Give a brief description of cavity QED experiments done in S. Laroche's group.....

Idem with circuit QED?.....

What are (weak) indirect measurements?

To formalize this scheme we consider a system interacting with probes. For simplicity we suppose that all probes have been prepared in a pure state $|\varphi\rangle$ (before interacting with the system). Let ρ be the system density matrix ρ. The system recursively interacts with a probe during some time interval. Let U be the interacting evolution operator - acting on the tensor product of the system and probe Hilbert spaces. After the interaction has taken place, some probe observable is measured (not a system observable). Let $|s\rangle$ be the complete basis of the observable eigen-states (we assume non-degeneracy for simplicity).

When interacting with a probe the evolution of the system density matrix decomposes in two steps. First there is the evolution due to the interaction. This evolution (of the system and the probe) is

$$\rho \otimes |\varphi\rangle\langle\varphi| \to U\,(\rho \otimes |\varphi\rangle\langle\varphi|)\,U^\dagger.$$

Second there is the probe measurement. Measuring the probe observable projects this state in one of the eigen-space of the observables. If the output measurement is s, the projection operator is $\mathbb{I} \otimes |s\rangle\langle s|$. As a consequence, the system state becomes

$$\rho \to \frac{F_s\,\rho\,F_s^\dagger}{\pi(s)}, \quad \text{with probability } \pi(s) = \mathrm{Tr}(F_s\,\rho\,F_s^\dagger),$$

with $F_s := \langle s|U|\varphi\rangle$ - which is an operator acting on the system Hilbert space only. As a consequence of the unitarity of U, the operators F_s satisfy

$$\sum_s F_s^\dagger\,F_s = \mathbb{I}.$$

This ensures that the probabilities sum to one: $\sum_s \pi(s) = 1$ because $\sum_s \mathrm{Tr}(F_s\,\rho\,F_s^\dagger) = \mathrm{Tr}(\rho) = 1$.

Definition "POVM":
A set of operators F_s satisfying the relation $\sum_s F_s^\dagger\,F_s = \mathbb{I}$ form, by definition, a so-called POVM - for Positive Operator Valued Measure.

Remark 1:
If the probes are prepared not in a pure state but in a mixed state $\rho_p = \sum_\sigma p_\sigma |\sigma\rangle\langle\sigma|$ with $\sum_\sigma p_\sigma = 1$ and $|\sigma\rangle$ the basis diagonalizing ρ_p. We may look at two transformations. First

we may apply the above procedure with $\rho \otimes \rho_p$ as system-probe initial state. If the probe is projected onto $|s\rangle$ by the measurement, then the system state is transformed into

$$\rho \to \sum_\sigma p_\sigma \, F_s^{(\sigma)} \rho F_s^{(\sigma)\dagger} / \pi(s), \text{ with probability } \pi(s) = \sum_\sigma p_\sigma \pi_s^{(\sigma)},$$

where $\pi_s^{(\sigma)} := \mathrm{Tr}(F_s^{(\sigma)} \rho F_s^{(\sigma)\dagger})$ with $F_s^{(\sigma)} := \langle s|U|\sigma\rangle$. Second we may look at applying randomly, with probability p_σ, the POVMs $F_s^{(\sigma)}$. Then conditioned on selecting σ, the system state is transformed into $F_s^{(\sigma)} \rho F_s^{(\sigma)\dagger} / \pi^{(\sigma)}(s)$ with probability $\pi^{(\sigma)}(s) = \mathrm{Tr}(F_s^{(\sigma)} \rho F_s^{(\sigma)\dagger})$. Its mean – over σ – differs from the above transformation. Let us instead condition on the output signal. The probability to find s as output is $\sum_\sigma p_\sigma \pi_s^{(\sigma)}$. Thus, conditioned on the signal to be s, the mean system state is now the same as above. As long as we are interested in the mean behavior, we are free to choose any of the two descriptions. In the following we shall restrict ourselves to probes in pure states.

Quantum channels and completely positive maps

If we don't record the measurement outputs (or, equivalently, if we trace over the probe degrees of freedom) the mean system density matrix $\bar{\rho}$ is transformed linearly by the map

$$\Phi : \; \bar{\rho} \to \Phi(\bar{\rho}) := \sum_s F_s \, \bar{\rho} \, F_s^\dagger.$$

This map preserves the positivity and the normalisation of the density matrix. It is actually an example of a completely positive map, with an explicit Kraus decomposition. To find such a dissipative evolution in case we trace over the probe degrees of freedom is not surprising because a series of probes behaves as a reservoir. Hence tracing over its degrees of freedom induces a dissipative evolution on the system.

Definition: "Completely Positive (CP) Maps"
Let $\mathcal{H}$ be a (finite dimensional) quantum system Hilbert space and let ρ denote quantum states – quantum density matrices – on $\mathcal{H}$. A map Φ acting linearly on quantum states is said to be completely positive if its extension $\Phi \otimes \mathbb{I}$ on states on tensor product extension of $\mathcal{H}$ of the form $\mathcal{H} \otimes \mathcal{K}_m$ with $\mathcal{K}_m$ a finite dimensional Hilbert space is positive. By convention, we shall always assume that CP-maps are trace preserving.

CP-maps represent operations on quantum states. As such they have to preserve the positivity of the density matrix. But completely positivity is stronger than positivity. We ask for this stronger condition because we may imagine operating on the system when it is next to another quantum system, but uncoupled with it. The operation is only going to act on the system, leaving its neighborhood untouched, and the result should still be consistent with state positivity. CP-maps are the most general (dissipative) transformations on quantum system and are alternatively call "quantum channels".

Given a CP-map Φ, there always exists a family of operators F_s on $\mathcal{H}$ such that the writing $\Phi(\rho) := \sum_s F_s \, \rho \, F_s^\dagger$ holds true. Of course such writing is not unique, it is called a "Kraus decomposition". If the CP-map is trace preserving, i.e. $\mathrm{Tr}\Phi(\rho) = 1$ for any state ρ, then the F_s's have to satisfied $\sum_s F_s^\dagger F_s = \mathbb{I}$. That is: the F_s's form a POVM.

Any CP-map can be viewed as coming from a unitary evolution on the system coupled to an auxiliary reservoir by tracing over the degree of freedom of the reservoir. This is expressed by the following theorem:

Theorem (Stinespring's theorem) "Auxiliary Reservoir":
Given a completely positive map Φ there exist a Hilbert space $\mathcal{K}$, a state ω on $\mathcal{K}$ and a unitary operator V on the tensor product $\mathcal{H} \otimes \mathcal{K}$ such that

$$\Phi(\rho) = \mathrm{Tr}_{\mathcal{K}}\big(V\,\rho \otimes \omega\, V^{\dagger}\big).$$

for any state ρ on $\mathcal{H}$.

In the case of the CP-map constructed via indirect measurement on probes the auxiliary reservoir is the probe system, the state is that in which the probes have been prepared and the unitary operator is that coding for the system-probe interaction. In formula this gives $\Phi(\rho) = \mathrm{Tr}(U\rho \otimes \rho_p U^{\dagger})$ because the kets $|s\rangle$ form a complete probe basis. This is the essence of the proof of the Stinepring's theorem in the finite dimensional case.

2.2 Repeated indirect measurements and progressive collapse

Repeated POVMs

We now repeat/iterate the previous indirect measurement, so that a series of probes recursively interacts with the system. Let $P_1, \cdots, P_n, \cdots$ be the series of probes. After measuring them, we got a series of measurement outputs $(s_1, \cdots, s_n, \cdots)$. Let ρ_n be the system density matrix after n step, that is after n probes have interacted with the system and have been measured. It is recursively - and randomly - updated according to

$$\rho_n \to \rho_{n+1} = \frac{F_s\,\rho_n\,F_s^{\dagger}}{\pi_n(s)}, \quad \text{with probability } \pi_n(s) = \mathrm{Tr}(F_s\,\rho_n\,F_s^{\dagger}).$$

Since the outputs s are random the series $(\rho_0, \rho_1, \cdots, \rho_n, \cdots)$ are also random. If one does not know the initial system state ρ_0, the only data/information available up to step n are the n first signals $(s_1, \cdots, s_n)$. These random sequences code for our knowledge on the system. If one knows ρ_0 and the sequences of outputs $(s_1, \cdots, s_n)$, one may then reconstruct all the intermediate system states $(\rho_0, \rho_1, \cdots, \rho_n)$.

Definition "(Discrete) Quantum Trajectories":
The random sequences of density matrices $(\rho_0, \rho_1, \cdots, \rho_n, \cdots)$ obtained by recursive applications of the random moves defined above are called "quantum trajectories". Quantum trajectories are specific to the chosen POVMs which may be modified at each step. Notice that quantum trajectories are classical random trajectories but in the space of quantum states.

Remark 1:
If we don't keep record of the measurement outputs we then have to look at the evolution of the mean density matrix $\bar{\rho}_n := \mathbb{E}[\rho_n]$. By construction this mean state is updated via the quantum channel Φ:

$$\bar{\rho}_n \to \bar{\rho}_{n+1} = \Phi(\bar{\rho}_n) = \Phi^{n+1}(\rho_0).$$

Convergence and asymptotic behaviors of such iteration depend on the spectral properties of the quantum channel. More below.

Remark 2:
In mathematical terms these data define a filtration on an appropriate probability space. The events of this probability space are sequences $(s_1, s_2, \cdots, s_n, \cdots)$, or equivalently the probability space $\mathcal{P}$ is the set of such infinite sequences. We can look at the sequences for which the n-th first entries are specified and define subspaces $\Omega_{s_1,\cdots,s_n}$ as the space of sequences starting with the specified sequences $(s_1, s_2, \cdots, s_n)$ for the first n data. Increasing n amounts to increase knowledge on the sequences. Hence, the σ-algebras $\mathcal{F}_n$ generated by these sets form an increasing series of σ-algebra, alias a filtration. Given the initial density matrix ρ_0, the space $\mathcal{P}$ is equipped with a probability measure by declaring that $\mathbb{P}[s_{n+1} = s|\mathcal{F}_n] = \mathrm{Tr}(F_s\,\rho_n\,F_s^\dagger)$.

The non-demolition condition

If we want the series of indirect measurements to be close to what a von Neumann measurement would do, we have to impose that it preserves a given basis of the system Hilbert space, which is then called the pointer state basis. Let us denote them by $|k\rangle$. That is: we have to demand that the updating preserves $|k\rangle$, that is $|k\rangle\langle k| \to |k\rangle\langle k|$ with probability one. It is easy to see that this demands that the interacting evolution operator to be of the following form:

$$U = \sum_k |k\rangle\langle k| \otimes U_k,$$

with the U_k's α-dependent unitary operator acting on the probes. Recall that U is a unitary operator acting on the tensor product of the system and probe Hilbert spaces.

Indeed such an operator clearly preserves the system pointer states $|k\rangle$, i.e. $U|k\rangle \otimes |\varphi\rangle = |k\rangle \otimes U_k|\varphi\rangle$, and U_k is the probe evolution conditioned on the system to be in the state k. Let $u_s(k) := \langle s|U_k|\varphi\rangle$. The F_s's are then diagonal in the pointer basis, $F_s|k\rangle = u_s(k)\,|k\rangle$, or equivalently

$$F_s = \sum_k |k\rangle\,\langle s|U_k|\varphi\rangle\,\langle k|.$$

Furthermore, $\mathrm{Tr}(F_s|k\rangle\langle k|F_s^\dagger) = |u_s(k)|^2$, so that, after the system-probe interaction, we get measurement output s with probability $p(s|k) := |u_s(k)|^2$ and $|k\rangle \to |k\rangle$ with probability 1.

Remark 1:
For k fixed, the numbers $p(s|k) := |\langle s|U_k|\varphi\rangle|^2$ specified a probability measure on the probe outputs, $\sum_s p(s|k) = 1$. These are the distributions of the outputs conditioned on the system to be in the pointer state $|k\rangle$. There is one such distribution for each pointer state.

Repeated non-demolition POVMs and convergence

Let us assume that the non-demolition condition is satisfied and let $\rho_n(j,k) := \langle j|\rho_n|k\rangle$ be the system density matrix in the pointer basis after n step. Let us give a special name for the diagonal matrix elements, $Q_n(k) := \rho_n(k,k)$. Because of the non-demolition condition the recursion formula is very simple. The diagonal elements transform as (we shall deal with the off diagonal a bit later)

$$Q_n(k) \to Q_{n+1}(k) = \frac{p(s|k)\,Q_n(k)}{\pi_n(s)},$$

with probability

$$\pi_n(s) = \mathrm{Tr}(F_s \rho_n F_s^\dagger) = \sum_k p(s|k)\, Q_n(k).$$

By construction – because $\mathrm{Tr}(\rho_n) = 1$ – the diagonal elements $Q_n(k)$ define a probability measure on the pointer states, that is $0 < Q_n(k) < 1$ and $\sum_\alpha Q_n(k) = 1$. This transformation thus specifies a random flow on probability measures. It is worth noticing to the updating simply consists in applying Bayes's rule for conditioned probabilities

We can now spell out the convergence theorem (which is a formalisation of cavity QED experiment on progressive collapse mentioned above):

Theorem (Bauer-Bernard) "Progressive collapse" :
Assume that the conditioned probabilities $s \to p(s|k)$ are all disjoints (that is: there does not exist a disjoint pair of pointer states k and k' such that $p(s|k) = p(s|k')$ for all outputs s). Then:
– The sequences $n \to Q_n(k)$ converge a.s. and in $\mathbb{L}^1$ for any k.
– The limit distribution is peaked: $Q_\infty(k) := \lim_{n\to\infty} Q_n(k) = \delta_{k;k_\infty}$ for some random target pointer k_∞.
– The random target k_∞ is distributed according to the initial distribution: $\mathbb{P}[k_\infty = k] = Q_0(k)$.
– The convergence to the target is exponential fast with

$$Q_n(k)/Q_n(k_\infty) \simeq \exp[-nS(k_\infty|k)],$$

with $S(k_\infty|k)$ the relative entropy $S(k_\infty|k) = -\sum_s p(s|k_\infty) \log\left[\frac{p(s|k)}{p(s|k_\infty)}\right]$.

Proof.
The proof is based on the fact that $Q_n(k)$ is a martingale. A process is (naively) a martingale if it is conserved in mean (demanding that it is conserved for each sample would be too much). Technically this property demands that the mean of the process at step $n+1$ conditioned on the information up to time n is its value at time n, i.e.

$$\mathbb{E}[Q_{n+1}(k)\,|\mathcal{F}_n] = Q_n(k), \quad \text{for all } k.$$

Let us compute baring in mind that the mean is computed by averaging on the possible output values at step $n+1$ with a distribution specified by the system state at time n, so:

$$\begin{aligned}\mathbb{E}[Q_{n+1}(k)\,|\mathcal{F}_n] &= \sum_s \frac{p(s|k)\, Q_n(k)}{\pi_n(s)}\, \pi_n(s) \\ &= \sum_s p(s|k)\, Q_n(k) = Q_n(k)\end{aligned}$$

A theorem in probability theory then says that a bounded martingale converges almost surely and in $\mathbb{L}^1$. The theorem is no so easy to prove but intuitively follows from the fact that the martingale property limits the possible fluctuations of the process. Hence, at k fixed the series $n \to Q_n(k)$ converges. Let $Q_\infty(k)$ be its limit. It has to satisfy the fixed point condition $p(s|k)Q_\infty(k) = Q_\infty(k)\,\pi_\infty(s)$ with $\pi_\infty(s) = \sum_k p(s|k)Q_\infty(k)$ for all s such that $\pi_\infty(s) \neq 0$. If the non-degeneracy condition imposing that all conditioned distribution $p(\dot{|}k)$ are different holds, the only solution to this fixed point condition is that $Q_\infty(k)$ is peaked. That is: there is a certain pointer state $|k_\infty\rangle$ such that $Q_\infty(k) = \delta_{k;k_\infty}$.

The target state $|k_\infty\rangle$ is random: it depends on the realization of the series of signals. Its statistics, its distribution, is again determined as a consequence of the martingale property because the probability that the target pointer has a given value k is:

$$\mathbb{P}[k_\infty = k] = \mathbb{E}[\delta_{k_\infty=k}] = \mathbb{E}[Q_\infty(k)] = Q_0(k),$$

where is the last equality we used, $Q_\infty(k) = \delta_{k_\infty=k}$, the martingale property and the fact that convergence is in $\mathbb{L}^1$.
The estimate for the large n behavior follows from an exact formula for the recursion relation defining the Q_n's:

$$Q_n(k) = Q_0(k)\,\frac{\prod_s p(s|k)^{N_n(s)}}{Z_n}$$

with $N_n(s)$ the number of the value s occurs in the n first output measurement and Z_n the normalisation factor $Z_n = \sum_k Q_0(k)\,\prod_s p(s|k)^{N_n(s)}$. Suppose that the sequence converges toward the target k_∞. Then,

$$\frac{Q_n(k)}{Q_n(k_\infty)} = \prod_s \big(\frac{p(s|k)}{p(s|k_\infty)}\big)^{N_n(s)} \simeq \prod_s \big(\frac{p(s|k)}{p(s|k_\infty)}\big)^{np(s|k_\infty)},$$

where in the last equality we used that asymptotically at large n, $N_n(s) \simeq np(s|k_\infty) + \cdots$ by the law of large numbers. This yields an exponential decrease with rate equal to the relative entropy. □

Remark 1:
Although we just argued that the system is projected onto one of the pointer state, one may wonder how to read on which pointer it is projected? The only available information is the series of outputs $(s_1, s_2, \cdots, s_n, \cdots)$. Thus the question is: how to read the target value from these sequences? The answer consists in looking at the histograms. Indeed, if the target is the state $|k_\infty\rangle$, the number $N_n(s)$ of time the value s appeared in the output signal after n step is $N_n(s) \simeq n\,p(s|k_\infty)$, asymptotically at large n. Hence, comparing the histogram of apparences of all possible output values s with the list of conditioned probabilities $p(s|k)$ allow to read what k_∞ is, provided that these conditioned probabilities are disjoint as we have assumed.

Remark 2:
The recursion relation used in the previous proposition started with the distribution $Q_0(k)$ read from the initial system state. Usually this initial state is unknown, since no information about the system has yet been extracted, and there is no way to recursively compute $Q_n(k)$. In absence of initial information, one may start with some a priori trial distribution, say $\widehat{Q}_0(k)$, which may differ from $Q_0(k)$, and update it recursively using Bayes' rules:

$$\widehat{Q}_n(k) \to \widehat{Q}_{n+1}(k) := \frac{p(s_{n+1}|k)\,\widehat{Q}_n(k)}{\widehat{Z}_n},$$

if s_{n+1} is the output signals at the $(n+1)$-the step. The distribution of the output signal is of course specified by the system state, hence by $Q_n(k)$ and not by the trial distribution $\widehat{Q}_n(k)$. As a consequence $\widehat{Q}_n(k)$ is not a martingale as is $Q_n(k)$. Nevertheless, one may prove that the trial distribution $\widehat{Q}_n(k)$ converges and possesses the same limit as the system state distribution $Q_n(k)$. In particular looking at the asymptotic behavior of the trial distribution provided another way to read what the target pointer state $|k_\infty\rangle$ is.

2.3 Decoherence, information and entropy

Decoherence or not?

If initially pure, the system state remains pure: not mixing and hence no decoherence although the off-diagonal matrix elements decrease exponentially. However decoherence is present if we don't record the probe outputs – because then the series of probes behaves a large reservoir. This can be made a bit more quantitative. Under the non-demolition hypothesis all matrix elements of the system density matrix evolve simply and multiplicatively according to

$$\rho_n(j,k) \to \rho_{n+1}(j,k) = \frac{u_s(j)\bar{u}_s(k)}{\pi_n(s)}\,\rho_n(j,k), \quad \text{with proba } \pi_n(s) = \sum_k p(s|k)Q_n(k).$$

If we do not keep track of the output signal we have to average over the output values s. The mean then evolves according to $\bar{\rho}_{n+1}(j,k) \to \langle\varphi|U_\alpha^* U_\beta|\varphi\rangle\,\bar{\rho}_n(j,k)$. Hence

$$\bar{\rho}_n(j,k) = [\langle\varphi|U_\alpha^* U_\beta|\varphi\rangle]^n\,\rho_0(\alpha,\beta) \to 0, \quad \text{exponentially.}$$

Remark 1:
The decoherence time $\bar{\rho}_n(j,k)$ is of order $-\frac{1}{2}\log|\langle\varphi|U_j^* U_k|\varphi\rangle|^2$. An upper bound, which we may take as an approximation, is $-\log(\sum_s \sqrt{p(s|j)p(s|k)}) = -\log(\vec{P}_j\cdot\vec{P}_k)$, with $\vec{P}_k$ the real vector with components $p(s|k)$. From usual estimates - or lore - one infers that the decoherence time is inversely proportional to the square distance between the constituents of the system. So, we could naively argue that $-1/\log[\vec{P}_j\cdot\vec{P}_k]$ provides a measure of the square distance between the pointer states $|j\rangle$ et $|k\rangle$, although these pointer states are not directly related to positions. Thsi distance may be viewed as defining a geometry over pointer states. Of course - may be unfortunately - the geometry induced by this notion of distance depends on the nature of the interaction with the surrounding (it depends on the interaction operator U).

Information and entropy balance

If initially pure, the system state remains pure and its von Neumann entropy vanishes at all step,

$$S_{\rm vN}(n) := -\mathrm{Tr}\,(\rho_n\log\rho_n) = 0.$$

However, we may define another entropy, which we call the extrinsic entropy, via

$$S_{\rm ext}(n) := -\sum_k Q_n(k)\log Q_n(k) \neq 0.$$

It is non zero unless the system is in a pointer state. It is of course relative to the pointer basis, but it possesses a nice interpretation in term of information theory.

Indeed, we have two sets of random variables, the k's, with probability distribution $Q_n(k)$ at step n, and the s's, the probe outputs at step $n+1$, with probability distribution $\pi_n(s) = \sum_k p(s|k)Q_n(k)$. The joint distribution is $P_n(s;k) := p(s|k)Q_n(k)$. A simple computation (detailed below) tells us that the extrinsic entropy monotically decreases in mean along repeated non-demolition measurements (it is a super-martingale) and that its variation is coded into the

so-called mutual information between the pointer state and the output distributions. Namely, we have

$$\mathbb{E}\big[S_{\rm ext}(n+1) - S_{\rm ext}(n)|\mathcal{F}_n\big] = -\mathcal{I}_n(s|k) \leq 0,$$

with $\mathcal{I}_n(s|k) = \sum_{s,k} P_n(s,k) \log\big[\frac{P_n(s,k)}{\pi_n(s)Q_n(k)}\big]$.

Recall that information theory naturally measures the correlation between two set of variables via the mutual information. Given two random variables X and Y with probability distribution $p(x,y)$ and marginales $p(x)$ and $p(y)$, the mutual information is defined as

$$\mathcal{I}(X|Y) := \sum_{x,y} p(x,y) \log\Big[\frac{p(x,y)}{p(x)p(y)}\Big] = S(X) + S(Y) - S(X,Y).$$

It is also equal to the relative entropie of X, Y with distribution $p(x,y)$ and X, Y distributed with the product distribution $p(x)p(y)$. It measures the amount of information that can be obtained from one variable by observing the other[1]. This is indeed what the previous equation codes for: the mean uncertainty on k at step $n+1$, measured by the entropy $S_{\rm ext}(n+1)$, equals that at step n, given by $S_{\rm ext}(n)$, minus the amount of information $\mathcal{I}_n(s|k)$ gained by knowing the $(n+1)$-th output value s.

Let us do the computation. At step $n+1$, we have $S_{\rm ext}(n+1) = -\sum_k Q_{n+1}(k) \log Q_{n+1}(k)$ with $Q_{n+1}(k) = P_n(s,k)/\pi_n(s)$ with probability $\pi_n(s)$. Its mean conditioned on the information $\mathcal{F}_n$ up to step n is $\mathbb{E}[S_{\rm ext}(n+1)|\mathcal{F}_n] = -\sum_{s,k} \pi_n(s) Q_{n+1}(k) \log Q_{n+1}(k)$. Since $Q_{n+1}(k) = P_n(s,k)/\pi_n(s)$ this may be rewritten as

$$\begin{aligned}
\mathbb{E}[S_{\rm ext}(n+1)|\mathcal{F}_n] &= -\sum_{s,k} P_n(s,k) \log[P_n(s,k)/\pi_n(s)] \\
&= -\sum_{s,k} P_n(s,k) \log\Big[\frac{P_n(s,k)}{\pi_n(s)Q_n(k)}\Big] + S_{\rm ext}(n)
\end{aligned}$$

where we used $\sum_s P_n(s,k) = Q_n(k)$ in the last step.

3 Open quantum walks

3.1 Open quantum walks

Repeated indirect measurements as random walks

Repeated indirect measurements can be seen as kinds of quantum random walks by associating values to the output signals. The walker moves randomly on a graph or a lattice, whose specification depends on which kind of information we keep. Contrary to usual classical random walks, the walker carries internal degrees of freedom which are encoded into the state of the

[1]Recall that the mutual information is positive because the log-function is concave. Indeed

$$\mathcal{I}(X|Y) = -\mathbb{E}\big[\log\big(\frac{p(x)p(y)}{p(x,y)}\big)\big] \geq -\log\big(\mathbb{E}\big[\frac{p(x)p(y)}{p(x,y)}\big]\big) = -\log\big(\sum_{x,y} p(x,y)\frac{p(x)p(y)}{p(x,y)}\big) = 0$$

monitored quantum system. The position of the walker and its internal state are both up-dated at each step with probability rules which depend on the value of the internal state but which are borrowed from indirect measurement experiments.

Let us give a few simple examples. Suppose that at each step of repeated indirect measurements we only retain the last digit s_n from the series of the first n outputs $(s_1, \cdots, s_n)$, we may then choose that the walk takes place on the complete graph with vertices indexed by the possible value s of outputs with the rule that if the n-th output is s_n then, by definition, the walker jumps on site s_n. The internal state is then up-dated as in repeated indirect measurements, $\rho \to F_{s_n}\rho F_{s_n}^\dagger/\mathrm{Tr}(F_{s_n}\rho F_{s_n}^\dagger)$. Suppose on contrary that we keep all information about the output signals so that the vertices of the graph are indexed by the sequences $(s_1, \cdots, s_n)$ of finite but arbitrary length. These naturally form an infinite tree, the descendants of a vertex corresponding to a given sequence of length n are all the sequences of length $n+1$ with their n-th first entrees being fixed by the original sequence. By definition the walker then moves randomly downwards on the tree and at each step its internal state is up-dated according to the rules of indirect measurements. Of course we can also imagine intermediate examples in which some partial information on the outputs (more than only the last digit but less than the complete series) is kept. Concrete examples will be given below.

Open random walks

Let us formalize the definition following ref.[**?**]. We need to introduce a graph on which the walker moves and a quantum system whose states describe the walker's internal degrees of freedom – sometimes called the walker's quantum gyroscope.

Definition (Attal et al) "(Quantum trajectories of) Open quantum walk":
Let Λ be an oriented graph. Let $\mathcal{H}$ be a Hilbert space. Let B_{xy} be (bounded) operators B_{xy} on $\mathcal{H}$ associated to any edge $x \to y$ of Λ such that $\sum_y B_{xy}^* B_{xy} = \mathbb{I}$ for all x. Let (x, ρ) be the position $x \in \Lambda$ and the internal state ρ of the walker. The open quantum walk (OQW) with transition matrices B_{xy} is the Markov chain defined by the moves

$$(x, \rho) \to (y, \frac{B_{xy}\,\rho\, B_{xy}^*}{\pi_{xy}}), \quad \text{with probability } \pi_{xy} = \mathrm{Tr}(B_{xy}\rho B_{xy}^*).$$

Let us give an example. Take $\mathbb{Z}$ as graph, $\mathbb{C}^2$ as internal space, and choose two matrices B_+ (resp. B_-) associated to steps to the right (resp. to the left), which are supposed to satisfy $B_+^* B_+ + B_-^* B_- = \mathbb{I}$. Let (x_n, ρ_n) be the position of the walker and its internal state at step n. The moves are:

$$(x_n, \rho_n) \to \Big(x_{n+1} = x_n + \epsilon_{n+1}, \rho_{n+1} = \frac{B_{\epsilon_{n+1}}\rho_n B_{\epsilon_{n+1}}^*}{\pi_n(\epsilon_{n+1})}\Big).$$

with

$$\epsilon_{n+1} = \pm \quad \text{with probability } \pi_n(\pm) = \mathrm{Tr}(B_\pm \rho_n B_\pm^*).$$

Note that the position of the walker is $x_n = \sum_j \epsilon_j$ and that $\epsilon = +$ (resp. $\epsilon = -$) corresponds to a move to the right (resp. left). This corresponds to a homogeneous open quantum walk on the line. Generalization to higher dimension, say to $\mathbb{Z}^d$, is simple.

The relation with repeated POVMs is simple. The probes are Qu-bits – two level systems – with basis $|\pm\rangle$. They interact recursively with the quantum system – which plays the role of the walker's internal quantum gyroscope. The walker moves to the left/right depending on the probe measurement output which can be either $+$ and $-$. Hence the information which is kept is the sum of the outputs.

The OQW trajectories may be considered from dual points of view: either one is interested in the internal quantum system behavior and focuses on the information extracted from the repeated POVMs, or one insists looking at the walker's position, say for geometrical purposes. This leads to two alternative presentations of OQWs, see below.

Remark 1:
Given x, the set of operators B_{xy} specifies a POVM, because they satisfy $\sum_y B_{xy}^* B_{xy} = \mathbb{I}$. Thus, we can view the OQW has an iteration of non homogeneous POVMs whose characteristics depend on the previous output signals. This is clearly reminiscent of feedback procedures: the action depends on the past information.

Remark 2:
To any OQW is associated a completely positive map acting on states in $\mathcal{H} \otimes \mathbb{L}^2(\Lambda)$. We restrict ourselves to (extended) density matrix of the form $\sum_x \rho_x \otimes |x\rangle\langle x|$. The normalization condition is $\sum_x \mathrm{Tr}(\rho_x) = 1$. The CP-map is then defined by

$$\mathfrak{P}\Big(\sum_x \rho_x \otimes |x\rangle\langle x|\Big) = \sum_{x;y} (B_{xy}\rho_x B_{xy}^*) \otimes |y\rangle\langle y|.$$

or equivalently

$$\mathfrak{P}(\rho)_x = \sum_y B_{yx}\rho_y B_{yx}^*.$$

It is trace preserving because $\sum_{x;y} \mathrm{Tr}(B_{xy}\rho_x B_{xy}^*) = \sum_x \mathrm{Tr}(\rho_x) = 1$.
Given an OQW (x_n, ρ_n) we may tautologically define density matrices $\rho_n \otimes |x_n\rangle\langle x_n|$. Their means are an (extended) density matrices defined through $\sum_x \bar{\rho}_x(n) \otimes |x\rangle\langle x| := \mathbb{E}[\rho_n \otimes |x_n\rangle\langle x_n|]$. Looking at one step forward for each sample and averaging we get that

$$\bar{\rho}_x(n+1) = \mathfrak{P}\big(\bar{\rho}(n)\big)_x.$$

Thus the CP-map $\mathfrak{P}$ is that governing the evolution of those mean (extended) density matrices.

Remark 3:
Alternatively, OQWs may be obtained by iterating alternatively position measurement and the map $\mathfrak{P}$. Indeed, let us start with the (extended) density matrix $\rho \otimes |x\rangle\langle x|$, localized at a given position x, and let us transform it with the map $\mathfrak{P}$ to obtain $\sum_y (B_{xy}\rho B_{xy}^*) \otimes |y\rangle\langle y|$. The probability to find y as result of a Von Neumann measurement of the position is $\mathrm{Tr}(B_{xy}\rho B_{xy}^*) = \pi_{xy}$ and the projected state after the measurement is $\frac{B_{xy}\rho B_{xy}^*}{\pi_{xy}} \otimes |y\rangle\langle y|$ if y is the result of the position measurement.

Remark 4:
To be more precise what we have defined is the notion of quantum trajectories for a specific quantum process – which will be described below, see "dilation of open quantum walks".

3.2 Basics: ergodicity, irreducibility, detailed balance

Many notions and properties of classical Markov chains apply to open quantum walks. We give here a few samples of those.

More examples

(i) Simple example of trapped trajectories:
Take $\Lambda = \mathbb{Z}$ and $\mathcal{H} = \mathbb{C}^2$ with canonical basis $|\uparrow\downarrow\rangle$, and transition matrices $B^*_{x;x-1} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $B^*_{x;x+1} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Because these matrices square to 0, the walk is very simple: a spin up $|\uparrow\rangle$ is transfered one step to the left into a spin down $|\downarrow\rangle$, and reciprocally. Hence, after one step any walker is trapped on two sites and oscillates between those two sites with its spin flipping at each step.

(ii) State engineering:
.....

(iii) Non-demolition iterated measurements as OQW:
In this example $\mathcal{H} = \mathbb{C}^2$, $\Lambda = \mathbb{Z}$ and the transition matrices $B_\pm := B_{x;x\pm1}$ are diagonal – this is the non-demolition hypothesis. Let us parametrize them as $B_\pm = \left(\begin{smallmatrix} \sqrt{p_\pm} & 0 \\ 0 & \sqrt{q_\pm} \end{smallmatrix}\right)$ with $p_+ + p_- = 1$ and $q_+ + q_- = 1$. Clearly $B^*_+B_+ + B^*_-B_- = \mathbb{I}$. We assume that $p_\pm \neq q_\pm$.

Suppose, for simplicity, that the initial density matrix ρ_0 is diagonal. Then all its iterations ρ_n are also diagonal. Let us parametrize them as $\rho_n = \left(\begin{smallmatrix} Q_n & 0 \\ 0 & 1-Q_n \end{smallmatrix}\right)$. Let (x_n, ρ_n) be an OQW trajectory. The up-datings are

$$(x_n, Q_n) \to \big(x_{n+1} = x_n \pm 1, Q_{n+1} = p_\pm\, Q_n/\pi_n(\pm)\big),$$

with probability $\pi_n(\pm) = p_\pm Q_n + q_\pm(1-Q_n)$. As in the first section on repeated non-demolition measurement, this recursion relation can be solved explicitly:

$$Q_n = Q_0\, p_+^{N_n(+)}\; p_-^{N_n(-)}/Z_n$$

where $Z_n = Q_0\, p_+^{N_n(+)} p_-^{N_n(-)} + (1-Q_0)\, q_+^{N_n(+)} q_-^{N_n(-)}$ with $N_\pm(n)$ the number of right/left steps after n iteration. By definition the sum $N_n(+)+N_n(-) = n$ and the difference $N_n(+)-N_n(-) = x_n - x_0$ is the walker displacement. The distribution of series of outputs $(\epsilon_1, \cdots, \epsilon_n)$ can be computed exactly by recursion:

$$\mathbb{P}[(\epsilon_1, \cdots, \epsilon_n)] = Q_0\, p_+^k p_-^{n-k} + (1-Q_0)\, q_+^k q_-^{n-k},$$

with k the number of + is the sequence. Alternatively, the law of $N_n(+)$ and $N_n(-)$ is:

$$\mathbb{P}\big[N_n(+) = k, N_n(-) = n-k\big] = Q_0 \tbinom{n}{k} p_+^k p_-^{n-k} + (1-Q_0) \tbinom{n}{k} q_+^k q_-^{n-k}.$$

Hence, the distribution of the number of steps is the sum of two binomial distributions, – i.e. the sum two classical random walks –, with respective parameters $p_\pm$ and $q_\pm$ and weights Q_0 and $1-Q_0$.

Ergodicity

Ergodicity theorems have been proven for CP-maps in a quite general setting. The theorem we are about to quote is a consequence of one of those proven in [**?**]. At a first naif level, ergodicity statements are consequences of the observation that the time average of a dynamical map is an invariant. In the case of OQWs, this amounts to consider the CP-map $\mathfrak{P}$, defined by $\mathfrak{P}(\rho_x \otimes |x\rangle\langle x|) = \sum_y (B_{xy}\rho_x B_{xy}^*) \otimes |y\rangle\langle y|$ and its iterations. The time average, with initial data $\rho \otimes |x_0\rangle\langle x_0|$, is defined by

$$S_n := \frac{1}{n}\sum_{k=0}^{n} \mathfrak{P}^k\big(\rho \otimes |x_0\rangle\langle x_0|\big).$$

It is clear that, if the sum converges, the limit is invariant: $\mathfrak{P}(S_\infty) = 0$. Since the CP-map $\mathfrak{P}$ governs the mean behavior of OQWs, this simple observation gives information on the time average of the mean behavior of OQWs. As it was realized for general CP-map, this information can be promoted into a path-wise ergodicity statement – i.e. sample by sample and not only in mean:

Theorem (Kummerer-Maassen) "Ergodicity":
Let (x_n, ρ_n) an OQW. Then, we have the almost sure convergence of its time average:

$$\frac{1}{n}\sum_{k=0}^{n} \rho_k \otimes |x_k\rangle\langle x_k| \to \sum_x \rho_x^{\rm inv} \otimes |x\rangle\langle x|, \quad \text{a.s.},$$

when $n \to \infty$, with $\mathfrak{P}(\rho^{\rm inv}) = 0$, or equivalent $\sum_x B_{xy}\rho_x^{\rm inv} B_{xy}^* = \rho_y^{\rm inv}$.

Of course when there is more than one invariant state, $\rho_y^{\rm inv}$ is a random variable, i.e. it depends of the realization. For finite OQWs the number of invariant state is finite, so that $\rho_y^{\rm inv}$ can be decomposed on a basis of invariant states. The coefficients of this decomposition are random. This is for instance what happens in the case of non-demolition measurements.

This theorem has different formulations or corollaries. First it is equivalent to claiming that

$$\frac{1}{n}\sum_{k=0}^{n} \rho_k \, \mathbb{I}_{x_k=y} \to \rho_y^{\rm inv} \quad \text{a.s.}$$

Taking the trace of the previous equation gives information on the frequency that a given site has been visited by the walk. Let $N_n(y)$ the number of time the walker visits the site y in the n first step, that is $N_n(y) = \sum_{k=0}^{n} \mathbb{I}_{x_k=y}$. Then,

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \mathrm{Tr}(\rho_y^{\rm inv}),$$

or equivalently $N_n(y) \simeq n\,\mathrm{Tr}(\rho_y^{\rm inv})$ at large n. As a consequence, one can reconstruct the normalized invariant state $\rho_y^{\rm inv}/\mathrm{Tr}(\rho_y^{\rm inv})$ by looking at the histogram of the visit of the site y. Namely,

$$\lim_{n\to\infty} \frac{1}{N_n(y)}\sum_{k=0}^{n} \rho_k \, \mathbb{I}_{x_k=y} = \frac{\rho_y^{\rm inv}}{\mathrm{Tr}(\rho_y^{\rm inv})}.$$

Proof.
A nice proof is given in [**?**]. It is based on a Doob decomposition. To simplify notations let

$\hat{\rho}_k = \rho_k \otimes |x_k\rangle\langle x_k|$. Consider the sum of its iteration $\sum_{k=0}^n \hat{\rho}_k$. Since $\mathbb{E}[\hat{\rho}_n|\mathcal{F}_{n-1}] = \mathfrak{P}(\hat{\rho}_{n-1})$, its Doob decomposition is

$$\sum_{k=0}^{n} \hat{\rho}_k = M_n + \sum_{k=0}^{n-1} \mathfrak{P}(\hat{\rho}_k),$$

with M_n a martingale, $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$. From the law of large for martingales and boundness arguments it follows that $\frac{1}{n} M_n \to 0$ and hence that $\frac{1}{n}\sum_{k=0}^n \hat{\rho}_k - \frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{P}(\hat{\rho}_k)$ converges to 0. By iteration, this implies that $\frac{1}{n}\sum_{k=0}^n \hat{\rho}_k - \frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{P}^m(\hat{\rho}_k)$ converges to 0 for any m. By summation this then implies that $\frac{1}{n}\sum_{k=0}^n \hat{\rho}_k - \frac{1}{n}\sum_{k=0}^{n-1} \left(\frac{1}{M}\sum_{m=0}^{M-1} \mathfrak{P}^m\right)(\hat{\rho}_k)$ also converges to 0. Now, the operation $\frac{1}{M}\sum_{m=0}^M \mathfrak{P}^m$ project on invariant states. Hence, $\frac{1}{n}\sum_{k=0}^n \hat{\rho}_k$ converges to an invariant state. □

Remark 1:
Notice that this theorem has a larger domain of application than the progressive collapse theorem that we quoted earlier because it is applicable to a more general setting than non-demolition measurements. But it is less precise than this previous theorem because it gives information about the convergences of the time average whereas the progressive collapse theorem provides detailed information about the convergence of the process and about the random limit invariant state.

<u>*Detailed balance*</u>

As for the classical random walks there is a simple notion of detailed balance. As in the classical case, this property implies the existence of a simple invariant state if normalizable.

Definition-Proposition "Detailed balance":
(i) Detailed balance is said to be fulfilled if there exists a family of operators μ_x, $x \in \Lambda$, acting on $\mathcal{H}$ such that $B_{yx}\mu_y = \mu_x B_{xy}^*$.
(ii) In such case, $\rho_x^{\rm inv} := \mu_x\mu_x^*$ is $\mathfrak{P}$-invariant: $\mathfrak{P}(\rho^{\rm inv}) = \rho^{\rm inv}$.
(iii) If $\mathfrak{z} := \sum_x {\rm Tr}(\mu_x\mu_x^*) < \infty$, then $\rho_x^{\rm inv} := \mu_x\mu_x^*/\mathfrak{z}$ is a normalized invariant state in $\mathcal{H}\otimes\mathbb{L}^2(\Lambda)$.
(iv) If μ_x fulfills of the detailed balance condition, so does μ_x^*. Hence, we may choose μ_x hermitian.

Proof.
The proof is very simple. The hypothetical relations are $B_{yx}\mu_y = \mu_x B_{xy}^*$ and thus $\mu_y^* B_{yx}^* = B_{xy}\mu_x^*$. This implies that μ_x^* is solution of the detailed balance condition. Furthermore

$$\mathfrak{P}(\mu\mu^*)_x = \sum_y B_{yx}\mu_y\mu_y^* B_{yx}^* = \sum_y \mu_x B_{xy}^* B_{xy}\mu_x^* = \mu_x\mu_x^*,$$

because $\sum_y B_{xy}^* B_{xy} = \mathbb{I}$. Hence $\mathfrak{P}(\mu\mu^*)_x = \mu_x\mu_x^*$ and similarly $\mathfrak{P}(\mu^*\mu)_x = \mu_x^*\mu_x$. □

As in the classical case, the detailed balance condition has a simple and natural interpretation in terms of time reversal. Consider a path $\Omega = (x_0, x_1, \cdots, x_{\ell-1}, x_\ell)$ from x_0 to x_ℓ. The probability for the walker, starting at x_0 with an internal state ρ_0, to follow the path Ω is

$$\mathbb{P}[\Omega_{x_0\to x_\ell}\big|x_0, \rho_0] = {\rm Tr}(B_\Omega\, \rho_0\, B_\Omega^*).$$

where B_Ω^* be the transfer matrix along these paths $B_\Omega^* = B_{x_0,x_1}^* \cdots B_{x_{\ell-1},x_\ell}^*$.
Consider now the reversed path $\overline{\Omega} = (x_\ell, x_{\ell-1}, \cdots, x_1, x_0)$ from x_ℓ to x_0, and its associated

transfer matrix $B^*_{\overline{\Omega}} = B^*_{x_\ell - 1, x_\ell} \cdots B^*_{x_0, x_1}$. The detailed balance intertwines them B_Ω and $B_{\overline{\Omega}}$

$$\mu_{x_0} B^*_\Omega = B_{\overline{\Omega}} \mu_{x_\ell}.$$

Hence, $\mathrm{Tr}(B_\Omega \mu^2_{x_0} B^*_\Omega) = \mathrm{Tr}(B_{\overline{\Omega}} \mu^2_{x_\ell} B^*_{\overline{\Omega}})$. Choosing μ_x to be hermitian, this translates into a statement about probabilities of visiting reversed paths:

$$\mathbb{P}[\Omega_{x_0 \to x_\ell} | x_0, \rho_0 = \mu^2_{x_0}/\mathrm{Tr}(\mu^2_{x_0})] \times \mathrm{Tr}(\mu^2_{x_0}) = \mathbb{P}[\overline{\Omega}_{x_\ell \to x_0} | x_\ell, \rho_\ell = \mu^2_{x_\ell}/\mathrm{Tr}(\mu^2_{x_\ell})] \times \mathrm{Tr}(\mu^2_{x_\ell}).$$

That is: the ratio of the probabilities to visit a path and its time reversed is proportional to the ratio the asymptotic frequencies of visits of the final and initial points, as usual with classical detailed balance.

Remark 1:
If it exists, μ_x intertwines $\mathfrak{P}$ and its dual $\mathfrak{P}^*$, in the sense that $\mathfrak{P}(\mu M \mu^*)_x = \mu_x \mathfrak{P}^*(M)_x \mu^*_x$.

Remark 2:
This notion of detailed balance can clearly be extended to any completely positive map. Suppose that we are giving ourself a completely positive map Φ whose action on states is given by the following Kraus decomposition $\Phi(\rho) = \sum_\alpha B_\alpha \rho B^*_\alpha$ with $\sum_\alpha B^*_\alpha B_\alpha = \mathbb{I}$. We say that detailed balance is fulfilled if there exists an operator μ and c-number matrix $C_{\beta\alpha}$ such that $\sum_\alpha C_{\beta\alpha} C^*_{\gamma\alpha} = \delta_{\beta\gamma}$ (i.e. $CC^\dagger = 1$) such that $B_\alpha \mu = \mu B^*_\beta C_{\beta\alpha}$. In such case, the operator μ intertwines Φ and its dual Φ^* via $\Phi(\mu O \mu^*) = \mu \Phi^*(O) \mu^*$ and, if $\mathrm{Tr}(\mu\mu^*) < \infty$, then $\rho_{\mathrm{inv}} := \mu\mu^*/\mathrm{Tr}(\mu\mu^*)$ is an invariant state for Φ.

Irreducibility

A relevant notion is the irreducibility of $\mathfrak{P}$ because it allows to decompose any OQW in elementary blocks.

Definition "Irreducibility of OQW":
Consider an open quantum walk on a graph Λ with transition matrices $B_{x,y}$ for $x, y \in \Lambda$ and let B_Ω be the transfer matrix associated to a path Ω as above. We say that this OQW is irreducible if and only if the map $\mathfrak{P}$ is irreducible in the sense of Davies [?], or equivalently (see [?]) if for any x, y in Λ and u, v in $\mathcal{H}$, there exists a path from x to y such that $\langle v, B_\Omega u \rangle \neq 0$.

This means that if we pick any pair of vectors in the internal walker space at two different positions on the graph, there always exists a path such the transport of the vector in the initial position to the final position has a non vanishing component along the chosen vector at the final position. In other words, an OQW is said to be irreducible if no internal subspace is left when transporting vectors along random paths.

For an irreducible OQW to possess an invariant state it is enough that the CP-map $\mathfrak{P}$ possesses an eigenvalue 1. In this case the invariant state is unique and faithful in the sense that its eigenvalues are all non zero – again no thing is left in the invariant state.

Irreducible OQWs are building blocks of non irreducible OQWs.

Proposition (Carbonne-Pautrat) "Decomposition of OQW":
Consider an open quantum walk on a graph Λ, and assume that it possesses a faithful invariant state. Then it may be decomposed in a collection of M irreducible open random walks as follows:

– Its Hilbert space decomposes in direct sum, $\mathcal{H}\otimes L^2(\Lambda)=\oplus_{m=1}^M\mathcal{H}^{(m)}$, such that the restriction $\mathfrak{P}^{(m)}$ of $\mathfrak{P}$ on $\mathcal{H}^{(m)}$ is irreducible;
– For any m, the space $\mathcal{H}^{(m)}$ decomposes as a collection of internal subspaces $\mathfrak{h}_x^{(m)}\subset\mathcal{H}$, with $\mathfrak{h}_x^{(m)}$ localized at point $x\in\Lambda$, such that $B_{xy}\mathfrak{h}_x^{(m)}\subset\mathfrak{h}_y^{(m)}$.

This means that we can decompose the total space $\mathcal{H}\otimes L^2(\Lambda)$ in pieces. Each component is made of collection of internal subspaces $\mathfrak{h}_x^{(m)}$ indexed by points of the original graph. These subspaces may be not identical at different points, that is we may have $\mathfrak{h}_x^{(m)}\neq\mathfrak{h}_x^{(m)}$ for $x\neq y$. They also may be not all non trivial. This means that a subgraph $\Lambda^{(m)}=\{x\in\Lambda\big|\mathfrak{h}_x^{(m)}\neq\{0\}\}$ is associated to each of the component. Each collection subspace $\mathfrak{h}_x^{(m)}$ is stable under transport along any path by the transfer matrices of the original OQW, and $\mathcal{H}^{(m)}=\oplus_{x\in\Lambda^{(m)}}\mathfrak{h}_x^{(m)}\otimes\mathbb{C}|x\rangle$.

Instead of proving this decomposition - but see ref.[**?**] -, let us take the very simple example of trapped trajectory we discussed before. Take $\Lambda=\mathbb{Z}$ and $\mathcal{H}=\mathbb{C}^2$ with canonical basis $|\uparrow\downarrow\rangle$, and transition matrices

$$B^*_{x;x-1}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad B^*_{x;x+1}=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

Because the transition matrices are so simple, a spin up $|\uparrow\rangle$ is transfer one step to the left into a spin down $|\downarrow\rangle$, and reciprocally. Hence we decompose this OQW into two irreducible components with identical sublattices $\Lambda^{(1)}=\Lambda^{(2)}=\Lambda$ and Hilbert spacs

$$\mathfrak{h}^{(1)}_{2n}=\mathbb{C}\,|\uparrow\rangle,\ \mathfrak{h}^{(1)}_{2n+1}=\mathbb{C}\,|\downarrow\rangle\text{ and }\mathfrak{h}^{(2)}_{2n}=\mathbb{C}\,|\downarrow\rangle,\ \mathfrak{h}^{(1)}_{2n+1}=\mathbb{C}\,|\uparrow\rangle.$$

Notice that this zigzag decomposition is not unique.

3.3 Geometrisation and dilation

Geometrisation

As it is hopefully clear from the previous discussion, open random walks may be viewed with different perspectives, one of those puts emphasis on the position related observables. This suggests that we can change gears and adopt a more geometrical point of view by transposing the geometric constructions of classical stochastic processes to the quantum ones, and in particular to open quantum walks. Of course such extensions of probability concepts to quantum processes has some flavors of non-commutative geometry. Natural questions are for instance: Can classical potential theory be transposed to open quantum walks? What are harmonic measures for quantum stochastic process? See however ref.[**?**] for answer in the context of open random walks. Etc... Another direction for the geometrization of quantum processes incorporating geometrical data, as OQW does, consists in using them to test the geometry of those data. For instance, it is well know that the 2D Brownian motion is conformally invariant. One may wonder [**?**] whether its open quantum analogue can be made conformally invariant?

Dilation and a simple quantum stochastic process

We now use the example of open quantum walks on the line (on $\mathbb{Z}$) to introduce the notion of quantum stochastic processes which can be seen as non-commutative extensions of classical stochastic processes. To make a long story short these processes correspond to the OQW

construction with series of probes interacting with a quantum walker but with the (important) modification that the probes are not measured after they have interacted with the walker.

So, we may imagine preparing the system in a state ρ and an (infinite) series of probes all in the identical pure state $|\varphi_p\rangle$. If $\mathcal{H}_{\rm sys}$ is the system Hilbert space and $\mathbb{H}_{\rm probe}$ the probe Hilbert space, the total Hilbert for the system and the collection of probes $\mathcal{H}_{\rm sys} \otimes \mathcal{H}_{\rm probe}^{\otimes\infty}$. In the case of OQW on the line $\mathcal{H}_{\rm sys} = L^2(\mathbb{Z}) \otimes \mathbb{C}^2$ and $\mathcal{H}_{\rm probe} = \mathbb{C}^2$. Such an infinite product does not make sense unless a specified state is singled-out. In our case, this state is the initial state of the system and the collection of probes, that is

$$\rho_{\rm sys}^0 \otimes \rho_{\rm p} \otimes \cdots \otimes \rho_{\rm p} \otimes \cdots = \rho_{\rm sys}^0 \otimes \rho_{\rm p}^{\otimes\infty} =: \Omega,$$

with $\rho_{\rm sys}^0$ and $\rho_{\rm p}$ the system and probes initial states respectively. For instance $\rho_{\rm p} = |\varphi_p\rangle\langle\varphi_p|$. When a finite, but arbitrary large, number of probes has interacted with the system the state Ω is locally modified – in the sense that there is still an infinite queue of probes in the state $\rho_{\rm p}$. The Hilbert space $\mathcal{H}_{\rm sys} \otimes \mathcal{H}_{\rm probe}^{\otimes\infty}$ describes such local modifications, in way analogous to the fact that states of a Fock space describe local excitations above a vacuum state.

By construction the evolution is given by the successive iteration of system-probe interactions, as for repeated POVM (but without probe measurements). To describe it we have to introduce a unitary operator U, acting on the tensor product $\mathcal{H}_{\rm sys} \otimes \mathcal{H}_{\rm probe}$, and we have to give a name for the operator describing the interaction of the system with the n-th probe. We call it $U_{0;n}$. It acts non trivially on the system Hilbert space and on the n-th copy of the probe Hilbert space and trivially on the other copy of $\mathcal{H}_{\rm probe}$. The evolution of the total state is thus given by the evolution

$$\Omega \to U_{0;1}\,\Omega\,U_{0;1}^\dagger \to U_{0;2}U_{0;1}\,\Omega\,U_{0;1}^\dagger U_{0;2}^\dagger \to \cdots .$$

After n steps the system and the n first probes are entangled and the density matrix of the total system is of the form

$$\rho_n^{\rm tot} \otimes \rho_{\rm p} \otimes \rho_{\rm p} \otimes \cdots ,$$

with $\rho_n^{\rm tot}$ the entangled state of the system and the first n probes produced by the n first successive interactions. At the next step, the evolution produces the state $\rho_{n+1}^{\rm tot} \otimes \rho_{\rm p} \otimes \cdots$ with

$$\rho_{n+1}^{\rm tot} = U_{0;n+1}\left(\rho_n^{\rm tot} \otimes \rho_{\rm p}\right) U_{0;n+1}^\dagger,$$

and all probes after the $(n+1)$-th remain untouched.

When successively increasing the number of probes having interacted with the system we increase the algebra of possible observables, that is we increase the algebra of possible observations hence of possible information gains. This series of increasing algebras is the quantum analogue of the notion of filtration in classical stochastic processes. If at each step we measure an observable on the last probes we are back to quantum trajectories.

Quantum stochastic processes with classical random walks

As an illustration, let us look at the case of a quantum walker with trivial internal space. The OQW trajectories are then that of classical random walks. What is the corresponding quantum process?

For a walk on the line, the system Hilbert space is $L^2(\mathbb{Z})$ with basis $|x\rangle$, $x \in \mathbb{Z}$. The probe Hilbert space is $\mathbb{C}^2$ with basis $|\pm\rangle$. Suppose that all probes as prepared in a pure state $|\varphi_p\rangle$, say $|\varphi_p\rangle = \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle]$. The system-probe interaction is supposed to be such that the unitary operator U is

$$U|x\rangle \otimes |\varphi_p\rangle = \frac{1}{\sqrt{2}}[|x+1\rangle \otimes |+\rangle + |x-1\rangle \otimes |-\rangle].$$

Suppose that initially the walker is in the state localized $|x_0\rangle$. Iterating the interaction between the walker and the probes produces an entangled state of the walker and the n first probes of the form

$$|\psi_n\rangle = \frac{1}{2^{n/2}} \sum_{\omega_n} |X(\omega_n)\rangle \otimes |\omega_n\rangle$$

where the sum is over all walks ω_n of length n, starting at x_0 and with position $X(\omega_n)$ at their n^{th} step, and $|\omega_n\rangle := |\pm\rangle \otimes ... \otimes |\pm\rangle$ are orthogonal vectors associated to each walk.

This state ressembles some kind of "quantum parallelism". It enables to book-keep track of all possible walks in an algebraic way. It also codes for the probability of realization of any given sample: the probability of occurrence of the walk ω_n is the modulus square of the coefficient in front of $|X_n(\omega_n)\rangle \otimes |\omega_n\rangle$. We may view the probes states $|\pm\rangle$ as quantum coins.

At this point there is no more information in this sum than in the classical description – just parallelism. But the information we can extract at step n by observing the n first probes is encoded into the non commutative algebra of observables on these probes. These non-commutative extension is the essence of quantum stochastic processes. It offers a way to 'observe quantum effects' by considering probe measurements that are not aligned with the basis vectors. The measurement results will then not be in one-to-one correspondence with walker trajectories and the position will not be well defined.

Let us assume that we measure the probe effective spin along a tilted direction $\mathbf{u}$, with observable $\sigma_u = \mathbf{u} \cdot \sigma$, with normalized eigen-vectors:

$$|\pm^u\rangle = e^{\pm i\varphi/2} \cos\vartheta/2\, |\pm\rangle \pm e^{\mp i\varphi/2} \sin\vartheta/2\, |\mp\rangle.$$

The outputs of these measurements are still be $\pm$.

Measuring $\sigma_u = \mathbf{u} \cdot \sigma$ successively on each probe yields repeated POVM and quantum trajectory, as explained before. If $\mathbf{u}$ is alined with the z-axis these trajectories are identical to the classical random walks - because the property for the walker to be on a well defined position is preserved by the process. But in all other cases with $\mathbf{u}$ generic, the walker quantum state is a superposition of different positions. Hence the quantum trajectories are not random walks. What are they?

The POVM associated to measuring $\sigma_u = \mathbf{u} \cdot \sigma$ is $F_\pm = \langle \pm^u | U | \varphi_p \rangle$, so that

$$F_\pm |x\rangle = \langle \pm^u | U \left(|x\rangle \otimes |\varphi_p\rangle \right) = \frac{1}{\sqrt{2}} \big[|x+1\rangle \langle \pm^u | + \rangle + |x-1\rangle \langle \pm^u | - \rangle \big].$$

Instead of working with the position basis $|x\rangle$ we can Fourier transform and work with the momentum basis $|k\rangle$. Then $F_\pm$ act diagonally on $|k\rangle$ and

$$F_\pm |k\rangle = u_\pm(k)\, |k\rangle, \text{ with } u_\pm(k) = \frac{1}{\sqrt{2}} \big[\langle \pm^u | + \rangle e^{ik} + \langle \pm^u | - \rangle e^{-ik} \big].$$

That is: we are in the non-demolition situation with the momentum basis states as pointer states. For $\mathbf{u}$ along the z-axis, the $u_\pm(k)$'s are degenerate in the sense that $|u_\pm(k)|^2 = 1$. But otherwise they are not. Hence we can apply the previous convergence theorem and claim that the walker state randomly collapses to a momentum state when iterating the process ad infinitum. If $|\psi_0\rangle$ is the initial walker state, the target state $|\psi_\infty\rangle$ at infinite time is

$$|\psi_\infty\rangle = |k_\infty\rangle, \text{ with probability } \mathbb{P}[k_\infty = p] = |\langle p|\psi_0\rangle|^2.$$

Of course these are von Neumann's rules for momentum measurement.

Remark 1:
To sum up this tiny exercice: Parallelising random walks yields the simplest possible quantum stochastic processes. That one is reduced to a classical stochastic process – quantum trajectories – when extracting information by measuring observables on the output coins. Measuring the spin coins in the preferred direction yields back the classical random walks. But measuring in any other direction produces quantum trajectories describing progressive non-demolition measurements of the momentum operator.

4 Continuous monitoring and quantum trajectories.

4.1 From repeated interactions and measurements

The continuous limit of repeated POVMs can be formulated when the time lapse between two successive POVMs is small compare to any other time scales. In such case, the successive information obtained via the repeated POVMs becomes a continuous random signal. Taking the continuous limit of repeated POVMs involves two facets:
– Writing the time continuous evolution equations for the system density matrix, these equations are driven by the random output signals;
– Describing what is the statistics of these random signals and what stochastic process they generate.

Recall the discrete evolution equation of the system density matrix under repeated POVMs:

$$\rho \to \frac{F_i\,\rho\,F_i^\dagger}{\pi(i)}, \quad \text{with probability } \pi(i) = \mathrm{Tr}(F_i\,\rho\,F_i^\dagger).$$

To simplify let us assume that we are dealing with a doublet of POVMs (i.e. the probes form two level systems) so that the probe measurement outputs are either $+$ or $-$ (i.e. the index i above takes only the two possible values $\pm$). We then label the elements of the POVM by $F_\pm$, with $F_+^*F_+ + F_-^*F_- = \mathbb{I}$. To take the continuous limit we have to assume that the system density matrix is only slightly modified at each step, so that there should exist a small parameter, say ε, such that when $\varepsilon = 0$ the density matrix is not modified. There is then two cases:
– either $F_\pm^*F_\pm$ are both non zero and proportional to the identity at $\varepsilon = 0$;
– or one of the two $F_\pm^*F_\pm$ vanishes and the other one is equal to the identity at $\varepsilon = 0$.
The first case leads to SDEs driven by a Brownian motion whereas the second case yields SDEs driven by a Poisson process. To simplify matter we shall deal with the first case only (which is actually more generic).

Scaling limit of POVM

Assuming that $F_\pm \propto \mathbb{I}$ at $\varepsilon = 0$, we look for an expansion of $F_\pm$ in power of $\sqrt{\varepsilon}$ (by convention), $F_\pm = f_\pm\big(\mathbb{I} + \cdots\big)$, with $|f_\pm|^2 = p_\pm$ and $p_+ + p_- = 1$, such that $F_+^\dagger F_+ + F_-^\dagger F_- = \mathbb{I}$. To simplify matter let us suppose that $f_+ = f_- = 1/\sqrt{2}$ and hence $p_+ = p_- = 1/2$ (the symmetric case). Solving pertubatively in $\sqrt{\varepsilon}$ the constraints $F_+^\dagger F_+ + F_-^\dagger F_- = \mathbb{I}$ (see below) yields the possible form of $F_\pm$:

$$F_\pm = \frac{1}{\sqrt{2}}\left[\mathbb{I} \pm \sqrt{\varepsilon} N - \varepsilon(iH \pm M + \frac{1}{2}N^\dagger N) + O(\varepsilon^{3/2})\right]$$

with H hermitian but not necessarily N and M. This is the most general expansion for $F_\pm$ solutions of the unitary constraint $F_+^\dagger F_+ + F_-^\dagger F_- = \mathbb{I}$ around the symmetric solution $F_\pm = \mathbb{I}/\sqrt{2}$ such that the scaling limit exists.

Indeed, the general solution to $F_+^\dagger F_+ + F_-^\dagger F_- = \mathbb{I}$ in the neighborhood of $F_\pm = \mathbb{I}/\sqrt{2}$ is $F_\pm = \frac{1}{\sqrt{2}}\left[\mathbb{I} \pm \sqrt{\varepsilon} N_\pm - \varepsilon(\pm M_\pm + \frac{1}{2}N_\pm^\dagger N_\pm) + \cdots\right.$ with $\Re\,(N_+) = \Re\,(N_-)$ and $\Re\,(M_+) = \Re\,(M_-)$ (here $2\,\Re\,(N) = N + N^\dagger$). Not all of these solutions lead to a consistent scaling limit. Existence of the scaling limit (see below) requires $N_+ = N_-$ and this limit only depends on the difference $M_+ - M_-$. Since $M_\pm$ have identical hermitian part, only the difference of their anti-hermitian components matters in the scaling limit, i.e. $M_\pm = K \pm iH_\pm$ with K and $H_\pm$ hermitian and the scaling limit depends only on $H := (H_+ + H_-)/2$.

Let us now see what happens to the one-step transformation $\rho \to F_\pm\,\rho\,F_\pm^\dagger/\pi(\pm)$, which occurs with probability $\pi(\pm) = \mathrm{Tr}(F_\pm\,\rho\,F_\pm^*\dagger)$. We have to compute the expansion $F_\pm\,\rho\,F_\pm^\dagger$ and $\mathrm{Tr}(F_\pm\,\rho\,F_\pm^\dagger)$ in power of $\sqrt{\varepsilon}$. The results are (for $M = 0$):

$$\begin{aligned} F_\pm\,\rho\,F_\pm^\dagger &= \frac{1}{2}\big[\rho \pm \sqrt{\varepsilon}\,\big(N\rho + \rho N^\dagger\big) + \varepsilon\big(-i[H,\rho] + L_N(\rho)\big) + O(\varepsilon^{3/2})\big] \\ \mathrm{Tr}(F_\pm\,\rho\,F_\pm^\dagger) &= \frac{1}{2}\big[1 \pm \sqrt{\varepsilon}\,\mathrm{Tr}\big(N\rho + \rho N^\dagger\big) + O(\varepsilon^{3/2})\big] \end{aligned}$$

with L_N, a so-called Lindbladian, is defined by

$$L_N(\rho) = N\rho N^\dagger - \frac{1}{2}(N^\dagger N\rho + \rho N^\dagger N).$$

Below, ε (not $\sqrt{\varepsilon}$) shall be identified with dt. Notice that the $\sqrt{\varepsilon}$-terms come with a $\pm$ sign, so that they depends on what is going to be the measurement outputs, or equivalently, they depend son the randomness of the signals. On contrary the ε-terms are independent of $\pm$ and hence independent of the output randomness.

Recall that by averaging the POVM, we get the completely positive map $\Phi : \rho \to \Phi(\rho) = F_+\rho F_+^* + F_-\rho F_-^*$. Hence its continuous limit is (one can check that the M-dependent terms cancel)

$$\Phi(\rho)|_{\varepsilon\to 0} = \rho + \varepsilon\big(-i[H,\rho] + L_N(\rho)\big) + \cdots$$

From this equation we learn that Lindbladians are the generators of completely positive maps (think about ε as the time interval dt). This is quite general.

Definition-Proposition (Lindblad, ...) "Lindladian":
Let $\mathcal{H}$ be a Hilbert space and let ρ denote quantum states on $\mathcal{H}$. Let H be hermitian and L_a be

set of bounded operators on $\mathcal{H}$. Lindblad operators are linear maps on quantum states defined by

$$L(\rho) = -i[H,\rho] + \sum_a \big[L_a \rho L_a^\dagger - \frac{1}{2}(L_a^\dagger L_a \rho + \rho L_a^\dagger L_a)\big].$$

– Lindblad operators are generators of CP-maps in the sense that the maps $\Phi_t := e^{tL}$ form a one parameter group of CP-maps.
– Reciprocally, any one parameter group of CP-maps, depending continuously on this parameter, can be written in such exponential form.
– Notice that the decomposition of L in terms of H and L_a is not unique: L is invariant under shifts $L_a \to L_a + m_a$ and $H \to H + \frac{i}{2}(m_a - m_a^\dagger)$ or under unitary linear transformations of the L_a's.

Remark 1:
Recall that POVM can be viewed as resulting from the system-probe interaction plus the measurement. If those probes are prepared in a state $|\varphi\rangle$, the operator $F_\pm$ are $F_\pm = \langle \pm|U|\varphi\rangle$ where U is the system-probe evolution operator. This can written as $U = e^{-i(\delta t)\hat{H}}$ with δt the interaction duration and $\hat{H}$ the system-probe hamiltonian which can be decomposed as $\hat{H} = H_s + H_p + H_{\rm int}$ with $H_{\rm int}$ the interaction hamiltonian and H_s (resp. H_p) the system (resp. probe) hamiltonian. If we take δt as small parameter, i.e. we choose $\varepsilon = \delta t$, we have to rescale the interaction hamiltonian according to $H_{\rm int} = \frac{1}{\sqrt{\varepsilon}} H_i$ for the continuous limit to exist (so that the interaction as a non-trial effect). Then (up to an irrelevant constant term)

$$F_\pm := \langle \pm|U|\varphi\rangle = \langle \pm|\varphi\rangle \big[\mathbb{I} - i\sqrt{\varepsilon}\,\langle \pm|H_i|\varphi\rangle - \varepsilon\,\big(iH_s + \frac{1}{2}\langle \pm|H_i^2|\varphi\rangle\big) + \cdots\big].$$

The continuous scaling limit exists only if $\langle +|H_i|\varphi\rangle + \langle -|H_i|\varphi\rangle = 0$. This corresponds to the previous expansion with the identification (when $\langle \pm|\varphi\rangle \neq 0$)

$$p_\pm = |\langle \pm|\varphi\rangle|^2 = 1/2,\ N = -i\langle +|H_i|\varphi\rangle = i\langle -|H_i|\varphi\rangle,\ H = H_s.$$

Remark 2:
One may of course generalizes the previous discussion to higher dimensional cases with POVMs defined by a bigger set of operators $\hat{F}_s$ satisfying $\sum_s \hat{F}_s^\dagger \hat{F}_s = \mathbb{I}$. Recall that CP-map is then defined by $\rho \to \Phi(\rho) = \sum_s \hat{F}_s \rho \hat{F}_s^\dagger$. Suppose that this CP-map is close to the identity and that the $\hat{F}_s$'s admit an expansion as the form $\hat{F}_s = u_s \mathbb{I} + \sqrt{\epsilon}\,\hat{N}_s + \epsilon\,\hat{M}_s$, with u_s complex numbers and N_s, M_s operators. It is then a simple algebraic exercice to find the conditions imposed by demanding that Φ is close to the identity and the POVM condition. The first relation is that $\sum_s |u_s|^2 = 1$, so that the numbers $|u_s|^2$ define a probability measure by setting $p_s := |u_s|^2$, with $\sum_s p_s = 1$. Let us denote by $\mathbb{E}^o$ this probability measure. The elements $\hat{F}_s$ of the POVM are then viewed as random operator $\hat{F}$ distributed according to $\mathbb{E}^o$. Let us assume for simplicity that none of these probabilities vanishes so that we can factorize them by defining $\hat{F}_s = u_s\, F_s$. The CP-map Φ then reads

$$\rho \to \Phi(\rho) = \sum_s p_s\,(F_s \rho F_s^\dagger) = \mathbb{E}^o[F\rho F^\dagger], \quad \text{with } \mathbb{E}^o[F^\dagger F] = \mathbb{I}.$$

For Φ close to the identity, $F = \mathbb{I} + \sqrt{\epsilon}\,N + \epsilon\,M + \cdots$, the conditions on N and M are

$$\mathbb{E}^o[N] = ib,\ b \in \mathbb{R}; \quad \mathbb{E}^o[M + \frac{1}{2}N^\dagger N] = -iH,\ H^\dagger = H.$$

The CP-map then becomes $\Phi(\rho) = \rho + \epsilon\,\mathcal{L}(\rho) + \cdots$ with a Lindbladian $\mathcal{L}$ given by

$$\mathcal{L}(\rho) = -i[H,\rho] + \mathbb{E}^o[N\rho N^\dagger - \frac{1}{2}(N^\dagger N\rho + \rho N^\dagger N)].$$

Recall that here the N are viewed as random operator distributed according to $\mathbb{E}^o$.

Remark 3:
The case in which one of the product $F_\pm^\dagger F_\pm$ vanishes (the case we didn't study) corresponds to preparing the probes in eigenstates of the measured observable. Suppose, to fix the idea, that the probes are prepared in the eigenstate $|+\rangle$. Then, the probe measurement outputs will be $+$ with very high probability because the probes are only slightly modified by the system-probe interaction. This means that the histogram of output signals we are going to observe are made of long series of pluses interrupted with a small number of $-$. As consequence the signals we get in the scaling limit are discontinuous, with random jumps. The evolution equations are going to be driven by these discontinuous random signals, not by Brownian motions. The case we did study correspond to the cases in which the probes are prepared in states which are not eigenstates of the measured observable.

Quantum trajectory SDEs

The aim is now to arrive at the evolution equations for quantum systems under continuous monitoring via repeated POVMs (which we assume to be a doublet for simplicity). These are known as quantum trajectory SDEs:

Proposition-Definition (Belavkin, Barchielli, Milburn-Wiseman,...) "Quantum Trajectories": *Let ρ be the system density matrix under continuous monitoring via doublet of POVMs. Its time evolution is governed by the following non-linear SDEs (in Itô form):*

$$d\rho_t = \Big(-i[H,\rho_t] + \sum_a L_{M_a}(\rho_t)\Big)dt + L_N(\rho_t)dt + D_N(\rho_t)dB_t$$

where B_t is a standard normalized Brownian motion, $dB_t^2 = dt$. Here $H^\dagger = H$ is a hamiltonian, L_{M_a} are Lindblad operators, $L_M(\rho) = M\rho M^\dagger - \frac{1}{2}(M^\dagger M\rho + \rho M^\dagger M)$, and L_N is a Lindblad operator and D_N a so-called stochastic innovation term, with

$$\begin{aligned} L_N(\rho) &= N\rho N^\dagger - \frac{1}{2}(N^\dagger N\rho + \rho N^\dagger N), \\ D_N(\rho) &= N\rho + \rho N^\dagger - \rho\,\mathrm{Tr}(N\rho + \rho N^\dagger) \end{aligned}$$

The operator N is called the measurement operator and the measured observable is $N + N^\dagger$. Solutions of these SDEs are called "quantum trajectories".

We can decompose $d\rho$ as

$$d\rho = (d\rho_t)_{\mathrm{syst}} + (d\rho_t)_{\mathrm{meas}}.$$

The first two terms $(d\rho_t)_{\mathrm{syst}} := \big(-i[H,\rho_t] + \sum_a L_{M_a}(\rho_t)\big)dt$ is a deterministic evolution which will be present without any monitoring (it depends whether the system evolution is hamiltonian or dissipative, i.e. it depends whether the system is isolated or open). The two last terms $(d\rho_t)_{\mathrm{meas}} := L_N(\rho_t)dt + D_N(\rho_t)dB_t$ code for the back action of the monitoring on the system evolution.

The quantum trajectories were originally introduced in quantum optics via continuous monitoring of electromagnetic modes coupled to atomic systems [**?**], there were independently introduced as efficient tools to simulate dissipative systems [**?**]. The approach based on iterated interactions was developed in [**?**, **?**, **?**].

The fact that the SDEs is driven by a Brownian motion is intuitively easy to grasp. The information we get on the system by the repeated POVMs is a series of pluses and minus (in the case of doublet of POVMs as we are deal with here). Series of $(+--+-+\cdots)$ are in one-to-one correspondance with classical random walks, whose scaling limits are Brownian samples. The statistics of the continuous signal obtained from the scaling limit of these series is not identical to that of a Brownian motion because it depends on the system state: its evolution is driven by a Brownian motion but contains extra drifts which are system dependent. However, any realization of the Brownian is in one-to-one correspondance with a sample of the output signal.

If we don't keep track of the information provided by the output signals, we only have access at the mean state. As consequence – since $\mathbb{E}[dB_t] = 0$ and because we are using the Itô form – the mean density matrix $\bar{\rho} = \mathbb{E}[\rho]$ satisfies:

$$d\bar{\rho}_t = \big(-i[H, \bar{\rho}_t] + \sum_a L_{M_a}(\bar{\rho}_t)\big)dt + L_N(\bar{\rho}_t)dt.$$

This is a linear noiseless equation in a Lindblad form. The net effect is to add an extra Lindblad term, namely L_N, to the bare system evolution. This was expected – say from the similar remark valid in the discrete setting – and it reflects that the series of probes – or the apparatus monitoring the system – behaves as a passive reservoir if we don't keep record of the information. Averaging over Brownian samples or tracing over probe degrees of freedom are equivalent.

We can actually reverse the logic: to get what equations describe the evolution of the system in contact with a Markovian reservoir one may look at the evolution of system under continuous monitoring and only keep track of part of the information thus extracted, say by looking only at a fraction of the outputs. This is the strategy we shall adopt in the following proof of the quantum trajectory equations.

Remark 1:
A few words on generalization with more Brownian motions, alias bigger probe Hilbert spaces.....

.....

Hint for a proof

Here we give hints for a proof, more rigorous proof can be found somewhere else [**?**, **?**]. The point we would like to describe is why and how randomness arises solely from Quantum Mechanics (i.e. its only encoded into the rules for randomness in Quantum Mechanics, no extra noise is added). If the POVM is close to the identity as above, its effect on the system is small and we can take the scaling limit $\varepsilon \to 0$ and $n \to \infty$ with $t = n\varepsilon$ fixed. Let ρ_t be the system density matrix at time t (naively such that $\rho_n = \rho_{t=n\varepsilon}$).

The information from the doublet of POVMs we get is a series of pluses and minuses. Let us denote the output of the n-th POVM by $s_n = \pm$ and define their scaled sum $X_n = \sqrt{\varepsilon}\, \sum_{k\leq n} s_k$. Their statistical behaviors are coded into the frequencies of occurrences $N_n(\pm)$ after n steps

– $N_n(\pm)$ is the number of times $\pm$ appeared as outputs. At large n, $N_n(\pm) \simeq n/2 + \cdots$, by the law of large number. The sub-leading term in $N_n(+)$ is opposite to that in $N_n(-)$ because $N_n(+) + N_n(-) = n$ and their difference is $(X_n - X_0)/\sqrt{\varepsilon}$. Hence, at large n,

$$\sqrt{\varepsilon}\,\big(2N_n(\pm) - n\big) = \pm(X_n - X_0) + \cdots$$

Recall that from step n to step $n+1$ the density matrix evolves randomly as:

$$\rho_n \to \rho_{n+1} = \frac{F_\pm\,\rho_n\,F_\pm^\dagger}{\pi_n(\pm)}, \quad \text{with proba } \pi_n(\pm) = \mathrm{Tr}(F_\pm\,\rho_n\,F_\pm^*).$$

or alternatively, because $X_{n+1} - X_n = s_{n+1}\sqrt{\varepsilon}$ with $s_{n+1} = \pm$,

$$\rho_{n+1} = \frac{1}{2}(\rho_n^{(+)} + \rho_n^{(-)}) + \frac{1}{2\sqrt{\varepsilon}}(\rho_n^{(+)} - \rho_n^{(-)})(X_{n+1} - X_n),$$

where we define $\rho_n^{(\pm)} := \frac{F_\pm\,\rho_n\,F_\pm^\dagger}{\pi_n(\pm)}$. In the scaling limit we may Taylor expand $\rho_n^{(\pm)}$:

$$\begin{aligned}
\rho_n^{(+)} + \rho_n^{(-)} &= 2\rho_n + 2\varepsilon\,\big[-i[H,\rho_n] + L_N(\rho_n) - D_N(\rho_n)\,U_N(\rho_n)\big] + \cdots \\
\rho_n^{(+)} - \rho_n^{(-)} &= 2\sqrt{\varepsilon}\,D_N(\rho_n) + \cdots
\end{aligned}$$

where $U_N(\rho) = \mathrm{Tr}(N\rho + \rho N^\dagger)$. Hence, $\rho_n^{(\pm)}$ are deterministic (non random) variables conditioned on the n first outputs. All the extra randomness at step $n+1$ is encoded into the variation $X_{n+1} - X_n$.

Because s_n are outputs of quantum measurement, the statistics of $X_{n+1} - X_n$, conditioned on the n first outputs, is determined by the system density at step n according to the rules of Quantum Mechanics. Its means and its second moment are thus:

$$\begin{aligned}
\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] &= \sqrt{\varepsilon}\,\mathbb{E}[s_{n+1}|\mathcal{F}_n] = \sqrt{\varepsilon}\,(\pi_n(+) - \pi_n(-)) = \varepsilon\,\mathrm{Tr}(N\rho_n + \rho_n N^\dagger) + \cdots, \\
\mathbb{E}[(X_{n+1} - X_n)^2|\mathcal{F}_n] &= \varepsilon\,\mathbb{E}[s_{n+1}^2|\mathcal{F}_n] = \varepsilon.
\end{aligned}$$

Recall now that we identified $\varepsilon = dt$. The first equation above gives the drift in the variation dX_t. The second gives the fluctuations which are Gaussian to leading order. This (naively/physically) implies that in the scaling limit the output signal X_t satisfies the SDEs

$$dX_t = \mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\,dt + dB_t.$$

Inserting this into the difference equation $\rho_{n+1} - \rho_n$ yields the SDE for the system density matrix given above.

Remark 1:
In the course of the above derivation we deduce the continuous equation for the (rescaled) output signal X_t. Hence, the quantum trajectory equations form actually a pair of SDEs, one giving the time evolution of the system density matrix, the other giving sense to the signal in the continuous limit:

$$\begin{aligned}
d\rho_t &= -i[H,\rho_t]\,dt + L_N(\rho_t)dt + D_N(\rho_t)dB_t, \\
dX_t &= \mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\,dt + dB_t.
\end{aligned}$$

Notice that X_t is slave to the density matrix ρ_t.

Remark 2:
The proof given above is not mathematically rigorous because it implicitly assumes the existence of the scaling limit with the property that the continuous process interpolates the discrete one such that $\rho_n = \rho_{(t=n\epsilon)}$ and $X_n = X_{(t=n\epsilon)}$ in law. Another way to obtain the scaling limit consists in decomposing the process ρ_n as a sum of a martingale M_n plus a predictable process O_n. This is called a Doob decomposition. It consists in writing

$$\rho_n = O_n + M_n,$$

with M_n a $\mathcal{F}_n$-measurable martingale – i.e. $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$ – and O_n a $\mathcal{F}_{n-1}$-measurable process. Such decomposition is always possible. Indeed it is enough to define $M_n := \sum_{k=1}^{n} \pi_k$ with $\pi_k := \rho_k - \mathbb{E}[\rho_k|\mathcal{F}_{k-1}]$, which by construction is a martingale, and to set $O_n := \rho_n - M_n$, which by construction is $\mathcal{F}_{n-1}$-measurable. In the scaling limit the martingale (resp. predictable) contribution is going to converge to the noisy source (resp. the drift) of the SDEs. This helps making the proof rigorous because there are theorems characterizing continuous martingales – they are integrals of the Brownian motion – and those theorems are enough to identify that the scaling limit of M_n is $\int_0^t dt' D_N(\rho_{t'}) dB_{t'}$.
Another way to make the proof rigorous consists at looking at the generator of the semi-groups associated to the discrete process and prove that it converges toward that of the time continuous process.

4.2 Basics examples and properties

We shall use simple Qu-bit systems to discover/present typical behaviors of quantum trajectories. The density matrix for a Qu-bit may be parametrized as

$$\rho = \begin{pmatrix} Q & U \\ U^* & 1-Q \end{pmatrix},$$

in a specified basis $|\downarrow\uparrow\rangle$. Let $\sigma^x,\ \sigma^y,\ \sigma^z$ be the Pauli matrices in that basis.

Continuous non-demolition measurements

Here we look at the non-demolition measurement of σ^z, with no extra evolution. The quantum trajectory SDE is then $d\rho = L_{\text{meas}}(\rho)\, dt + D_{\text{meas}}(\rho)\, dB_t$, with B_t a normalized Brownian motion, and

$$L_{\text{meas}}(\rho) = -\frac{\gamma^2}{32}[\sigma_z, [\sigma_z, \rho]] \quad \text{and} \quad D_{\text{meas}}(\rho) = \frac{\gamma}{4}\big(\{\sigma_z, \rho\} - 2\rho\,\mathrm{tr}(\rho\sigma_z)\big),$$

with γ^2 the measurement rate. We did not write a term associated to a hamiltonian evolution, but we could have provided that the hamiltonian is proportional to σ^z. Actually, non-demolition measurements are characterized by the fact that the measured observable commutes with hamiltonian.

For the diagonal matrix element Q this SDE reads

$$dQ_t = \gamma\, Q_t(1 - Q_t)\, dB_t,$$

with dB_t a standard Brownian motion. This equation is simply the scaling limit of the discrete equations for repeated indirectement measurements. It is clear that Q_t is a martingale, because there is no drift term in its evolution. By the convergence martingale theorem, it converges to one of the fixed points of the SDE, i.e to $Q_\infty = 0$ or $Q_\infty = 1$. Since Q_t is a martingale and the convergence is in $\mathbb{L}^!$, we have $\mathbb{E}[Q_\infty] = Q_0$ and hence

$$Q_\infty = 1 \text{ with proba } Q_0, \quad Q_\infty = 0 \text{ with proba } 1 - Q_0.$$

The convergence is exponentially fast with a time scale of order γ^{-2}.

Actually, this SDE can be solved explicitly. The solution is

$$Q_t = \frac{Q_0\, e^{\gamma B_t - \frac{\gamma^2}{2}t + A_t}}{Q_0\, e^{\gamma B_t - \frac{\gamma^2}{2}t + A_t} + (1 - Q_0)},$$

with $A_t = \gamma^2 \int_0^t ds\, Q_s$. This is the scaling limit of the exact solution we wrote for discrete repeated non-demolition POVMs.

Remark 1:
The parameter γ, and hence the typical time to collapse, code for - or is directly related to - the interaction strength between the system and the monitoring system (the probes).

Qu-bit in thermal contact

Here we monitor a system in contact with a thermal bath by observing continuously its hamiltonian. We model the contact with the thermal bath by a Lindblad equation (i.e. we assume Markovianity). The SDE is then $d\rho = (d\rho)_{\text{syst}} + (d\rho)_{\text{meas}}$. For a Qu-bit, with hamiltonian $h = \omega\, \sigma^z$ diagonal in the basis $|\downarrow\uparrow\rangle$, the thermal relaxation is $(d\rho)_{\text{syst}} = -i[h, \rho]dt + L_{\text{therm}}(\rho)\, dt$ with

$$L_{\text{therm}}(\rho) = \lambda p\big(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\big) + \lambda(1-p)\big(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\big),$$

where the first term codes for transitions from $|\downarrow\rangle$ to $|\uparrow\rangle$ at rate λp and the second for transitions from $|\uparrow\rangle$ to $|\downarrow\rangle$ at rate $\lambda(1-p)$. The term associated to the σ^z-measurement at a rate γ^{-2} is $(d\rho)_{\text{meas}} = L_{\text{meas}}(\rho)\, dt + D_{\text{meas}}(\rho)\, dB_t$ with L_{meas} and D_{meas} as above.

For the diagonal matrix element Q this reads

$$dQ_t = \lambda(p - Q)dt + \gamma\, Q_t(1 - Q_t)\, dB_t.$$

This equation decouples for the other ones so that we can restrict ourselves to diagonal density matrices $\rho = Q|\downarrow\rangle\langle\downarrow| + (1-Q)|\uparrow\rangle\langle\uparrow|$. The convention is that the system is in state $|\downarrow\rangle$ for $Q = 1$ and in state $|\uparrow\rangle$ for $Q = 0$.

There is a competition between the drift term $\lambda(p - Q)dt$, which codes for the thermal relaxation, and the noisy term $\gamma\, Q_t(1-Q_t)dB_t$, which codes the effect of the monitoring. There are two natural time scales: the thermal relation time $\tau_{\text{therm}} := \lambda^{-1}$ and the typical measurement time $\tau_{\text{meas}} := \gamma^{-2}$. Hence the control parameter is their ratio $tau_{\text{therm}}/\tau_{\text{meas}} = \gamma^2/\lambda$. Efficient monitoring corresponds to γ large.

The change in the typical quantum trajectories, solution of the above SDE, when γ increases – that is when the strength of monitoring procedure increases – is depicted in **Figure...**. We see an evolution from almost unperturbed exponential relaxation towards thermal equilibrium for γ small, with $Q_t \simeq p$, to jumpy trajectory for γ large, with Q_t jumping and hesitating between the extreme values 0 and 1.

We are interested in the limit of strong measurement $\gamma \to \infty$, that is $\tau_{\rm meas} \ll \tau_{\rm therm}$.

Proposition "Thermal quantum jumps":
In the limit $\gamma \to \infty$ of very efficient monitoring, we have :
— The invariant measure of the thermal quantum trajectory SDE has a limit:
. $\lim_{\gamma\to\infty} dP_{\rm stat} = [(1-p)\delta(Q) + p\delta(1-Q)]\, dQ.$
— The limits of the mean time $T_\downarrow$ (resp. $T_\uparrow$) the trajectories spend near $Q \simeq 1$ (resp. $Q \simeq 0$) are:
. $\lim_{\gamma\to\infty} T_\downarrow = 1/\lambda(1-p)$ *and* $\lim_{\gamma\to\infty} T_\uparrow = 1/\lambda p.$

The first statement means that at strong monitoring the long time mean behavior of the quantum trajectory coincides with the Gibbs state. The second statement expresses an ergodicity property since $T_\uparrow/T_\downarrow = p/(1-p) = e^{-\beta}$ in the large γ limit, as expected. These statements are true only at large γ.

Proof.
This proposition can be proved using standard tools from classical probability theory. Every relevant quantity can be exactly computed, with a grain of brute force, because the SDE is one dimensional. For a SDE of the form $dQ_t = f(Q_t)dt + g(Q_t)dB_t$, the invariant measure – if normalizable – is $dP_{\rm stat} = e^{-2h(Q)}dQ/g^2(Q)$ with $h(Q)$ such that $\partial h(Q) = -f(Q)/g^2(Q)$. We can apply this formula for the quantum trajectory SDE with $f(Q) = \lambda(p-Q)$ and $g(Q) = \gamma Q(1-Q)$. We then have to take the limit $\gamma \to \infty$. Since the limit is a distribution this is done by testing this stationary measure against test functions. Choosing tests function with support on one of the two half sides of the interval $[0,1]$ allows to prove that the support of the stationary measure localises at the two extremities of the interval.
Let $0 < Q_i < Q_f < 1$. Let $T_{i\to f}$ be the first instance the process started at Q_i hits Q_f before hitting 0. This is a stopping time. Standard formula from probability theory based on using martingales tells that, in the case the process escapes the interval $[0, Q_f]$ from Q_f with probability one, its mean is

$$\mathbb{E}[T_{i\to f}] = 2\int_{Q_i}^{Q_f} dQ\, e^{2h(Q)} \int_0^Q dP_{\rm stat}.$$

The mean time $T_\downarrow$ (resp. $T_\uparrow$) is then obtained from this formula by taking the limit $Q_i \to 0^+$ (resp. 1^-) and $Q_f \to 1^-$ (resp. 0^+) after the large γ limit. □

Remark 1:
One may get finer information than the second statement by looking for the distribution of the random stopping time $T_{i\to f}$. For $0 < Q_i < Q_f < 1$, this distribution is the following at large γ:

$$\lim_{\gamma\to\infty} \mathbb{P}[\lambda\, T_{i\to f} \in B] = \frac{Q_i}{Q_f}\, \mathbb{I}_{0\in B} + \Big(1 - \frac{Q_i}{Q_f}\Big)\frac{p}{Q_f}\int_B e^{-s\frac{p}{Q_f}}\, ds,$$

For $Q_i \to 0^+$ and $Q_f \to 1^-$ we get that this limit distribution has $e^{-sp}pds$ for density. Hence

the time to go from $|\downarrow\rangle$ to $|\uparrow\rangle$ is exponentially distributed with mean $1/\lambda p$. That is: the jumps are Poisson like.
However, for finite Q_i and Q_f we see that the distribution possesses two contributions: a Dirac peak at zero and a exponential tail with a Q_f-dependent slop. The meaning of this formula is the following: in the large γ limit, starting from Q_i, either with probability $\frac{Q_i}{Q_f}$ it takes no time for the trajectory to reach Q_f, or with probability $1-\frac{Q_i}{Q_f}$ it takes an exponential time with parameter $\frac{\lambda p}{Q_f}$ to reach Q_f. As we shall explain later, this is linked to aborted quantum jumps surviving the strong monitoring limit.

Remark 2:
Notice that the jumps are not built in the SDE but are generated by it. They appear at large measurement rate. For finite γ the jumps are not instantaneous. They last a typical time of order $4\gamma^{-2}\log(\gamma^2/\lambda)$. They become instantaneous only in the infinite γ limit.

Rabi oscillations on a Qu-bit

Here we monitor the system observable σ^z for a Qu-bit with hamiltonian $\omega\sigma^x$. That is: we observe continuously an observable not commuting with the system hamiltonian. The system evolution is again $d\rho = (d\rho)_{\rm sys} + (d\rho)_{\rm meas}$ with the measurement part as before, $(d\rho)_{\rm meas} = L_{\rm meas}(\rho)\,dt + D_{\rm meas}(\rho)\,dB_t$ relative to the measurement of σ^z at a rate γ^{-2}. The system evolution is hamiltonian

$$(d\rho)_{\rm sys} = -i\omega[\sigma^x,\rho_t]\,dt$$

with Rabi frequency ω. The quantum trajectory SDEs are:

$$\begin{aligned} dQ_t &= \omega\,U_t\,dt + \gamma Q_t(1-Q_t)\,dB_t, \\ dU_t &= -\omega\,(Q_t-\frac{1}{2})\,dt - \frac{\gamma^2}{8}U_t\,dt - \gamma U_t(Q_t-\frac{1}{2})\,dB_t \end{aligned}$$

where Q_t and U_t parametrize the density matrix as above. The first term is the effect of the Rabi oscillation, the second that of the monitoring. It is easy to verify that these equations imply an exponential purification of the state. For instance, if $\Delta_t = Q_t(1-Q_t) - |U_t|^2$ is the determinant of the density matrix, then its expectation decreases exponentially fast: $\mathbb{E}[\Delta_t^{1/2}] = \Delta_0^{1/2}\,e^{-\gamma^2 t/8}$.

There are again two processes in competition: the hamiltonian evolution which yields to a precession of the spin and the monitoring which projects the spin on the pointer states. If $\gamma^2 \ll \omega$ the system undergoes Rabi oscillations – because the measurement process is not efficient enough to project the state during a Rabi period. On contrary, if $\omega \ll \gamma^2$ the measurement process wins over the Rabi oscillation so that the system state is projected onto one of the two σ^z-eigenstates $|\uparrow\downarrow\rangle$ with jumps from to the other induced by the Rabi hamiltonian.

By solving explicitly the quantum trajectory SDEs one can show that, asymptotically for large γ, the mean time between successive jumps is

$$\bar{T}_{\rm jump} = \gamma^2/4\omega^2 \propto \tau_{\rm Rabi}^2/\tau_{\rm meas}.$$

It diverges when $\gamma\to\infty$, inversionally to the measurement laps time $\tau_{\rm meas}=\gamma^{-2}$ as expected from the Zeno effect.

Remark 1:
The system evolution $d\rho$ preserves pure states – actually mixed states are exponentially purified. So we can restrict the analysis to pure state. The system evolution also preserves a reality condition so we restrict to states of the form $|\psi_t\rangle = \cos(\theta_t/2)|\uparrow\rangle + \sin(\theta_t/2)|\downarrow\rangle$, which correspond to $Q_t = \frac{1}{2}(1+\cos\theta_t)$ and $U_t = \frac{1}{2}\sin\theta_t$. This reduces the evolution equation to a single SDE for Q_t or θ_t:

$$d\theta_t = -(\omega + 2\gamma \sin\theta_t \cos\theta_t)\, dt - 2\gamma \sin\theta_t\, dB_t.$$

Again, as for any one variable SDEs one can compute exactly many of its properties. The mean time between jumps, say from $|\uparrow\rangle$ to $|\downarrow\rangle$ is defined as the first time the system reaches $|\downarrow\rangle$, i.e. $\theta = \pi$, starting from the state $|\uparrow\rangle$, i.e. from $\theta = 0$. Standard tools of probability theory allow to compute exactly its probability distribution, and in particular its mean, in a way similar to those use in case of thermal quantum jumps. This is the result quoted above.

4.3 Applications to quantum control and feedback

By construction, and because we continuously get information, system monitoring obviously opens the route to feedback and quantum control. One can find many examples of quantum control in the literature, and we will here only give a few examples.

Dissipative driving

Dissipative dynamics can be used to drive quantum system towards a given target state because, at late time, a quantum system under the influence of dissipative dynamics converges towards invariant states. Hence one may imagine coupling the quantum system to an auxiliary reservoir manufactured so that the dynamical interaction possesses the target state as only invariant state. This type of control does not use any information extracted from the system. It only requires preparing the dynamics appropriately. Actually most of old processes for quantum state preparation may be viewed as examples of this procedure with the macroscopic preparation apparatus driving the system to its target.

More clever construction.....

Flux control via indirect measurement

Another way to control quantum systems may be based on adapting the monitoring process in a way depending on the information acquired on the system. This is peculiar to quantum mechanics and is grounded into the fact that any extraction of information from a quantum system back-act on it. For instance, the quantum Zeno effect provides a way to freeze a system into a given sub-space – an eigen-space of the measured observable.

Another example consists in using that hamiltonian and dissipative channels do not react in the same way to quantum monitoring (one is Zeno frozen, the other not). Hence by changing the strength at which we monitor a quantum system we may open or close a hamiltonian channel while leaving free any dissipative channels. Let us give an example. Consider an electronic double quantum dot in contact with two reservoirs, one coupled to the left dot the other to the right dot. See **Figure...**. The respective dot-reservoir couplings are dissipative. The two dots are coupled unitarly though the tunnel effect. To simplify, let us assume that the Coulomb

interaction is strong enough that the double dot system is occupied by one electron at most. So the electron can be either on the left or on the right dot or absent, and any system state is combinaison of those three states. Imagine now that we monitor the number of electron on the left and the right dot. We may decide to change the strength of the monitoring depending on the information we get on the electron position. Say: if, based on this information, we estimate that there is an electron on the left dot, we may decide to measure more strongly and, on contrary, if we estimate that there is no electron on the left dot we keep measuring (to still get some information) but mildly. When measuring strongly we Zeno freeze the hamiltonian tunnel channel in-between dots but not the dissipative channel from dot to reservoir. The effect is then to reduce the probability to jump from the left to the right dot when an electron is present on the left dot, while preserving its probability to jump from the dot to the reservoir, and without modifying the probability of inter-dot transition when the electron is on the right dot. As a consequence, this produces a net electron flux from the right reservoir to the left reservoir, even if the two reservoir were at identical chemical potential.

This double dot measurement control is an example of mesoscopic Maxwell daemon. Closing or not the tunnel channel in-between dots by changing the monitoring strength is analogous to opening or closing the inter-compartiment door in Maxwell daemon thought experiments.

Stochastic feedback using measurement signals

Another feedback procedure consists in back acting on the system in a manner depending on the information gathered by the monitoring process. If during a time interval dt we extract the data dX_t we may act on the system, either unitarly or dissipatively, with a CP map depending on dX_t. The complete evolution of the system during a time laps dt then decomposes into two sucessive steps:
– First, the system state ρ_t evolves under monitoring into the state $\hat{\rho}_{t+dt}$, and the data dX_t is extracted;
– Second, a dX_t-dependent feedback map acts on $\hat{\rho}_{t+dt}$ to produce the system state ρ_{t+dt}.
Notice that this way of processing avoid any acausality because we back-act on the system after we have got the information on dX_t.

Let $L_{\rm sys}$ be the Lindbladian coding for the evolution in absence of monitoring and feedback, and N the measurement operator of the monitoring process. Then $\hat{\rho}_{t+dt} = \rho_t + (L_{\rm sys}(\rho_t) + L_N(\rho_t))dt + D_N(\rho_t)dB_t$. Let now Φ_{dX_t} be the feedback CP-map. Then the system state at time $t + dt$ reads:

$$\rho_{t+dt} = \Phi_{dX_t}\big(\rho_t + (L_{\rm sys}(\rho_t) + L_N(\rho_t))dt + D_N(\rho_t)dB_t\big).$$

When dealing with this formula one has to recall that dX_t is driven by the Brownian motion dB_t through $dX_t = U_N(\rho_t)dt + dB_t$. Hence, $dX_t dB_t = dt$ so that one has to pay attention to expand it to sufficient order to take Itô's rules into account.

The simplest case is when the feedback action is unitary with a hamiltonian depending linearly on dX_t. That is: $\Phi_{dX_t}(\rho) = e^{-i\mathfrak{h}dX_t}\,\rho\,e^{i\mathfrak{h}dX_t}$ with $\mathfrak{h}$ hermitic. Then, because $dX_t^2 = dt$,

$$\Phi_{dX_t}(\rho) = \rho - i[\mathfrak{h},\rho]\,dX_t + L_{\mathfrak{h}}(\rho)\,dt,$$

with $L_{\mathfrak{h}}$ the Lindblad operator associated to $\mathfrak{h}$. Hence the system state evolution is $\rho_{t+dt} - \rho_t =:$

$d\rho_t$ with

$$d\rho_t = \Big((L_{\rm sys}(\rho_t) + L_N(\rho_t))dt + D_N(\rho_t)dB_t\Big) + \Big(L_{\mathfrak{H}}(\rho)\,dt - i[\mathfrak{h},\rho]\,dB_t\Big) - i[\mathfrak{h}, N\rho + \rho N^\dagger]dt.$$

It naturally splits in three parts: The first two are the evolutions induced by the monitoring and the feedback map, respectively, and the third is due to the correlations between the feedback and the monitoring.

– A few words on examples of applications.....

– An example based on (semi-classical) gravity. Just a few words.....

Remark 1:
It is clear that this simple example can be generalized to more complicated feedback procedures. The feedback map Φ_{dX_t} has to be a CP-map – a quantum channel – and has to be equal to the identity map in absence of information, that is $\Phi_{dX_t}(\rho) = \rho$ for $dX_t = 0$. Hence, assuming that it depends smoothly on dX_t we may Taylor expand: $\Phi_{dX_t}(\rho) = \rho + dX_t\, L_{feed}(\rho) + \frac{1}{2}dt\,\Phi^{(2)}(\rho) + \cdots$, using $dX_t^2 = dt$. The first term in this Taylor expansion has to be a Lindbladian because Φ is a CP-map.

Remark 2:
Here the feedback loop is classical in the sense that we use the classical information extracted from the system to back act on it. There exists another class of quantum control procedure, sometimes called coherent control, in which one control or drive a quantum system by coupling it to another quantum system, called the ancilla, without reading any classical information. The coupling to the ancilla can be hamiltonian or dissipative.

4.4 Open quantum Brownian motion

In the same way as the Brownian motion can be viewed as the scaling limit of classical random walks, the open quantum Brownian motion (OQBM) is obtained from a limit of open quantum walks in which space and time are simultaneously scaled in the diffusive way: $\delta x^2 \sim \delta t \to 0$.

Continuous limit of OQWs

Recall that the dynamics of the mean (extended) density matrix of OQW is coded into the CP-map $\mathfrak{P}$. Namely, recall that if $\sum_x \bar\rho_n(x) \otimes |x\rangle\langle x| := \mathbb{E}[\rho_n \otimes |x_n\rangle\langle x_n|]$ is the mean density matrix of an OQW (x_n, ρ_n), then its evolution reads

$$\bar\rho_{n+1}(x) = \mathfrak{P}(\bar\rho_n)(x) = \sum_y B_{yx}\bar\rho_n(y)B_{yx}^*,$$

with B_{xy} the transition matrices of the OQW.

As before we first restrict to homogeneous OQW on the line. In the continuous limit, we write $\bar\rho_t = \int dx\, \bar\rho_t(x) \otimes |x\rangle\langle x|$, with $\bar\rho_t(x)$ the mean density matrix on the internal Hilbert space and $\mathrm{Tr}(\bar\rho_t(x))$ the probability density to find the quantum walker at position x at time t. At each time step δt, it is updated according to the above open OQW rule which in this simple case reads,

$$\bar\rho_{t+\delta t}(x) = F_-\,\bar\rho_t(x+\delta x)\,F_-^\dagger + F_+\,\bar\rho_t(x-\delta x)\,F_+^\dagger,$$

We are interested in the scaling limit. The transition matrices are $B_{x;x\pm1} =: F_\pm$ with $F_+^* F_+ + F_-^* F_- = \mathbb{I}$, and we assume that the follwing Taylor expansion around the trivial symmetric solution (with $M = 0$) holds true:

$$F_\pm = \frac{1}{\sqrt{2}} \left[\mathbb{I} \pm \sqrt{\varepsilon} N - \varepsilon (iH + \frac{1}{2} N^\dagger N) + O(\varepsilon^{3/2}) \right]$$

As before, we identify ε with δt, and as for classical walks, we scale $\delta x^2 = \delta t$. Taylor expansion then yields :

$$\partial_t \bar{\rho}_t(x) = -i\big[H, \bar{\rho}_t(x)\big] + \frac{1}{2}\partial_x^2 \bar{\rho}_t(x) - \big(N\partial_x \bar{\rho}_t(x) + \partial_x \bar{\rho}_t(x) N^\dagger\big) + L_N(\bar{\rho}_t(x)),$$

with $L_N(\rho) := N\rho N^\dagger - \frac{1}{2}(N^\dagger N \rho + \rho N^\dagger N)$ as before. This is the equation defining the open quantum Brownian motion map. It mixes pieces from diffusive Fokker-Planck equation and from Lindbladian quantum evolution. It is a natural extension of the heat equation incorporation internal degrees of freedom. It has interesting consequences with, in particular, a cross-over between ballistic and diffusive behavior depending on the form of transition matrix N.

The OQBM dynamical map can of course be written in an operator form:

$$\partial_t \bar{\rho}_t = \mathcal{L}(\bar{\rho}_t) := -i[H, \bar{\rho}_t] - \frac{1}{2}[P, [P, \bar{\rho}_t]] - i\big(N[P, \bar{\rho}_t] + [P, \bar{\rho}_t] N^\dagger\big) + L_N(\bar{\rho}_t),$$

with $P = -i\partial_x$ the momentum operator (which here commutes with N) which is the generator of translations. It is valid for non-diagonal density matrices.

A few words on the microscopic derivation.....

A few words on possible 'physically realistic' realization.....

Remark 1:
The above OQBM dynamical map can be written in a Lindblad form by presenting $\mathcal{L}$ in the form

$$\mathcal{L}(\bar{\rho}_t) = -i\big[\mathcal{H}, \bar{\rho}_t\big] + \big(\mathcal{P}\,\bar{\rho}_t\,\mathcal{P}^\dagger - \frac{1}{2}(\mathcal{P}^\dagger \mathcal{P}\,\bar{\rho}_t + \bar{\rho}_t\,\mathcal{P}^\dagger \mathcal{P})\big),$$

with $\mathcal{P} := P + iN$ and $\mathcal{H} := H + \frac{1}{2}(PN + N^\dagger P)$. As a consequence the OQBM equation is well defined in the sense that the Linbldad operator $\mathcal{L}$ has the required positivity property to formally generate a completely positive map (to make this statement more precise mathematically would require talking about domain operators, etc). In this sense it does not suffer from problems with complete positivity as do Markovian approximations of Caldeira-Leggett models.

Remark 2:
In the discrete setting, we saw that repeated POVMs and OQWs are one and the same. One may wonder if this identification resists to the scaling limit? That is: one may wonder what is relation between quantum trajectories of quantum states under continuous monitoring and the OQBM dynamical map? Recall that when deriving the SDEs for quantum trajectories we also derive the SDEs for the output signals X_t:

$$\begin{aligned} d\rho_t &= -i[H, \rho_t]\, dt + L_N(\rho_t) dt + D_N(\rho_t) dB_t, \\ dX_t &= \mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\, dt + dB_t. \end{aligned}$$

If one goes back to the derivation of those equations, one sees that the scaling used to defined X_t is the same diffusive scaling as for OQBM. Hence (X_t, ρ_t) are the quantum trajectories of the open quantum Brownian motion. Hence, the OQBM mean density matrix is obtained by averaging over quantum trajectory samples. That is: starting from quantum trajectories (X_t, ρ_t) – including both the system state ρ_t and the output signal X_t – we get the mean OQBM density matrix through an averaging procedure:

$$\bar{\rho}_t = \int dx\, \bar{\rho}_t(x) \otimes |x\rangle\langle x| := \mathbb{E}\big[\, \rho_t \otimes |X_t\rangle\langle X_t| \,\big].$$

or alternatively, $\int dx\, \bar{\rho}_t(x) f(x) = \mathbb{E}[\, \rho_t\, f(X_t)\,]$, for any test function $f(x)$. It is a simple matter of computing using Itô rules to prove that the quantum trajectory SDEs imply the OQBM equations for the mean density matrix. In other words, OBQM and simple quantum trajectory, including the output signal, are different reincarnation of one and the same entity.

Generalization to higher dimensions

Of course OQBM can generalized to any dimension and with inhomogeneous hamiltonian and transition matrices. The coordinates x^ν, $\nu = 1, \cdots, D$, then refer to points in a D dimensional space, that we take be the standard flat space $\mathbb{R}^D$ for simplicity. The total Hilbert space is $\mathcal{H} \otimes \mathbb{L}^2(\mathbb{R}^D)$ with $\mathcal{H}$ the internal Hilbert space as before. The hamiltonian H is supposed to be x-dependent. There is one transition matrix N^ν per direction and those are also x-dependent.

As for OQW there are different descriptions, either at the level of quantum trajectories, sample by sample, or at the mean level with a mean density matrix obtained by averaging over quantum trajectories. Similarly to the 1D case, the Lindblad equation for the density matrix $\bar{\rho}_t = \int dx\, \bar{\rho}_t(x) \otimes |x\rangle\langle x|$ is:

$$\partial_t \bar{\rho}_t(x) = -i\big[H(x), \bar{\rho}_t(x)\big] + \frac{1}{2} G^{\mu\nu} \partial_\mu \partial_\nu \bar{\rho}_t(x) - \partial_\mu \big((N^\mu \bar{\rho}_t)(x) + (\bar{\rho}_t N^{\mu\dagger})(x) \big) + L_{N(x)}(\bar{\rho}_t(x)),$$

with Lindbladian $L_N(\rho) = G_{\mu\nu}\big(N^\mu \rho\, N^{\nu\dagger} - \frac{1}{2}(N^{\nu\dagger} N^\mu \rho + \rho\, N^{\nu\dagger} N^\mu)\big)$. Here $G^{\mu\nu}$ is a metric which we take to be constant and $G_{\mu\nu} G^{\nu\sigma} = \delta^\sigma_\mu$. In operator form this Lindblad equation reads

$$\partial_t \bar{\rho}_t = -i\big[H, \bar{\rho}_t\big] - \frac{1}{2} G^{\mu\nu} \big[P_\mu, [P_\nu, \bar{\rho}_t]\big] - i\big[P_\mu, N^\mu \bar{\rho}_t + \bar{\rho}_t N^{\mu\dagger}\big] + L_N(\bar{\rho}_t).$$

with $P_\mu := -i\partial/\partial x^\mu$, the translation operator in the μ-direction, and N^μ operator acting diagonally on position state with $N^\mu(|\phi\rangle \otimes |x\rangle) = (N^\mu(x)|\phi\rangle) \otimes |x\rangle$. Notice that one has to pay attention to the order of the operators which matters if H and N are inhomogeneous.

The quantum trajectories are

$$\begin{aligned} d\rho_t &= \big(-i[H(X_t), \rho_t] + L_{N(X_t)}(\rho_t) \big) dt + D_\mu(\rho_t)\, dB^\mu_t, \\ dX^\mu_t &= U^\mu(\rho_t)\, dt + dB^\mu_t, \end{aligned}$$

with B^ν Brownian motions, $dB^\mu_t\, dB^\nu_t = G^{\mu\nu}\, dt$, and non-linear diffusion coefficient $G^{\mu\nu} D_\nu(\rho) = (N^\mu \rho + \rho\, N^{\mu\dagger}) - \rho\, U^\mu(\rho)$ with potential $U^\mu(\rho) = \mathrm{Tr}(N^\mu \rho + \rho\, N^{\mu\dagger})$, similarly as in 1D. They of course correspond to quantum trajectories for monitoring with higher dimensional multiplets of POVMs.

Notice that the Hamiltonian, the Lindbladian and the diffusion coefficients entering the SDEs for the internal density matrix is evaluated the position X_t of the quantum trajectories. This is very reminiscent of feedback procedure.

5 Strong indirect continuous monitoring

As seen on simple examples, quantum trajectories jump from one pointer states to another when the measure strength is big enough. The presence of jumps has a simple explanation: even though the measurement is strong enough to recursively project the system state onto one of the pointer states, the system evolution still acts to push away the system from those pointer states.

When the measurement strength increases two phenomena appears:
– The quantum states are projected on the pointer states, the quantum trajectories jump from one pointer states to another and the system evolution naively reduces to a quantum Markov chain on pointer states;
– These jumpy trajectories are always dressed with aborted jumps, called spikes, and those survive the limit of infinitely strong measurement.

These two results may seem to be incompatible: The first one implies that under strong monitoring the system state is one of the pointer states with probability one, whereas the second one says that time to time the system goes away from the pointer states. The resolution of this paradox is that the aborted jumps have a time duration vanishing in the strong monitoring limit (and don't form a dense set). Hence, at a fixed given time, the quantum trajectory is not going to be on one of the aborted jumps and the system state is one of the pointer state with probability one. This indicates that the limit to the Markov chain is a weak limit: only the correlation functions of the quantum trajectories converge to those of a Markov chain but the trajectories themselves do not converge almost surely to those of Markov chain.

5.1 The quantum jump Markov chains

We here explain in which sense quantum trajectories reduce to quantum Markov chains on pointer states and how to compute the jump rates from the microscopic data.

The strong monitoring limit

We consider a general quantum system – not a Qu-bit as before – but with a finite dimensional Hilbert space, whose dynamics are prescribed by a Lindbladian $L_{\rm sys}$. In absence of monitoring the system evolution would be $d\rho_t = L_{\rm sys}(\rho_t)\,dt$. Suppose now that an observable is continuously measured at a rate γ^2. As previously explained the density matrix then evolves according to:

$$d\rho_t = L_{\rm sys}(\rho_t)\,dt + \gamma^2 L_N(\rho_t)\,dt + \gamma\, D_N(\rho_t)\,dB_t,$$

where B_t is a standard Wiener process, and N is the measurement operator with $\mathcal{O} = N + N^\dagger$ the measured observable. For simplicity we assume that N is hermitian so that $\mathcal{O} = 2N$, but the generalization is simple. In such case $L_N(\rho) = -\frac{1}{2}[N,[N,\rho]]$ and $D_N(\rho) = N\rho + \rho N - 2\rho\,\mathrm{Tr}(N\rho)$.

We shall write everything in the basis where N is diagonal, $N = \sum_k \nu_k |k\rangle\langle k|$. The states $|k\rangle$ are the pointer states. Let Q_i be the diagonal coefficients of ρ in the pointer basis – the so-called probabilities – and U_{ij} be the non diagonal coefficients of ρ – the so-called phases –,

$$Q_i := \langle i|\rho|i\rangle, \quad U_{ij} := \langle i|\rho|j\rangle, \ i \neq j.$$

We are interested in the limit of very efficient monitoring, that is $\gamma \to \infty$. In this limit, and as observed in the case of a Qu-bit, the quantum trajectories spend much of their time close to the pointer states, and jump time to time in-between pointer states. The jumps from state $|i\rangle$ to $|j\rangle$ occur at a rate depending on the microscopic dynamics. These rates, which we denote m^i_j, may be characterized by the looking at the behavior of the mean probabilities $\overline{Q}_i := \mathbb{E}[Q_i]$ obtained by averaging over quantum trajectories through

$$\partial_t \overline{Q}_j = \sum_i \overline{Q}_i \, m^i_j.$$

The jump rates may be evaluated by looking at the mean probabilities, but we can be more precise and claim that the times in between jumps are exponentially distributed, so that the quantum trajectories are statistically equivalent to a Markov chain. This says something about the multipoint correlations of the quantum trajectories Q_i at different times – not just about the mean. However, a finer structure survives the strong monitoring limit beyond the Markov chain description.

Some care has to be taken to formulate this statement. Indeed, as seen in the case of a Qu-bit, dissipative and hamiltonian dynamical channels do not behave in the same way in the large γ limit. If $L_{\rm sys}$ is generated by a simple hamiltonian, a continuous strong measurement will tend to Zeno freeze the system in one of the pointer states for an arbitrary long time, i.e. when $\gamma \to \infty$ all the jump rates will go to 0. Hence, to get meaningful predictions in this limit we need to adequately rescale the different parts of the hamiltonian dynamics to keep finite jump rates in the large γ limit. Such a rescaling is not required for all parts of the dynamics because jumps that emerge from a dissipative coupling cannot be Zeno frozen.

We consequently split the Lindbladian in four different super-operators: A that sends the probabilities to the probabilities, B the phases to the probabilities, C the probabilities to the phases and D the phases to the phases.

$$\partial_t \rho_t = L_{\rm sys}(\rho_t) \underset{\text{notation}}{\Longleftrightarrow} \left\{ \begin{array}{c} \partial_t Q_i = A(\mathbf{Q})_i + B(\mathbf{U})_i \\ \partial_t U_{ij} = C(\mathbf{Q})_{ij} + D(\mathbf{U})_{ij} \end{array} \right. ,$$

with $A(\mathbf{Q})_i = A^k_i Q_k$, $B(\mathbf{U})_i = B^{kl}_i U_{kl}$, $C(\mathbf{Q})_{ij} = C^k_{ij} Q_k$, and $D(\mathbf{U}) = D^{kl}_{ij} U_{kl}$ (summation over repeated indices is implicit). A good rationale for the differences in their rescaling is that as the strong measurement will tend to shrink the phases they will obviously need a differentiated treatment from the probabilities. We now claim that A needs no rescaling, that C and B need to scale like γ and D like γ^2. In what follow, we thus write :

$$A = \mathcal{A}, \ B = \gamma \mathcal{B}, \ C = \gamma \mathcal{C}, D = \gamma^2 \mathcal{D}.$$

For such a scaling to be consistent with the complete positivity of the map generated by $L_{\rm sys}$ in the large γ limit, $\mathcal{B}$ and $\mathcal{C}$ have to be associated to an Hamiltonian flow and $\mathcal{D}$ needs to be

diagonal: $\mathcal{D}_{ij}^{kl} = -d_{ij}\,\delta_i^k\delta_j^l$. We should also add that the above equation only gives the dominant terms in an expansion in power of γ and that the sub-leading corrections may in general be needed for compatibility with the complete positivity of the map associated to $L_{\rm sys}$. We just claim that they have no impact on the large γ limit as expected and omit them for clarity.

Proposition (Bauer-Bernard-Tilloy) "Markov Quantum Jumps":
With the previous notations, when $\gamma \to \infty$ the finite dimensional distributions of the conditioned density matrix ρ_t converge to those of a finite state Markov process on the projectors associated to the measurement eigenvectors. The jump rate from state $|i\rangle$ to state $|j\rangle$ reads in terms of the rescaled coefficients as:

$$m_j^i = \mathcal{A}_j^i + 2\,\mathfrak{Re}\sum_{k<l}\frac{\mathcal{C}_{kl}^i\mathcal{B}_j^{kl}}{\Delta_{kl}} \tag{1}$$

with $\Delta_{kl} := \frac{1}{2}|\nu_k - \nu_l|^2 + d_{kl}$.

Proof
The proof of this proposition is interesting because it requires dealing with SDEs in the strong noise limit which is not so familiar – the weak noise limit is more familiar. It may have applications in other physical noisy contexts. Here we only give hints on the strategy of the proof. This strategy consists in analysis the large γ limit of the probability kernel $K_t(\rho_0, d\rho)$ to go from a given density matrix ρ_0 to another density matrix ρ, up to $d\rho$, after a time t. As for any process driven by SDEs, this kernel verifies a Kolmogorov or Fokker-Planck equation $\partial_t K_t = K_t\mathfrak{D}$ where $\mathfrak{D}$ is the second order differential operator associated to the SDEs. This differential operator codes for the Itô derivative of any function of the process,

$$df(\rho_t) = (\mathfrak{D}f)(\rho_t)\,dt + (\cdots)dB_t.$$

First, without writing $\mathfrak{D}$ explicitly one can notice that because of Itô's formula the coefficients in front of the noise terms come squared, so that $\mathfrak{D}$ only contains terms of order 0 and 2 in γ. As a result we may write

$$\mathfrak{D} = \mathfrak{D}_0 + \gamma^2\mathfrak{D}_2,$$

where $\mathfrak{D}_2$ is the differential operator associated to the measurement process.
Second, to compute $K_t = e^{t\mathfrak{D}_0 + t\gamma^2\mathfrak{D}_2}$ for large γ we may use the following argument. Any second order operator associated to well defined SDEs is a non-positive operator, so that when γ is very large, even after a small amount of time, only the functions in the kernel of $\mathfrak{D}_2$ are going to play a role in $e^{t\gamma^2\mathfrak{D}_2}$ at very large γ. These functions, which are martingales for the measurement process (because they are in the kernel of $\mathfrak{D}_2$) are in one-to-one correspondance with the pointer states.
Third, the idea is then to perform a perturbative expansion around those remaining eigenfunctions and compute the jump rate between them. □

Remark 1:
It may be worth noticing that the previous proposition only talks about the multipoint correlation functions. Only these correlation functions converge to those of a Markov chains. The proposition does not claim that sample by sample the quantum trajectories converge to those of a Markov chain, a statement which is incorrect.

Remark 2:
The reader may wonder how these results, written in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$, may be related to the generators of the Lindbladian. It is actually easy to verify that the most general scaling that can be written is the following:

$$L_{\text{sys}}(\rho) = -i[\gamma H + \gamma^2 H^{diag}, \rho] + \sum_a L_{M_a}(\rho) + \gamma^2 \sum_b L_{M_b^{diag}}(\rho) + \text{subleading terms}$$

where L_M denotes as above the Lindblad generator associated to M and where the superscript *"diag"* means that the corresponding matrix has to be diagonal, H (without superscript) is any self-adjoint matrix and the N_a (without superscript) can be any matrix. The subleading terms in γ are irrelevant for the jump rate computation.

5.2 A finer structure: quantum spikes

Let us consider a Qu-bit in thermal contact with a diagonal density matrix $\rho = Q|\downarrow\rangle\langle\downarrow| + (1-Q)|\uparrow\rangle\langle\uparrow|$ with $0 \leq Q \leq 1$. The SDEs for a continuous monitoring of the energy, diagonal in the pointer states $|\downarrow\uparrow\rangle$, is:

$$dQ = \lambda(p - Q)\, dt + \gamma\, Q(1-Q)\, dW_t,$$

with λ the thermal relaxation rate and γ^2 the measurement rate.

What are quantum spikes?

As previously explained, the multipoint correlations of these quantum trajectories coincide with a two state Markov chain, with rates λp and $\lambda(1-p)$. However, as it is clear from the figure **Figure...** those quantum trajectories are dressed with multiples aborted jumps, which are not present in Markov chain trajectories. We call them "quantum spikes". The aim of the following is to give a more precise characterization of those spikes.

Quantum spikes are aborted quantum jumps, that is they are excursions of quantum trajectories starting close to 0 say and moving rapidly upwards before returning close to 0, or starting close to 1 and moving downwards. A more precise way to define them consists in first defining the excursions at finite measurement rate γ and then taking the limit $\gamma \to \infty$. Excursions emerging from 0 are defined as the limit when $\epsilon \to 0$ of portions of quantum trajectories between its starting point at ϵ and the first time it reaches $\epsilon/2$. That is: they are portions of quantum trajectories between two points approaching 0, but we have to take two different initial and final points because, as for Brownian motion, quantum trajectories visiting a given point at time t_o visit this point an infinite number of times in any time open time interval including t_o.

Proposition (Bauer-Bernard-Tilloy) "Quantum Spikes":
At large measurement strength γ, the trajectories of Q_t can be reconstructed from the data of two Poisson point processes on $[0,1] \times \mathbb{R}_+$, one attached to the point $Q = 0$ and the other to the point $Q = 1$, with respective intensity

$$\begin{cases} d\nu_0 = \lambda p\, dt \cdot \Big[\delta(1-Q)dQ + \dfrac{dQ}{Q^2}\Big], \\ d\nu_1 = \lambda(1-p)\, dt \cdot \Big[\delta(Q)dQ + \dfrac{dQ}{(1-Q)^2}\Big]. \end{cases}$$

These processes code for the positions of the local maxima (ν_0) and the local minima (ν_1) of the quantum spikes.

Proof.
A few elements of proof are given below. More precisely, what is presented is a set of characterizing consistent checks based on computing directly relevant statistical properties of quantum spikes and comparing them with know results obtained from the large γ limit of the quantum trajectory SDEs. □

Let us explain how the reconstruction procedure works in this simple case. **Figure...** Suppose that the trajectory starts at $Q = 0$ at $t = 0$ and one wants to construct all spikes (of heights bigger than a given threshold σ) which emerge from $Q = 0$. Given a realization of the two Poisson processes one first looks only at the process attached to $Q = 0$. On a finite time interval $[0, T]$ there is a finite number of points above σ, i.e. there is finite number of points in $[\sigma, 1] \times [0, T]$. By construction, these points are the positions and the time coordinates of the maxima of the spikes emerging from 0. The first instance at which one of this point is at 1 corresponds to the first jump from $Q = 0$ to $Q = 1$. One then reconstructs the spikes emerging from $Q = 1$ by implementing the same procedure but using the points of the Poisson process associated to $Q = 1$ as the minima of the spikes. Again the first instance at which one of these points is at 0 corresponds to the next jump from $Q = 1$ to $Q = 0$. And the procedure starts again.

Remark 1:
Also true in the Rabi case..... And this is more surprising.....
And more difficult to prove for higher dimensional quantum trajectories.....

Remark 2:
Although we do not present the complete proof, there is a simple argument telling why the intensity of the Poisson process in the variable Q has the same form in all cases, independently of the nature of the Liouvillian dynamics. This intensity is determined, up to a proportionality coefficient, by the probability that Q_t, the diagonal component of the density matrix, starting at $Q_i \in [\sigma, Q]$, escapes at Q from the interval $[\sigma, Q]$, in the limit $\sigma \to 0$ (after the limit $\gamma \to \infty$ has been taken). Now, at fixed σ, Q_i, Q away from 0 and 1 and large γ, the process dominating these events is that generated by the measurement process. Because they are martingales for the measurement process, the diagonal components of the density matrix Q_t are Brownian motion up to a random time parameterisation. These escape probabilities are thus that of the Brownian motion and equal to $\frac{Q_i - \sigma}{Q - \sigma}$. They converge to $\frac{Q_i}{Q}$ in the limit $\sigma \to 0$. These escape probability are universal and so are the intensities of the Poisson processes.

Comparing the spikes and quantum trajectories statistics

Instead of proving this proposition we show that how properties of quantum trajectories valid in the large γ limit can be recovered using the spike Poisson processes. Actually the computation we are about to present are enough to fully determine the intensities of these processes.

Recall that when studying quantum trajectories of a Qu-bit in thermal contact we pointed out two results: one about the distribution of the time interval between two jumps and another about the distribution of the maximum height of the excursions. Namely:

(i) The time duration between two successive jumps is a Poisson variable with mean $1/\lambda p$ for jumps from $Q_i = 0$ to $Q_f = 1$, and $1/\lambda(1-p)$ for jumps from $Q = 1$ to $Q = 0$;
(ii) For $0 < \sigma < Q_i < Q < 1$, the probability for a quantum trajectory starting at Q_i to reach the height Q before going back to σ is Q_i/Q in the limit $\sigma \to 0$.

The last statement is a claim about the distribution of the maximum height of the spikes starting from 0 conditioned to be bigger than Q_i. These statements have been proved directly from analyzing the quantum trajectory SDEs. Their proofs in [**?**] involve a limit $\gamma \to \infty$ at fixed σ, Q_i and Q. We are thus extending them by then taking the limit $\sigma \to 0$ after the limit $\gamma \to \infty$.

Recall that a Poisson point process on measurable space $\mathcal{M}$ with intensity ν is characterised by the following properties: (a) For any measurable set $U \subset \mathcal{M}$, the number $\mathcal{N}_U$ of points in U is a Poisson variable with mean $\nu(U)$; (b) For any family $U_1, U_2, \cdots, U_n$ of disjoint measurable sets in $\mathcal{M}$, the random numbers $\mathcal{N}_{U_1}, \mathcal{N}_{U_2}, \cdots, \mathcal{N}_{U_n}$ are independent random variables.

The first point (i) is easy to verify. Suppose that the trajectory is at $Q = 0$ at time $t = 0$. The probability that the first jump occurs at time t, up to dt, is the probability that the sample of the process associated to $Q = 0$ contains no point in $\{1\} \times [0, t]$ and a point in $\{1\} \times [t, t + dt]$ with dt with small. This probability is:

$$\begin{aligned} \mathbb{P}\big[\mathcal{N}_{\{1\}\times[0,t]} = 0,\ \mathcal{N}_{\{1\}\times[t,t+dt]} = 1\big] &= e^{-\nu_0(\{1\}\times[0,t])} \cdot \nu_0(\{1\} \times [t, t + dt]) e^{-\nu_0(\{1\}\times[t,t+dt])} \\ &= e^{-\lambda p t} \lambda p \, dt. \end{aligned}$$

To prove the second point (ii) consider a sample for which a spike started at zero goes above Q_i and compute the probability that this spike goes further above Q. This is the probability that, conditioned on having a point on $\{[\sigma, 1] \times [0, \delta t]\}$ with δt vanishingly small, the sample of the process associated to the vertex $Q = 0$ contains no point in $[Q_i, Q] \times [0, \delta t]$ and one point in $[Q, 1] \times [0, \delta t]$. This probability is:

$$\mathbb{P}\big[\mathcal{N}_{[Q_I,Q]\times[0,\delta t]} = 0,\ \mathcal{N}_{[Q,1]\times[0,\delta t]} = 1 \big| \mathcal{N}_{[Q_i,1]\times[0,\delta t]} = 1\big]\big|_{\delta t\to 0} = \frac{\nu_0([Q, 1] \times [0, \delta t])}{\nu_0([Q_i, 1] \times [0, \delta t])}\Big|_{\delta t\to 0} = \frac{Q_i}{Q}$$

Notice that, reciprocally, these two computations completely fix the intensity of the point process to be $d\nu_0 = \lambda p[\delta(Q) + dQ/Q^2]$, including the relative weight of the $\delta(1-Q)dQ$ and dQ/Q^2 terms.

The third propriety of the quantum trajectory of thermal Qu-bit which we mentioned earlier and that we would like to verify using the description in terms spikes is the following:
(iii) For $0 < Q_i < Q < 1$, the limit for $\gamma \to \infty$ of the distribution of the first instance that a trajectory starting at Q_i reaches the point Q is

$$d\mathbb{P}_{\text{excur}} := \frac{Q_i}{Q}\, \delta(t) dt + (1 - \frac{Q_i}{Q}) \frac{\lambda p}{Q} e^{-\frac{\lambda p}{Q} t} dt.$$

This result is appropriate to deal with the maxima emerging from 0, and there is a twin formula for the minima emerging from 1. The formula (iii) can be derived from the spikes' description. One needs to compute the distribution of the first time the trajectory reaches Q starting at Q_i. Conditioning to start at Q_i amounts to condition the spikes' process to have

a point in $[Q_i, 1] \times [0, \delta t]$ with δt vanishingly small. We aim at computing the distribution of the random time it takes before a next spike going above Q occurs. There are two possibilities. Either the initial spike above Q_i is actually above Q, so that the occurrence time is zero. This happens with probability

$$\mathbb{P}\big[\mathcal{N}_{[Q,1]\times[0,\delta t]} = 1 \big| \, \mathcal{N}_{[Q_i,1]\times[0,\delta t]} = 1\big]\big|_{\delta t\to 0} = \frac{\nu_0([Q,1]\times[0,\delta t])}{\nu_0([Q_i,1]\times[0,\delta t])} = \frac{Q_i}{Q}.$$

Or the next spike above Q differs from the initial spike above Q_i. The probability that its occurrence time is t, up to dt, is then

$$\begin{aligned}&\mathbb{P}\big[\mathcal{N}_{[Q_i,Q]\times[0,\delta t]} = 1, \, \mathcal{N}_{[Q,1]\times[0,t]} = 0, \, \mathcal{N}_{[Q,1]\times[t,t+dt]} = 1 \big| \, \mathcal{N}_{[Q_i,1]\times[0,\delta t]} = 1\big]\big|_{\delta t\to 0}\\ &= \frac{\nu_0([Q_i,Q]\times[0,\delta t])\,\nu_0([Q,1]\times[t,t+dt])}{\nu_0([Q_i,1]\times[0,\delta t])}\, e^{-\nu_0([Q,1]\times[0,t])}\Big|_{\delta t\to 0} = e^{-\frac{\lambda p}{Q}t}\,\big(1-\frac{Q_i}{Q}\big)\frac{\lambda p}{Q}\,dt.\end{aligned}$$

Summing up the two above contributions gives the distribution $d\mathbb{P}_{\rm excur}$ for this stopping time, as it should.

Hence, the spikes' description of the quantum trajectories at infinitely strong monitoring gives an efficient way to compute fine structures of those trajectories beyond the Markov chain description.

An explicit construction at infinite γ

Here we take an alternative route and present a different approach to directly reconstruct the process in the infinite strong measurement limit. This construction is based on using the reflected Brownian motion parametrized not with the usual time but with its local time – i.e. the time the Brownian motion spends in 0 or 1. This provides another way to take the large noise limit in SDE, which may have potential applications in other contexts.

We have to deal with the SDE $dQ_t = \lambda(p - Q_t)dt + \gamma Q_t(1 - Q_t)dB_t$ when $\gamma \to \infty$. Let us first look at the linear version of it,

$$dX_t = \lambda p dt + \gamma X_t dB_t.$$

From what we know about quantum trajectories, we expect that X_t is always positive, and that its trajectories for $\gamma \to \infty$ are formed with spikes emerging from the real axis – these are simply obtained by zooming the quantum trajectories of Q_t close to zero.

Alternatively, for γ large, the drift term is negligible as soon as $X_t \neq 0$. Hence away from zero, X_t is a martingale – because $dX_t = \gamma X_t dB_t$ – and hence a Brownian motion up time reparametrisation. Let us use that time to parametrize X_t. That is, let us define the new 'time' τ by $d\tau := \gamma^2 X_t^2 dt = (dX_t)^2$ and set $Z_\tau = X_t$ with the time t viewed as an implicit function τ. By construction

$$dZ_\tau = \frac{\lambda p}{\gamma^2 Z_\tau^2} d\tau + dW_\tau = \lambda p\, dt_\tau + dW_\tau,$$

where $dW_\tau = \gamma X_t dB_t$ is a Brownian motion with respect to τ. Again, in the limit $\gamma \to \infty$ the first term does not matter when $Z_\tau \neq 0$ and away from zero Z_τ behaves like a Brownian motion. This first term plays a role when Z_τ is close to zero and, because it is repulsive, it reflects Z_τ away from the real axis and ensures that Z_τ stays positive.

So we have to find a process which stays positive, is reflected on the real axis, and behaves like the Brownian motion away from zero. This process has to be the reflected Brownian motion which is equivalent to its absolute value, i.e. it is to be $|\hat{W}_\tau|$ for some Brownian motion $\hat{W}_\tau$. Recall the Tanaka formula for the Itô derivative of the absolute value of the Brownian motion

$$d|\hat{W}_\tau| = \delta(\hat{W}_\tau)d\tau + \mathrm{sign}(\hat{W}_\tau)d\hat{W}_\tau =: dL_\tau^{(0)} + dW_\tau,$$

where we defined W_τ by $dW_\tau = \mathrm{sign}(\hat{W}_\tau)d\hat{W}_\tau$, and the "local time" at zero $L_\tau^{(0)}$ by $dL_\tau^{(0)} = \delta(\hat{W}_\tau)d\tau$, which is the time spend at the origin by the Brownian motion $\hat{W}_\tau$.

Hence, by identifying Z_τ with the reflected Brownian $|\hat{W}_\tau|$ we got the identification $dL_\tau^{(0)} = \lambda p\, dt = \frac{\lambda p}{\gamma^2 Z_\tau^2} d\tau$. That is: up to a proportionality coefficient, the original time t is the local time of the Brownian motion $\hat{W}_\tau$. Since we started from $X_t = Z_\tau$, we get that X_t is equivalent (in law) to a reflected Brownian but parametrized by its local time at zero.

It is clear pictorially **Figure...** that this produces spiky trajectories because the time t is flat when the trajectory is away from the zero.

It is also clear that the distribution of these spikes form a Poisson process with intensity $d\nu = \lambda p\, dtdz/z^2$. Indeed, in the τ-parametrization, a spike of height at least z_o corresponds to a Brownian excursion stasted at the origin and going above z_o before coming back to the origin. The probability for such an event is proportional to $1/z_o$ and hence the intensity – which is the z_o-derivative of this probability – is proportional to dz_o/z_o^2.

For the complete quantum trajectories, not the linear version, we have to consider a Brownian motion $\hat{W}_\tau$ on the trip $[0,1] \times \mathbb{R}_+$ reflected on the two boundaries of the strip, the lower at 0 and the upper at 1. We can then claim

Claim "Reconstructed spiky trajectories":
The quantum trajectories Q_t, for a Qu-bit in contact with a thermal bath, are equivalent in law to a Brownian motion $\hat{W}_\tau$ reflected at 0 *and* 1 *but parametrized by its local times via the relation*

$$t = \frac{1}{\lambda p}\, L_\tau^{(0)} + \frac{1}{\lambda(1-p)}\, L_\tau^{(1)},$$

with $L_\tau^{(0)}$ and $L_\tau^{(1)}$ the local times at 0 *and* 1 *respectively.*

The arguments presented above give good (physical) evidences for this proposition but a complete proof would require a more mathematically rigorous approach. The generalization of this proposition for higher dimensional system (bigger than a Qu-bit) is yet unclear.

Remark 1:
One may naively argue in favor of the proposition as follows. Let us change time parametrization in the quantum trajectory SDE in a manner similar as for the linearized version by defining a new time τ by $d\tau := (dQ_t)^2 = \gamma^2 Q_t^2(1-Q_t)^2 dt$. Let $Z_\tau := Q_t$ and W_τ the Brownian motion, with respect to τ, defined via $dW_\tau := \gamma Q_t(1-Q_t)dB_t$, with $dW_\tau^2 = d\tau$. We have $dZ_\tau = \frac{\lambda p}{\gamma^2 Z_\tau^2 (1-Z_\tau)^2}\, d\tau + dW_\tau$. In the limit $\gamma \to \infty$, the drift has a relevant contribution only for Z_τ close to 0 or 1, so that we may write

$$dZ_\tau \simeq \frac{\lambda p}{\gamma^2 Z_\tau^2}\, d\tau - \frac{\lambda(1-p)}{\gamma^2(1-Z_\tau)^2}\, d\tau + dW_\tau = dL_\tau^{(0)} - dL_\tau^{(1)} + dW_\tau,$$

with the identification $dL_\tau^{(0)} \simeq \frac{\lambda p}{\gamma^2 Z_\tau^2}\, d\tau$ and $dL_\tau^{(1)} \simeq \frac{\lambda(1-p)}{\gamma^2(1-Z_\tau)^2}\, d\tau$ as in the linear version. Now recall the relation between the new and original time parametrisation, $d\tau = \gamma^2 Q_t^2 (1-Q_t)^2 dt$, or equivalently $dt = \frac{d\tau}{\gamma^2 Z_\tau^2 (1-Z_\tau)^2}$. Again this is non flat only for Z_τ close to 0 or 1, and we get:

$$dt \simeq \frac{d\tau}{\gamma^2 Z_\tau^2} + \frac{d\tau}{\gamma^2 (1-Z_\tau)^2} = \frac{1}{\lambda p}\, dL_\tau^{(0)} + \frac{1}{\lambda(1-p)}\, dL_\tau^{(1)},$$

as suggested in the proposition.

Remark 2:
In the linear case, the construction can be made rigorous using a result due to Skorohod. This result claims that given a continuous function $x(t)$ defined for $t \geq 0$, with $x(0) = z \geq 0$, there exists a unique non-decreasing continuous function $l(t)$ such that the function $y(t) := x(t) + l(t)$ is positive and such that $l(t)$ is constant outside the set $\{t;\ y(t) = 0\}$. That is: $l(t)$ increases only when $y(t) = 0$. An explicit solution is given by

$$l(t) = \max\big[0, \max_{0 \leq s \leq t}[-x(s)]\big].$$

In the case in which $x(t)$ is the Brownian motion B_t started at z, another result due to Levy states that the process $y(t)$ is equivalent in law to the reflected Brownian motion $|B(t)|$, and $l(t)$ is the local time $L_t^{(0)}$ at 0 thanks to the Tanaka relation $d|B_t| = dL_t + \mathrm{sign}(B_t)\, dB_t$.

5.3 Applications/illustrations

– Operator localised on spikes [**à voir?...**]...
– Entropy balance....

6 Quantum stochastic processes

6.1 Quantum noises and Fock spaces

6.2 Quantum stochastic differential equations

6.3 From quantum SDE to quantum trajectories

6.4 An out-of-equilibrium application.....