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ON THE WESS–ZUMINO–WITTEN MODELS ON RIEMANN SURFACES*

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We give a formulation of the Wess–Zumino–Witten models on Riemann surfaces of arbitrary genus in which the Ward identities for the current algebras become complete. It requires twisting the models in a non-abelian way by Lie group elements. The Ward identities are written in terms of twisted Poincaré series for Schottky groups and the zero modes of the currents are defined by a Lie derivation acting on the twists. Furthermore, we identify the denominator of the chiral partition function and we argue that both the numerator and the denominator of the partition function satisfy a heat equation on the moduli space.

1. Introduction

Two-dimensional conformal field theories [1] are of great interest not only as models of critical phenomena in statistical physics but also as classical solutions to a quantum string theory. Even though there has been great progress in the last few years, we are still far from a complete understanding. Except for the formulation of general principles [2–5], most of the progress so far has centered around free conformal field theories [6] – especially in connection with bosonization phenomena. As a first step toward a better understanding, it is useful to deal with solvable examples of conformal field theories, and in particular with the Wess–Zumino–Witten (WZW) models [7] on Riemann surfaces of arbitrary genus. Most of the solvable conformal field theories can be reached from the WZW models via a coset construction [8].

In the WZW models [7, 9], in addition to the conformal symmetry, there are two conserved currents, $J_z^a(z)$ and $\bar{J}_{\bar{z}}^a(\bar{z})$, which generate two commuting current algebras, $\mathscr{G}^{(1)}$ and $\bar{\mathscr{G}}^{(1)}$. $\mathscr{G}^{(1)}$ is the affinization of the semi-simple Lie algebra $\mathscr{G}$ of the Lie group G on which the models are defined. There are Ward identities

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associated with both of these symmetries [9]. As pointed out in ref. [10], the "standard" formulation of the WZW models on Riemann surfaces is not complete; i.e. the Ward identities do not determine the correlation functions with insertions of currents in terms of those without insertions.

Let us recall how the Ward identities for the current algebras on a Riemann surface of genus h were formulated in ref. [4]. Consider N affine primary fields, $\Phi_1(\xi_1)\dots\Phi_N(\xi_N)$ (affine primary fields are characterized through eq. (4.10), see sect. 4 below). They belong to some representations $\rho_{(n)}$ of the finite semi-simple Lie algebra $\mathscr{G}$. The Ward identities for one insertion of the current $J_z^a(z)$ are [4]

$$\begin{aligned}\langle J_z^a(z)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle = &\sum_{n=1}^{N}\partial_z\log E(z,\xi_n)t^a_{(n)}\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle\\ &+\sum_{j=1}^{h}\omega_j(z)\langle J^a_{0;\,j}\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle.\end{aligned}\tag{1.1}$$

Here, $t^a_{(n)}=\rho_{(n)}(t^a)$ is the representation of the Lie algebra $\mathscr{G}$ which acts on the fields $\Phi_n(\xi_n)$. The matrices t^a satisfy $[t^a,t^b]=f_c^{ab}t^c$ where f_c^{ab} are the structure constants of $\mathscr{G}$. To write eq. (1.1), we have chosen a basis of canonical cycles on the Riemann surface, (a_j,b_j), $j=1,\dots,h$. $E(z,\xi)$ is the prime form on the Riemann surface; it depends on the choice of the canonical cycles (a_j,b_j). The $\omega_j(z)$'s are the holomorphic differential forms dual to the canonical cycles (a_j,b_j) [11]. The operators $J^a_{0;\,j}$ are the zero modes of the currents

$$J^a_{0;\,j}=\oint_{a_j}J_z^a(z).\tag{1.2}$$

The proof of eq. (1.2) either relies on the use of appropriate (non-chiral) Green functions [4], or on the properties of meromorphic differential forms on Riemann surfaces [10]. But none of this demonstration gives a meaning to the correlation functions with insertions of the zero modes $J^a_{0;\,j}$.

The lack of a precise definition of the action of the zero modes $J^a_{0;\,j}$ on the correlation functions is very bad. First, because the zero modes contain all the information concerning the solitonic sector of the theory. For example, the partition function is completely determined from the expectation values of these zero modes [10]. Second, because with Ward identities written as in eq. (1.1), the study of the WZW models stop at this point; we cannot deduce linear differential equations satisfied by the correlation functions, and thus we cannot solve the models.

In this article, we present a complete formulation of the WZW models on Riemann surfaces. In this version of the WZW models, the affine currents $J_z^a(z)$ are twisted by elements of the Lie group G. These twists define new moduli which are valued in the Lie group. The zero modes are defined by Lie derivation with respect to these new moduli. Therefore, the Ward identities and the linear differential equations satisfied by the correlation functions involve derivatives with respect to all the moduli parameters – the moduli of the Riemann surface and the new moduli parameters.

This article is organized as follows. In sect. 2 we briefly recall how the WZW models on the torus were described in ref. [10]. It gives us enough clues to extend this construction to higher genus. In sect. 3, we introduce the mathematical ingredients needed in the formulation of the Ward identities as they are presented in sect. 4. Sect. 5 deals with the Sugawara construction and the associated Virasoro × Kac–Moody Ward identities. In particular, we argue that both the denominator and the numerator of the chiral partition functions satisfy a kind of heat equation on the moduli space. An appendix is devoted to the Virasoro Ward identities.

2. Back to the torus

Before going deeply into the WZW models on Riemann surfaces of arbitrary genus, let us recall how we describe these models on the torus. As always, once the case of the torus is correctly interpreted, the generalization to arbitrary genus is more or less straightforward. We first present the results obtained in ref. [10], and then give the correct interpretation.

To give a precise meaning to the zero modes J_0^a – on the torus each current component has only one zero mode – we introduced the so-called "character valued expectation values" which are denoted by $\langle \cdots \rangle_g$. In the operator formalism, they correspond to inserting an element g of the Lie group G

$$\langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g = \frac{1}{Z(\tau;\, g)} \operatorname{Tr}\big(g \Phi_1(\xi_1) \dots \Phi_N(\xi_N) q^{L_0} \big). \tag{2.1}$$

The trace is taken in the physical Hilbert space and L_0 is the zero component of the Sugawara stress-tensor. Inside the trace, the element g of G is represented by an exponentiation of an element of the Lie algebra $\mathscr{G}$ generated by the zero modes J_0^a. $Z(\tau;\, g)$ is the partition function (twisted by g)

$$Z(\tau;\, g) = \operatorname{Tr}\big(g q^{L_0} \big), \tag{2.2}$$

with $q = \exp(i2\pi\tau)$, where τ, with $\operatorname{Im}\tau > 0$, is the moduli parameter of the torus.

All the correlation functions depend on g. The action of the zero modes J_0^a on the correlation functions is defined by

$$\langle J_0^a \rangle_g = \mathscr{L}^a \log Z(\tau; g), \tag{2.3}$$

$$\langle J_0^a \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g - \langle J_0^a \rangle_g \langle \Phi_1(\xi_1) \dots (\Phi_N(\xi_N) \rangle_g$$
$$= \mathscr{L}^a \langle \Phi_1 \xi_1) \dots \Phi_N(\xi_N) \rangle_g . \tag{2.4}$$

where the $\mathscr{L}^a$ are the left invariant Lie derivatives on G.

Since the correlation functions now depend on g, the contraction function – i.e. the analogue of the derivative of the logarithm of the prime form in eq. (1.1) – also depends on g. It is no longer a scalar but a matrix; i.e., it is 1-form valued in End($\mathscr{G}$), the endomorphisms of the Lie algebra $\mathscr{G}$. The Ward identities for one and two insertions of affine currents take the following forms

$$\langle J_z^a(z) \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g - \langle J_z^a(z) \rangle_g \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_n) \rangle_g$$
$$= \sum_{n=1}^{N} \omega(z, \xi_n | g)^{ab} t^b_{(n)} \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g$$
$$+ z^{-1} \mathscr{L}^a \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g , \tag{2.5}$$

$$\langle J_z^a(z) J_w^b(w) \Phi_1(\xi_1) \dots \Phi_N(\xi_n) \rangle_g - \langle J_z^a(z) \rangle_g \langle J_w^b(w) \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g$$
$$= -K \, \partial_w \omega(z, w | g)^{ab} \langle \phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g$$
$$+ \left\{ \omega(z, w|g)^{ad} f_c^{db} + \delta_c^b \sum_{n=1}^{N} \omega(z, \xi_n | g)^{ad} t^d_{(n)} \right\} \langle J_w^c(w) \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g$$
$$+ z^{-1} \mathscr{L}^a \langle J_w^b(w) \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g . \tag{2.6}$$

In ref. [10], these Ward identities were proved by using the Kubo–Martin–Schwinger (KMS) condition [12]

$$\langle J_z^a(z) J_w^b(w) \dots \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g$$
$$= q \langle J_w^b(w) \dots \Phi_1(\xi_1) \dots \Phi_N(\xi_N) J_z^d(zq) \rangle_g \mathrm{Ad}(g)^{da} \tag{2.7}$$

where $\mathrm{Ad}(g)$ is the adjoint representation of G. Eq. (2.7) is equivalent to the cyclic property of the trace. Its proof uses the fact that the currents $J_z^a(z)$ are Virasoro primary fields with conformal weight one and that they belong to the adjoint representation of G. Thus we have

$$q^{L_0}J_z^a(z)q^{-L_0}=qJ_z^a(zq) \quad \text{and} \quad gJ_z^a(z)g^{-1}=J_z^d(z)\mathrm{Ad}(g)^{da}. \tag{2.8}$$

Together with the commutation relations between the primary fields and the modes of the currents $J_z^a(z)$, the KMS condition (2.7) allowed us to determine the function $\omega(z,w|g)$

$$\omega(z,w|g)=\sum_{n=1}^{\infty}\frac{q^n}{q^nz-w}\mathrm{Ad}(g^{-n})+\sum_{n=0}^{\infty}\frac{w/z}{q^{-n}z-w}\mathrm{Ad}(g^n). \tag{2.9}$$

The function $z\omega(z,w|g)$ is the meromorphic continuation of the function given in eq. (3.8) of ref. [10].

The interpretation goes as follows. First, to write the KMS condition as in eq. (2.7) corresponds to defining the torus by identifying the points z and $\gamma_0(z)=zq$ in the complex plane. The torus is obtained by gluing together the two sides of the annulus $A=\{z\in\mathbb{C}, |q|<|z|<1\}$. In other words, if Γ_0 denotes the cyclic group generated by the Mobius transformation γ_0, $\Gamma_0=\{\gamma_0^n, n\in\mathbb{Z}\}$, then the torus is identified as $\mathbb{C}/\Gamma_0$. The fundamental domain of $\mathbb{C}/\Gamma_0$ is the annulus A. The canonical cycles of the torus, (a,b), can be choosen as the circle C_q, $C_q=\{z\in\mathbb{C}, |z|=|q|\}$, for the cycle a, and as a path from C_q to the other side of the annulus A for the cycle b.

Second, we have implicitly supposed that the currents $J_z^a(z)$ possess only integral modes. It means that we have choosen to describe the affine algebra $\mathcal{G}^{(1)}$ in one of its homogeneous gradations. Therefore, the currents $J_z^a(z)$ are single valued around the cycle a. But, to insert the element g in the correlation functions, eq. (2.1), implies that in the path-integral formalism the currents $J_z^a(z)$ are twisted along the cycle b by the group element g. To be precise, for $\gamma_0(z)=zq$, the twist is

$$\gamma_0^*J_z^a(z)\equiv\gamma_0'(z)J_z^a(\gamma_0(z))=\mathrm{Ad}(g)^{ab}J_z^b(z). \tag{2.10}$$

This twisted boundary condition is implicit in the KMS condition (2.7). The zero modes J_0^a are defined by Lie derivation acting on this twist.

Third, the contraction function $\omega(z,w|g)$ possesses a geometrical interpretation. The "double pole contraction" reads

$$\partial_w\omega(z,w|g)=\sum_{n\in\mathbb{Z}}\frac{q^n}{(q^nz-w)^2}\mathrm{Ad}(g^{-n}). \tag{2.11}$$

It is a Poincaré series for the cyclic group Γ_0 twisted by the elements g^{-n} of the Lie group G [13].

The strategy for studying the WZW models at arbitrary genus is now obvious. It decomposes into three steps.

(i) We first describe the Riemann surface Σ as the quotient of a covering $\mathscr{S}$ of Σ by a kleinian group $\Gamma, \Sigma = \mathscr{S}/\Gamma$. For concreteness, we will choose the Schottky parametrization in which $\mathscr{S}$ is the complex plane (with a point at infinity) and Γ a Schottky group [13–15]. This choice will allow us to develop a chiral analysis. Schottky groups have already been widely used in string theory [16, 17].

(ii) The affine currents $J_z^a(z)$ must be twisted along the non-trivial cycles by elements of the Lie group G. They will therefore belong to a flat vector bundle defined over $\mathscr{S}$ whose rank is dim G. The expectation values will depend on these twists. As in eqs. (2.3) and (2.4), the zero modes will be defined by the Lie derivatives acting on these twists.

(iii) The contraction functions – as well as the zero modes – will be explicitly defined in terms of twisted Poincaré series. They are twisted automorphic forms for the kleinian group Γ [13].

3. Mathematical tools

In this section, we describe the mathematical framework used in sect. 4. By the uniformization theorem [14, 15], all Riemann surfaces Σ can be described as the quotient of a covering $\mathscr{S}$ by a discontinuous kleinian group Γ: $\Sigma = \mathscr{S}/\Gamma$. There are many possible choices of kleinian groups Γ. Among them, the most popular choices are either those where Γ is a fuchsian group or those where Γ is a Schottky group. The first case – Γ a fuchsian group and $\mathscr{S}$ the upper-half complex plane – corresponds to describing the Riemann surface Σ via its "cut surface" drawn in fig. 1. This choice facilitates the description of the twisted boundary conditions i.e., the

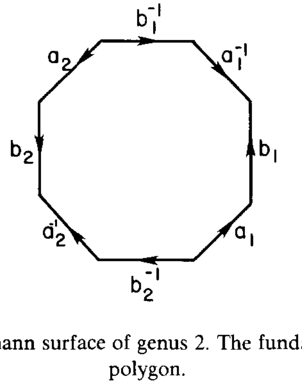

Fig. 1. The "cut surface" of a Riemann surface of genus 2. The fundamental region is the interior of the polygon.

spin structures. But the complex structure of the Teichmüller space is somewhat obscure in this fuchsian parametrization. On the contrary, the second choice gives rise to a complex parametrization of the moduli space, the so-called Schottky parametrization. In the Schottky description of the Riemann surfaces the covering space is the complex plane $\mathbb{C}$ and Γ is a discontinuous subgroup of $\mathrm{PSL}(2,\mathbb{C})$.

The elements γ of $\mathrm{PSL}(2,\mathbb{C})$ are the Mobius – or homographic – transformations, $z \to \gamma(z)$

$$\gamma(z)=\frac{az+b}{cz+d}, \qquad \gamma^{-1}(z)=\frac{-dz+b}{cz-a}, \tag{3.1}$$

with $a, b, c, d \in \mathbb{C}$ and $ad-bc=1$. Instead of parametrizing γ with the coefficients a, b, c, d, it is useful to characterize it via its two fixed points, denoted by u_γ and v_γ, and via its multiplier q_γ, $|q_\gamma|<1$. They are defined by the following equations:

$$\begin{gathered}\gamma(u_\gamma)=u_\gamma, \qquad \gamma(v_\gamma)=v_\gamma,\\ q_\gamma+1/q_\gamma=(\operatorname{tr}\gamma)^2-2.\end{gathered} \tag{3.2}$$

In terms of these parameters, the Mobius transformation γ can be written as follows (if $u_\gamma \neq v_\gamma$)

$$\frac{\gamma(z)-u_\gamma}{\gamma(z)-v_\gamma}=q_\gamma\frac{z-u_\gamma}{z-v_\gamma}. \tag{3.3}$$

The isometric circle a_γ of γ is the circle $|cz+d|=1$. On this circle, γ preserves the distances, $|\gamma'(z)|=1$. The isometric circle a'_γ of γ^{-1} is the circle $|cz-a|=1$. Notice that $a'_\gamma=\gamma(a_\gamma)$, and that γ maps the interior of a_γ in the exterior of a'_γ, and the exterior of a_γ into the interior of a'_γ.

The Schottky groups are constructed as follows [13–15]. Consider h Mobius transformations $\gamma_1,\dots,\gamma_h$ whose pairs of isometric circles, $a_1, a'_1;\dots; a_h, a'_h$, are external to one another. The Schottky group Γ is the group generated by the transformations γ_j, $j=1,\dots,h$: $\Gamma=\langle\gamma_1;\dots;\gamma_h\rangle$. That is to say, it is the infinite group of Mobius transformations obtained by taking any combination of products of positive or negative powers of the γ_j's.

The Riemann surface $\Sigma=\mathbb{C}/\Gamma$ is defined by identifying points on the complex plane which are conjugated by an element of Γ. Identifying the circles a_j and a'_j produces h handles on the sphere. Since the transformation γ_j maps the exterior of a_j in the interior of a'_j, the fundamental domain of $\mathbb{C}/\Gamma$ is the region F outside the $2h$ circles (a_j, a'_j), $j=1,\dots,h$. The complex plane $\mathbb{C}$ is covered by the Γ-images of F.

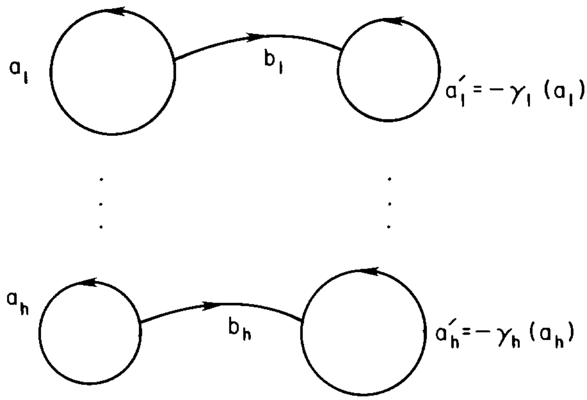

Fig. 2. For Γ a Schottky group of genus h the fundamental domain is the region exterior to the circles a_j and a'_j. The canonical cycles of the Riemann surface $\Sigma = \mathbb{C}/\Gamma$ are the cycles a_j and b_j.

As a canonical homology basis of the Riemann surface Σ, we can choose the circles $a_1, \ldots, a_h$ and the paths $b_1, \ldots, b_h$ from the circles a_j to the circles a'_j (see fig. 2).

Each homographic transformation γ_j is parametrized by the three complex parameters $(q_j; u_j; v_j)$. But two Schottky groups, Γ and $M\Gamma M^{-1}$, conjugated by a Mobius transformation M, define two conformally equivalent Riemann surfaces. Therefore, the $3h$ moduli parameters $(q_j; u_j; v_j)$, $j = 1, \ldots, h$, modulo this equivalence relation, give rise to $(3h-3)$ complex moduli parameters of the Riemann surface Σ. Note that the parameters q_j remain unchanged during the conjugation $\Gamma \to M\Gamma M^{-1}$ whereas the parameters $(u_j; v_j)$ are transformed into $(M(u_j); M(v_j))$.

Besides producing an explicit complex parametrization of the moduli space, this parametrization has the advantage of giving a concrete framework for applying Friedan and Shenker's program [2]. For example, it is enlightening in fig. 2 that the limit in which the radii of the circles a_h and a'_h vanish corresponds to inserting fields located at the centers of these circles. In other words, by applying this limit one relates correlation functions between N fields in genus h to correlation functions between $(N+2)$ fields in genus $(h-1)$*. Taking this limit consists of decomposing the coset of the Schottky group $\Gamma = \langle \gamma_1; \ldots; \gamma_h \rangle$ by its subgroup $\tilde{\Gamma} = \langle \gamma_1; \ldots; \gamma_{h-1} \rangle$ generated by the remaining $(h-1)$ Mobius transformations.

Let us now define the twisted flat vector bundle that we will be interested in. The fiber of this vector bundle is the Lie algebra $\mathscr{G}$ on which the Lie group G acts by its adjoint representation. To define a flat vector bundle over Σ whose twists are elements of the Lie group G, we have to specify a homomorphism from $\pi_1(\Sigma)$ to G [14]. Once the canonical homology basis (a_j, b_j) is given, this homomorphism is

* I thank J.-B. Zuber for a discussion concerning this remark.

completely specified by the data of the group elements $(g(a_j), g(b_j))$ provided that they satisfy the following non-linear constraint

$$\prod_{j=1}^{h} g^{-1}(b_j) g^{-1}(a_j) g(b_j) g(a_j) = e, \tag{3.4}$$

where e is the identity in G.

As explained in sect. 3 in the case of the torus, to describe the affine algebra $\mathscr{G}^{(1)}$ in its homogeneous gradation corresponds to currents single-valued around the cycle a of the torus. Therefore, we define the *homogeneous gradation on higher genus* by requiring that

$$g(a_j) = e \quad \text{for} \quad j = 1, \dots, h. \tag{3.5}$$

In this way, eq. (3.4) is fulfilled whatever the group elements $g(b_j)$ are. On the other hand, since the cycles b_j are in one-to-one correspondence with the generators γ_j of the Schottky group Γ, the data of the $g(b_j)$'s define a homomorphism from Γ to G

$$\gamma \in \Gamma \to g(\gamma) \in \mathrm{G}, \tag{3.6}$$

such that $g(\gamma_j) = g(b_j) \equiv g_i$, and that $g(\gamma \circ \mu) = g(\gamma) g(\mu)$ for γ and μ in Γ.

Notice that to a homomorphism, $c \to g(c)$ from $\pi_1(\Sigma)$ to G and an element g_0 of G, there is associated another homomorphism $g_0 g(c) g_0^{-1}$ from $\pi_1(\Sigma)$ to G. If $g(c)$ satisfies eqs. (3.4) and (3.5), the conjugated homomorphism $g_0 g(c) g_0^{-1}$ also does. Therefore, they define conjugated homomorphisms $g(\gamma)$ and $g_0 g(\gamma) g_0^{-1}$ from Γ to G. Furthermore, two conjugated homomorphisms $g(c)$ and $g_0 g(c) g_0^{-1}$ define the same flat vector bundle.

Let us now mix the two previous definitions – the Schottky parametrization and the appropriate homomorphism $g(\gamma)$ – and discuss the twisted Poincaré series. The Poincaré series of the Schottky group Γ are automorphic forms for this group [13]. They will play a crucial role in the construction of the complete Ward identities presented in sect. 4. We define a Poincaré series $\Theta_z(z, w|g)$ twisted by the homomorphism $g(\gamma)$ by

$$\Theta_z(z, w|g) = \sum_{\gamma \in \Gamma} \frac{\gamma'(z)}{\gamma(z) - w} \operatorname{Ad} g^{-1}(\gamma). \tag{3.7}$$

For Γ a Schottky group, the series (3.7) converges, except when z is conjugated to w [13]. Observe that for the cyclic group Γ_0 defining the torus in sect. 2, the isometric circle are not external each another; therefore the series (3.7) is not convergent in this case. The function $\omega(z, w|g)$ in eq. (2.9) is a "normal ordered" version of the Poincaré series (3.7). Another advantage of the Schottky description of the surface Σ is that the series (3.7) is not always convergent if Γ is a fuchsian group.

The series (3.7) is valued in End($\mathscr{G}$). It possesses a simple pole at $z=w$ with residue one. The other poles of $\Theta_z(z,w|g)$ are simple poles located at the images $\gamma^{-1}(w)$ of w and at the images $\gamma^{-1}(\infty)$ of the point at infinity.

The series $\Theta_z(z,w|g)$ is a twisted automorphic 1-form in z

$$\gamma_z^*\Theta_z(z,w|g) \equiv \gamma'(z)\Theta_z(\gamma(z),w|g) = \mathrm{Ad}\, g(\gamma)\Theta_z(z,w|g), \tag{3.8}$$

for any element γ of Γ.

The a-periods of $\Theta_z(z,w|g)$ vanish if w is inside the fundamental domain F – i.e. outside the circles a_j and a'_j

$$\oint_{a_j} \mathrm{d}z\, \Theta_z(z,w|g) = 0. \tag{3.9}$$

To prove eq. (3.9), we only have to enumerate the simple poles of $\Theta_z(z,w|g)$ which are inside the circle a_j. These are the images $\gamma^{-1}(w)$ of w and the images $\gamma^{-1}(\infty)$. The reside at $\gamma^{-1}(w)$ is $[\mathrm{Ad}\, g^{-1}(\gamma)]$, whereas the residue at $\gamma^{-1}(\infty)$ is $[-\mathrm{Ad} g^{-1}(\gamma]$. But since the image under γ^{-1} of the fundamental domain F is a connected domain inside one of the circles a or a', either $\gamma^{-1}(w)$ and $\gamma^{-1}(\infty)$ are both inside the circle a_j or both outside a_j. Therefore, the sum over the residues vanishes.

The Poincaré series $\Theta_z(z,w|g)$ fails to be an automorphic zero-form in w, but transforms as follows

$$\Theta_z(z,\gamma(w)|g) = \left[\Theta_z(z,w|g) - \Theta_z(z,\gamma^{-1}(\infty)|g)\right]\mathrm{Ad}\, g^{-1}(\gamma), \tag{3.10}$$

for any γ in Γ. Note that the shift in eq. (3.10) is independent of w. In particular, this implies that the symmetric 1-form $\partial_w\Theta_z(z,w|g)$

$$\partial_w\Theta_z(z,w|g) = \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{(\gamma(z)-w)^2}\mathrm{Ad}\, g^{-1}(\gamma), \tag{3.11}$$

is a twisted automorphic 1-form both in z and w.

The proof of eq. (3.10) only relies on the following relation

$$\frac{\gamma'(z)}{\gamma(z)-\gamma_0(w)} = \frac{\gamma'(z)}{\hat{\gamma}(z)-w} - \frac{\hat{\gamma}'(z)}{\hat{\gamma}(z)-\gamma_0^{-1}(\infty)}, \tag{3.12}$$

with $\hat{\gamma} = \gamma_0^{-1}\gamma$.

The readers familiar with the theory of holomorphic differential forms would have recognized in eqs. (3.8)–(3.11) the twisted version of the properties of the derivative of the logarithm of the prime form, $\partial_z \log E(z, w)$. Hence, it should not be a surprise that $\Theta_z(z, w|g)$ will later play the role of the twisted Green function in the Ward identities of the current algebras.

Let us now study the Poincaré series $\Theta_z(z, \gamma^{-1}(\infty)|g)$. They are evidently twisted automorphic 1-forms. They are holomorphic on the fundamental domain F of $\mathbb{C}/\Gamma$. By twice applying the relation (3.10), one proves the following composition law for γ and μ in Γ

$$\Theta_z\big(z, (\gamma\mu)^{-1}(\infty)|g\big) = \Theta_z\big(z, \gamma^{-1}(\infty)|g\big)\mathrm{Ad}\, g(\mu) + \Theta_z\big(z, \mu^{-1}(\infty)|g\big), \quad (3.13)$$

$$\Theta_z\big(z, \gamma(\infty)|g\big) = -\Theta_z\big(z, \gamma^{-1}(\infty)|g\big)\mathrm{Ad}\, g^{-1}(\gamma). \quad (3.14)$$

Eqs. (3.13) and (3.14) imply that not all the series $\Theta_z(z, \gamma^{-1}(\infty)|g)$ are independent but that only h among them are. The independent series can be canonically associated to the generators γ_j of Γ; they are

$$\omega_z^j(z|g) = \frac{1}{2i\pi}\Theta_z\big(z, \gamma_j^{-1}(\infty)|g\big). \quad (3.15)$$

The holomorphic 1-forms $\omega_z^j(z|g)$ are the twisted version of the holomorphic differentials $\omega_j(z)$. They satisfy

$$\gamma^*\omega_z^j(z|g) \equiv \gamma'(z)\,\omega_z^j(\gamma(z)|g) = \mathrm{Ad}\, g(\gamma)\,\omega_z^j(z|g), \quad (3.16)$$

for any γ in Γ. They are dual to the cycles a_k

$$\oint_{a_k} \mathrm{d}z\, \omega_z^j(z|g) = \delta_k^j. \quad (3.17)$$

Indeed, in addition to the simple pole at $z = \gamma_j^{-1}(\infty)$, $\omega_z^j(z|g)$ has simple poles at $z = \gamma^{-1}\gamma_j^{-1}(\infty)$ and $z = \gamma^{-1}(\infty)$, with residue $[\mathrm{Ad}\, g^{-1}(\gamma)]$ and $[-\mathrm{Ad}\, g^{-1}(\gamma)]$, respectively. But both domains $\gamma^{-1}(F)$ and $\gamma^{-1}\gamma_j^{-1}(F)$ are always in the same circle a_i or a_i', therefore the sum over the residues coming from the poles at $\gamma^{-1}(\infty)$ and at $\gamma^{-1}\gamma_j^{-1}(\infty)$ cancels. On the other hand, the pole at $z = \gamma_j^{-1}(\infty)$ is inside a_j; it therefore gives a non-vanishing contribution only if $j = k$.

To be more precise, for the 1-forms $\omega_z(z|g)$ to be holomorphic on the Riemann surface, they also have to be regular at infinity. This means that they must behave like $\omega_z(z|g) \sim 1/z^2$ at infinity. The differentials $\omega_z^j(z|g)$ do not satisfy this condition. The holomorphic differentials (regular at infinity) are the set of 1-forms

$$\omega_z^{\nu}(z|g)^a = \sum_{j=1}^{h} \omega_z^j(z|g)^{ab} \nu_j^b, \tag{3.18}$$

where ν_j, $j = 1, \ldots, h$, are h vectors of the Lie algebra $\mathscr{G}$ satisfying

$$\sum_{j=1}^{h} \left(\nu_j - (\mathrm{Ad}\, g_j)\nu_j\right) = 0. \tag{3.19}$$

It is easy to show that for generic values of the twists g_j there are $(h-1)\dim \mathrm{G}$ independent solutions of eq. (3.19). This means that there is $(h-1)\dim \mathrm{G}$ independent twisted holomorphic 1-forms. Hence, from the Riemann–Roch theorem, in general there is no holomorphic function in the dual flat bundle. The physical implication of this fact is that for generic values of the twists there is no extra global-gauge conservation law besides those described in eqs. (4.4) and (4.5) below.

4. Ward identities for the current algebras

As already explained, to have complete Ward identities for the WZW models, we must consider a twisted formulation of these models. The fundamental group-valued fields of the WZW models [7–9], denoted by $G(z, \bar{z})$, can be twisted by acting either on the right or on the left. In the *homogeneous gradation* defined in sect. 3, the twists are specified by two homomorphisms $g(\gamma)$ and $h(\gamma)$ from Γ to G. On the covering space, these homomorphisms act on $G(z, \bar{z})$ as follows

$$G\left(\gamma(z), \overline{\gamma(z)}\right) = g(\gamma) G(z, \bar{z}) h^{-1}(\gamma). \tag{4.1}$$

In the case of the torus, the chiral invariance of the WZW allows us to map any twisted versions of the WZW models to a particular version which is untwisted along the a-cycle. We have not been able to generalize this statement on higher genus. But if this property remains true on arbitrary genus, it is then enough to study the WZW in their homogeneous gradations defined by eq. (4.1).

Due to the chiral nature of the WZW models, the left and the right twists act separately on the left and right conserved currents. On the left conserved current $J_z^a(z)$

$$J_z^a(z) t^a = -\tfrac{1}{2} K (\partial_z G) G^{-1}, \qquad \bar{\partial}_{\bar{z}} J_z^a(z) = 0, \tag{4.2}$$

the twists $g(\gamma)$ act according to

$$\gamma^* J_z^a(z) \equiv \gamma'(z) J_z^a(\gamma(z)) = \mathrm{Ad}\, g(\gamma)^{ad} J_z^d(z). \tag{4.3}$$

The currents $J_z^a(z)$ are unaffected by the right twist $h(\gamma)$. Similar relations hold for the right conserved currents $\bar{J}_{\bar{z}}^a(\bar{z})$. Eq. (4.3) means that the currents $J_z^a(z)$ belong to the flat vector bundle specified by the homomorphism $g(\gamma)$ as described in sect. 3.

Because the currents – or the fundamental field $G(z, \bar{z})$ – are twisted by the group elements $g(\gamma)$, the partition function $Z(\Gamma; g)$ and all the correlation functions, denoted by $\langle \cdots \rangle_g$ as in the case of the torus, depend on the group elements $g(\gamma)$. But the group formed by the $g(\gamma)$'s is generated by the h independent group elements $g_j \equiv g(\gamma_j)$, $j = 1, \ldots, h$. Therefore, the expectation values $\langle \cdots \rangle_g$ and the partition function $Z(\Gamma; g)$ can be understood as functions from G^h to some-finite dimensional vector spaces. The arguments of these functions are the h independent group elements g_j.

Covariance of the correlation functions $\langle \cdots \rangle_g$ under global gauge transformations arises from the fact that a conjugation of the homomorphism $g(\gamma)$ by a constant element g_0 of G, $g(\gamma) \to g_0 g(\gamma) g_0^{-1}$, does not change the vector bundle. Thus, the partition function is invariant under a conjugation of $g(\gamma)$

$$Z(\Gamma; g) = Z\left(\Gamma; g_0 g g_0^{-1}\right), \tag{4.4}$$

and the correlation functions of the affine primary fields transform covariantly

$$\langle \Phi_1(\xi_1) \ldots \Phi_N(\xi_N) \rangle_{g_0 g g_0^{-1}} = \rho_{(1)}(g_0) \ldots \rho_{(N)}(g_0) \langle \Phi_1(\xi_1) \ldots \Phi_N(\xi_N) \rangle_g, \tag{4.5}$$

if $\Phi_n(\xi_n)$ belongs to the representation $\rho_{(n)}$ of G.

Being functions of the h group elements g_j, we can derive the correlation functions $\langle \cdots \rangle_g$ with respect to any one of the g_j's. Thus, *we define* the action of the zero modes $J_{0;\,j}^a$, eq. (1.2), on the correlation functions by*

$$\langle J_{0;\,j}^a \rangle_g = 2i\pi \mathscr{L}_j^a \log Z(\Gamma; g), \tag{4.6}$$

$$\begin{aligned} &\langle J_{0;\,j}^a \Phi_1(\xi_1) \ldots \Phi_N(\xi_N) \rangle_g - \langle J_{0;\,j}^a \rangle_g \langle \Phi_1(\xi_1) \ldots \Phi_N(\xi_N) \rangle_g \\ &\quad = 2i\pi \mathscr{L}_j^a \langle \Phi_1(\xi_1) \ldots \Phi_N(\xi_N) \rangle_g, \end{aligned} \tag{4.7}$$

* One may wonder if the definitions (4.6) and (4.7) are consistent with the other properties of the WZW models. But as shall be explained later, the Ward identities for the current algebras on Riemann surfaces completely determine the definition of the zero modes $J_{0;\,j}^a$. To be precise, eq. (4.23) below, which follows from the consistency of the Ward identities, implies that the definition (4.6) and (4.7) are the unique consistent ones.

On the other hand, twisted WZW models as defined by eq. (4.1) can be studied via a path-integral formalism. The Ward identities described below can then be rederived from the properties of the WZW action of the twisted models. From this demonstration, left to the reader, it becomes apparent that the zero modes are associated to the variation of the twists as emphasized in the definitions (4.6) and (4.7).

where the $\mathscr{L}_j^a$ are the left invariant Lie derivatives acting on the group elements g_j. To be precise, if $F(g_1,\dots,g_h)$ is a function depending on $g_1,\dots,g_h$, then, for any vector v of the Lie algebra $\mathscr{G}$, the Lie derivatives $\mathscr{L}_j^v$ act as follows

$$\left(\mathscr{L}_j^v F\right)(g_1,\dots,g_h) = \frac{\mathrm{d}}{\mathrm{d}t} F\left(g_1,\dots,g_j\,\mathrm{e}^{tv},\dots,g_h\right)\bigg|_{t=0}. \tag{4.8}$$

We are now in a position to derive the Ward identities for the twisted correlation functions $\langle\cdots\rangle_g$. In contrast with the case of the torus, at higher genus there is no KMS condition because the expectation values are not traces. We cannot use the commutation relations of the currents to derive the Ward identities. Therefore, we will prove them by using the method of the operator-valued differential forms described by Witten [3]. Hence, instead of assuming the commutation relations of the currents, we suppose that inside the correlation functions the affine currents $J_z^a(z)$ satisfy the following operator product expansion (OPE)

$$J_z^a(z)J_w^b(w) \sim \frac{-K\,\delta^{ab}}{(z-w)^2} + \frac{f_c^{ab}J_w^c(w)}{(z-w)}. \tag{4.9}$$

Here, K is the central charge of the current algebra $\mathscr{G}^{(1)}$. The primary fields $\Phi(\xi)$ for the affine algebra $\mathscr{G}^{(1)}$ are defined by the OPE

$$J_z^a(z)\Phi(\xi) \sim \frac{\rho(t^a)\Phi(\xi)}{(z-\xi)}, \tag{4.10}$$

if the field $\Phi(\xi)$ belongs to the representation ρ of $\mathscr{G}$. Regular terms are omitted in eqs. (4.9) and (4.10).

To prove the Ward identities via the method of the operator valued differential forms, one has to judiciously define a 1-form $\Omega_x(x)$ and has to apply to it the residue theorem

$$\sum_{P\in\Sigma} \mathrm{Res}_P(\Omega_x(x)) = \frac{1}{2i\pi}\sum_{j=1}^{h}\oint_{b_j^{-1}a_j^{-1}b_ja_j} \mathrm{d}x\,\Omega_x(x). \tag{4.11}$$

The l.h.s. in eq. (4.11) is the contour integral on the "cut sufface" drawn in fig. 1. Eq. (4.11) is slight modification of the identities presented in ref. [3] which allows us to take into account the existence of zero modes.

It is worthwhile to do the demonstration in details. To derive the Ward identities for one insertion of a current $J_z^a(z)$, the appropriate 1-form is

$$\Omega_x^a(x) = \Theta_z(z,x|g)^{ab}\langle J_x^b(x)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g. \tag{4.12}$$

By assumption, the expectation values $\langle J_z^b(x)\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g$ possess only simple poles at $x=\xi_n$, $n=1,\ldots,N$, whose residues are given by the OPE (4.10). Besides these poles, $\Omega_x^a(x)$ also has a simple pole at $z=x$ through the Poincaré series $\Theta_z(z,x|g)$. $\Omega_z^a(x)$ is single valued along the a-cycles; i.e. $\Omega_x^a(x)$ is well-defined on the fundamental domain F. Therefore, since $\gamma_j(a_j)=-a_j'$, eq. (4.11) becomes

$$\sum_{n=1}^{N}\Theta_z(z,\xi_n|g)^{ab}t_{(n)}^b\langle\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g-\langle J_z^a(z)\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g$$

$$=-\frac{1}{2i\pi}\sum_{j=1}^{h}\left(\oint_{a_j}+\oint_{a_j'}\right)\mathrm{d}x\,\Omega_x^a(x)$$

$$=\frac{1}{2i\pi}\sum_{j=1}^{h}\oint_{a_j}\mathrm{d}x\left[\gamma_j^*\Omega_x^a(x)-\Omega_x^a(x)\right]. \tag{4.13}$$

Note the difference of the orientations of the a-cycles in fig. 1 and fig. 2.

From eqs. (3.10) and (4.3), it follows that $\Omega_x^a(x)$ fails to be single valued along the cycle b_j by

$$\gamma_j^*\Omega_x^a(x)-\Omega_x^a(x)=-2i\pi\omega_z^j(z|g)^{ab}\langle J_x^b(x)\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g. \tag{4.14}$$

Therefore, the l.h.s. of eq. (4.13) reduces to the insertion of the zero modes in the correlation functions

$$\frac{1}{2i\pi}\oint_{a_j}\mathrm{d}x\left(\gamma_j^*\Omega_x^a(x)-\Omega_x^a(x)\right)=-\omega_z^j(z|g)^{ab}\langle J_{0;\,j}^b\Phi_1(\xi_1)\ldots\Phi_n(\xi_N)\rangle_g. \tag{4.15}$$

Using the definition (4.6) and (4.7) of the zero modes, we finally obtain the complete form of the Ward identities

$$\langle J_z^a(z)\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g-\langle J_z^a(z)\rangle_g\langle\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g$$

$$=\sum_{n=1}^{N}\Theta_z(z,\xi_n|g)^{ab}t_{(n)}^b\langle\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g$$

$$+\mathscr{L}_z^a(z|g)\langle\Phi_1(\xi_1)\ldots\Phi_N(\xi_N)\rangle_g, \tag{4.16}$$

where we have introduced the following notation

$$\mathscr{L}_z^a(z|g) = 2i\pi \sum_{j=1}^{h} \omega_z^j(z|g)^{ab}\mathscr{L}_j^b . \tag{4.17}$$

With this notation, the expectation values of the currents $J_z^a(z)$ become

$$\langle J_z^a(z)\rangle_g = \mathscr{L}_z^a(z|g)\log Z(\Gamma;\, g). \tag{4.18}$$

One can check that the global conservation law (4.4) implies that $\nu_j^a = \mathscr{L}_j^a \log Z(\Gamma;\, g)$ satisfy eq. (3.19), and hence that $\langle J_z^a(z)\rangle_g$ are effectively twisted holomorphic 1-forms on the Riemann surface.

In the same way Ward identities for an arbitrary number of insertions of currents can be derived by the same techniques if one chooses the 1-form $\tilde{\Omega}_x^a(x)$

$$\tilde{\Omega}_x^a(x) = \Theta_z(z, x|g)^{ad}\langle J_x^d(x) J_w^b(w)\dots \Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g. \tag{4.19}$$

For example, for two insertions the OPE's (4.9) and (4.10) imply that

$$\begin{aligned}
&\langle J_z^a(z) J_w^b(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g - \langle J_z^a(z)\rangle_g \langle J_w^b(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g \\
&\quad = -K\,\partial_w \Theta_z(z, w|g)^{ab}\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g \\
&\qquad + \left\{\Theta_z(z, w|g)^{ad} f_c^{db} + \delta_c^b \sum_{n=1}^{N} \Theta_z(z, \xi_n|g)^{ad} t_{(n)}^d\right\} \\
&\qquad\quad \times \langle J_w^c(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g \\
&\qquad + \mathscr{L}_z^a(z|g)\langle J_w^b(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g.
\end{aligned} \tag{4.20}$$

Note the similarity between the Ward identities (2.5) and (4.6). The consistency of the Ward identities described above implies stringent constraints on the twists of the primary fields, but above all it determines the allowed definitions of the zero modes $J_{0;\,j}^a$.

If the primary fields are twisted linearly, (i.e., if they are twisted automorphic forms for the Schottky group)

$$(\gamma^*\Phi)(\xi) = \mathscr{T}(g|\gamma)\Phi(\xi) \tag{4.21}$$

then, an examination of the pull-back of the Ward identities (4.16) shows that the dependence of the twist $\mathscr{T}(g|\gamma)$ on the homomorphism $g(\gamma)$ is fixed through the relation

$$\mathscr{T}(g|\gamma) = \eta(\gamma)\rho(g(\gamma)), \tag{4.22}$$

if the primary fields $\Phi(\xi)$ belongs to the representation ρ of G. $\eta(\gamma)$ may evidently depend on Φ; it is a character for the Schottky group Γ.

Moreover, the same analysis shows that we must have

$$\rho(g^{-1}(\gamma))[\mathscr{L}_z^a(z|g)\rho(g(\gamma))] = \Theta_z(z,\gamma^{-1}(\infty)|g)^{ab}\rho(t^b). \tag{4.23}$$

This latter constraint is strong enough to prove that the definition of the zero modes given in eqs. (4.6) and (4.7) is unique. The proof assumes that the zero modes act on the correlation functions through a first-order differential operator acting on the group element $g(\gamma)$. Actually, eq. (4.23) completely specifies the operator $\mathscr{L}_z^a(z|g)$. In particular, it is this equation which fixes the factors $(2i\pi)$ in eqs. (4.6) and (4.7).

Finally, let us recall that the algebraic structure is locally preserved [4, 9], i.e. the local modes $J_n^a(w)$ of the currents $J_z^a(z)$

$$J_z^a(z) = \sum_{n\in Z} J_n^a(w)(z-w)^{-n-1}, \tag{4.24}$$

satisfy the commutation relations of the affine Kac–Moody algebra $\mathscr{G}^{(1)}$. Therefore, as extensively discussed in ref. [1] for the case of the sphere, and in refs. [4, 19] on higher genus, correlation functions obey linear differential equations. These differential equations arise from the null-vectors which occur in the affine Verma modules [18]. Schematically, if $\Phi(\xi)$ is a primary field for an affine representation whose Dynkin indices are D_s, $s = 0,\dots,\mathrm{rank}\,G$, then the differential equations are

$$[J(-\tilde{\alpha}_s)(w)]^{D_s+1}\Phi(w) = 0, \tag{4.25}$$

where $J(\tilde{\alpha}_s)$ are the current components associated with the simple roots $\tilde{\alpha}_s$ of the affine algebra $\mathscr{G}^{(1)}$.

To write down explicitly these differential equations is not very enlightening. We will therefore discuss the Virasoro × Kac–Moody Ward identities which have turned out to be more powerful.

5. The mixed Virasoro × Kac–Moody Ward identities

For completeness, before describing the Virasoro × Kac–Moody Ward identities, we recall the Virasoro Ward identities on Riemann surfaces. They can be proved as in ref. [4] by using an appropriate Green function together with the method of operator valued differential forms [3]. The demonstration is similar to the one for the current algebras. Thus we omit it and just report some details in the appendix. But because we will need it, we will discuss the dependence on the moduli parameters of the Riemann surfaces. As above, we describe the Riemann surfaces in the Schottky parametrization.

The defining relations of the Virasoro algebra and of its highest weight representations are encoded in the OPE of the stress tensor $T(z)$

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \left[\frac{2}{(z-w)^2} + \frac{1}{(z-w)}\partial_w\right]T(w), \tag{5.1}$$

where c is the Virasoro central charge. For the current algebras (4.9), the Virasoro central charge is $c = K\dim \mathrm{G}/(K+h^*)$. The OPE characterizing the Virasoro primary fields are

$$T(z)\Phi(\xi) \sim \left[\frac{\Delta}{(z-\xi)^2} + \frac{1}{(z-\xi)}\partial_\xi\right]\Phi(\xi), \tag{5.2}$$

where Δ is the conformal weight of the primary field $\Phi(\xi)$.

Let us denote by $\mathbb{G}_{zz}(z,w)$ the following Green function

$$\mathbb{G}_{zz}(z,w) = \sum_{\gamma\in\Gamma} \frac{[\gamma'(z)]^2}{\gamma(z)-w} - \sum_{\alpha=1}^{3h-3} h^{\alpha}_{zz}(z)\,p_\alpha(w). \tag{5.5}$$

Here $h^{\alpha}_{zz}(z)$, $\alpha = 1,\dots,(3h-3)$, form a basis of holomorphic quadratic differentials, and the $p_\alpha(w)$'s are polynomials defined in the appendix. For w in F, $\mathbb{G}_{zz}(z,w)$ has only a simple pole in the fundamental domain located at $z=w$ with residue one. It is an automorphic 2-form in z: $\gamma_z^*\mathbb{G}_{zz}(z,w) = \mathbb{G}_{zz}(z,w)$ for any γ in Γ. It fails to be a vector field in w by a shift which is a holomorphic quadratic differential in z. (See the appendix for further details.)

The Ward identity for one insertion of the stress tensor reads [4]

$$\begin{aligned}
&\langle T(z)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g - \langle T(z)\rangle_g\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g \\
&\quad= \sum_{n=1}^{N}\left[\Delta_n\,\partial_{\xi_n}\mathbb{G}_{zz}(z,\xi_n) + \mathbb{G}_{zz}(z,\xi_n)\,\partial_{\xi_n}\right]\langle\phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g \\
&\qquad+ \sum_{\alpha=1}^{3h-3} h^{\alpha}_{zz}(z)\frac{\partial}{\partial m^\alpha}\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g .
\end{aligned} \tag{5.4}$$

Here m^α, $\alpha = 1, \dots, h$ are the moduli parameters dual to the holomorphic quadratic differentials $h^\alpha_{zz}(z)$.

As for the affine algebras, the expectation values involving the zero modes are defined by a derivation with respect to the moduli parameters. Namely, the variation of the partition function is [2–4]

$$h^\alpha_{zz}(z)\frac{\partial}{\partial m^\alpha}\log Z(\Gamma;\,g) = \langle T(z)\rangle_g. \tag{5.5}$$

Similar expressions hold for the correlation functions of the primary fields (see the appendix). Notice that the last equation is only valid in an atlas of local coordinates such that the coordinate transformations between neighbouring patches belong to PSL(2, $\mathbb{C}$). That is to say, eq. (5.5) is only valid once we have chosen a projective structure $\mathcal{P}$ on the Riemann surface. From one projective structure, with local coordinate z, to another one, with local coordinate w, the stress tensor transforms inhomogeneously [1]

$$\langle T(w)\rangle_g \,\mathrm{d}^2 w = \langle T(z)\rangle_g \,\mathrm{d}^2 z + \tfrac{1}{12}c\{z;\,w\}\,\mathrm{d}^2 w, \tag{5.6}$$

where $\{z;\,w\}$ is the schwarzian derivative. Accordingly, the partition function will transform inhomogeneously. This transformation reflects the projective nature of the line bundle defined by Friedan and Shenker [2]. The partition function is a function not only of the moduli parameters of Σ but also of the projective structure $\mathcal{P}$ defined over Σ. For example, the torus can be described as the complex plane with the points z and $z + \tau$ identified, or as the complex plane with the identification of the points z and zq, ($q = \exp(2i\pi\tau)$). From the first to the second description the partition function loses a factor $\exp(-2i\pi\tau c/24)$.

Eq. (5.5) is one of the key equations of conformal field theories because it allows us to evaluate the partition functions. As noticed by Martinec [17], it is effectively possible to carry out the integration of eq. (5.5) because the r.h.s. reduces to a contour integral on the covering space. Indeed, a deformation of the complex structure is specified by the change of complex coordinate, $z \to S(z, \bar{z})$ [15]

$$S(z, \bar{z}) = z + \varepsilon(z, \bar{z}), \tag{5.7}$$

where $\varepsilon(z, \bar{z})$ is not globally defined on the covering space. The transformation (5.7) is called a quasi-conformal mapping. It modifies the kleinian group Γ defining the Riemann surfaces according to $\Gamma \to \Gamma^S$

$$\Gamma^S = S \circ \Gamma \circ S^{-1}. \tag{5.8}$$

The generators γ_j^S of the group Γ^S are $\gamma_j^S = S\gamma_j S^{-1}$. For Γ^S a group of Mobius transformations, the Beltrami coefficient $\mu_S(z, \bar{z})$ of the transformation

$$\mu_S(z, \bar{z}) = \left(\frac{\partial S}{\partial \bar{z}}\right) \Big/ \left(\frac{\partial S}{\partial z}\right), \tag{5.9}$$

must transform covariantly [15]

$$(\gamma^* \mu_S)(z, \bar{z}) \equiv \frac{\overline{\gamma'(z)}}{\gamma'(z)} \mu_S\left(\gamma(z), \overline{\gamma(z)}\right) = \mu_S(z, \bar{z}). \tag{5.10}$$

As infinitesimal transformation $\mu(z, \bar{z}) = \bar{\partial}_{\bar{z}} \varepsilon(z, \bar{z})$ induces a change of the partition function

$$2\pi \delta_\varepsilon \log Z(\Gamma; g) = \int_\Sigma \mu(x, \bar{x}) \langle T(x) \rangle_g . \tag{5.11}$$

By integrating eq. (5.11) by parts, one finds the contour integral [17]

$$\delta_\varepsilon \log Z(\Gamma; g) = \frac{1}{2i\pi} \sum_{j=1}^{h} \oint_{a_j} \mathrm{d}x \, \chi_j[\varepsilon](x) \langle T(x) \rangle_g , \tag{5.12}$$

where $\chi_j[\varepsilon](x) = \varepsilon\left(\gamma_j(z), \overline{\gamma_j(x)}\right)[\gamma_j'(x)]^{-1} - \varepsilon(x, \bar{x})$. Note that the Beltrami equation (5.10) implies that $\chi_j[\varepsilon](x)$ is holomorphic.

The crucial point resides in the fact that to evaluate the variation of the partition function there is no need to explicitly know the non-analytic function $\varepsilon(z, \bar{z})$. To be precise, the shift function $\chi_j(x)$ associated with a change of the moduli parameters $(q_j; u_j; v_j)$ is completely determined by eqs. (5.9) and (3.3). For example, the shift functions corresponding to the variation of the moduli parameters q_j, $(q_j \to q_j + \delta q_j)$, are defined by

$$\frac{S\gamma_j S^{-1}(z) - u_j}{S\gamma_j S^{-1}(z) - v_j} = \left(q_j + \delta q_j\right) \frac{z - u_j}{z - v_j} . \tag{5.13}$$

Similar equations hold for the variation of the moduli parameters u_j and v_j.

By solving these relations, one expresses the variation of the partition function as follows

$$q_j \frac{\partial}{\partial q_j} \log Z(\Gamma; g) = \frac{1}{2i\pi} \oint_{a_j} \mathrm{d}x \, \frac{(x-u_j)(x-v_j)}{(u_j - v_j)} \langle T(x) \rangle_g , \tag{5.14}$$

$$\frac{\partial}{\partial v_j} \log Z(\gamma; g) = \frac{1}{2i\pi} \oint_{a_j} \mathrm{d}x \, \frac{(\gamma_j(x) - x)(x - u_j)}{(\gamma_j(x) - v_j)(u_j - v_j)} \langle T(x) \rangle_g , \tag{5.15}$$

with similar equations with u_j and v_j exchanged.

Let us now apply this formalism to the WZW models in order to derive the differential equations satisfied by the partition function and the correlation functions. In the WZW models, the stress tensor is defined inside the correlation functions by the Sugawara construction

$$T^{\mathrm{Sug.}}(z) = \lim_{w \to z} \left\{ -\frac{1}{2(K+h^*)} \left[J_z^a(z) J_w^a(w) + \frac{K \dim \mathrm{G}}{(z-w)^2} \right] \right\} . \tag{5.16}$$

Here, h^* in the dual Coxeter number of the Lie algebra $\mathscr{G}$.

From the Ward identities (4.20), the expectation value of the Sugawara operator is

$$\begin{aligned} \langle T^{\mathrm{Sug.}}(z) \rangle_g = &+ \frac{K}{2(K+h^*)} \operatorname{tr} S(z|g) \\ &- \frac{1}{2(K+h^*)} \operatorname{tr}(\Xi(z|g) A\, dt^a) \mathscr{L}_z^a(z|g) Z(\Gamma; g) \\ &- \frac{1}{2(K+h^*)} \frac{1}{Z(\Gamma; g)} \mathscr{L}_z^a(z|g) \mathscr{L}_z^a(z|g) Z(\Gamma; g) . \end{aligned} \tag{5.17}$$

The traces are taken in the adjoint representation of G. $S(z|g)$ is the twisted projective connection

$$S(z|g) = \sum_{\gamma \neq e} \frac{\gamma'(z)}{(\gamma(z) - z)^2} \mathrm{Ad}\, g^{-1}(\gamma) , \tag{5.18}$$

and $\Xi(z|g)$ is defined by

$$\Xi(z|g) = \sum_{\gamma \neq e} \frac{\gamma'(z)}{\gamma(z) - z} \mathrm{Ad}\, g^{-1}(\gamma) . \tag{5.19}$$

Neither the quadratic operator $\mathscr{L}_z^a(z|g)\mathscr{L}_z^a(z|g)$ nor the operator $\mathrm{tr}(\Xi(z|g)A\,dt^a)\cdot \mathscr{L}_z^a(z|g)$ are single valued under the pull-back by an element γ of Γ, but the sum is. Thus, $T^{\mathrm{Sug.}}(z)$ is an automorphic 2-form

$$\gamma^*\langle T^{\mathrm{Sug.}}(z)\rangle_g \equiv [\gamma'(z)]^2\langle T^{\mathrm{Sug.}}(z)\rangle_g = \langle T^{\mathrm{Sug.}}(z)\rangle_g . \tag{5.20}$$

We now want to integrate the differential equations (5.14) and (5.15). The analysis decomposes into two steps. First, by an integration on the moduli space, we identify the quantum part of eq. (5.6) as coming from the denominator of the partition function. We then argue that both the denominator and the numerator of the partition function satisfy a kind of heat equation on the moduli space. The projective connection arises in eq. (5.17) from the quantum fluctuations of the affine currents. It is the stress tensor of dim G free bosons twisted by the homomorphism $g(\gamma)$. The stress tensor $T^{\mathrm{qu.}}(z)$ of such bosons is

$$\langle T^{\mathrm{qu.}}(z)\rangle_g = \tfrac{1}{2}\mathrm{tr}\, S(z|g) . \tag{5.21}$$

To find the partition function of these twisted bosons, whose inverse is denoted by $\Pi(\Gamma;\, g)$, we have to integrate the differential equation (5.5)

$$\sum_{\alpha=1}^{3h-3} h^{\alpha}_{zz}(z)\frac{\partial}{\partial m^{\alpha}}\log \Pi(\Gamma;\, g) = -\langle T^{\mathrm{qu.}}(z)\rangle_g . \tag{5.22}$$

Following Martinec [17], since the contour integral (5.14) picks out the poles of $S(z|g)$ which are the fixed points of Γ, we find that

$$q_j\frac{\partial}{\partial q_j}\log \Pi(\Gamma;\, g) = \sum_{\gamma\,\mathrm{prim.}}\sum_{n=1}^{\infty}\frac{q_\gamma^n}{\left(1-q_\gamma^n\right)^2}\mathrm{tr}(\mathrm{Ad}\, g(\gamma))\frac{q_j}{q_\gamma}\frac{\partial q_\gamma}{\partial q_j} . \tag{5.23}$$

The sum is over the primitive elements of Γ, i.e. those which are not power of other elements, with each conjugacy class counted only once. Therefore, we have

$$\Pi(\Gamma;\, g) = \prod_{\gamma\,\mathrm{prim.}}\prod_{n=1}^{\infty}\det\left[1-q_\gamma^n\,\mathrm{Ad}\, g(\gamma)\right] . \tag{5.24}$$

The determinant is taken in the adjoint representation. This infinite product is the denominator of the chiral partition function. It is the simple twisted version of the determinant which appears in the loop measure of the dual string models [16].

After having inserted the relation (5.22) in eq. (5.17), the differential equations satisfied by the partition function become

$$\left[\sum_{\alpha=1}^{3h-3} h_{zz}^{\alpha}(z)\frac{\partial}{\partial m^{2}}+\frac{1}{2(K+h^{*})}\left(\nabla_z^a-2\nabla_z^a\log \Pi(\Gamma;g)\right)\nabla_z^a\right] Z(\Gamma;g)\Pi(\Gamma;g)$$

$$=\frac{h^{*}}{K+h^{*}}Z(\Gamma;g)\left[\sum_{\alpha=1}^{3h-3} h_{zz}^{\alpha}(z)\frac{\partial}{\partial m^{\alpha}}+\frac{1}{2h^{*}}\left(\nabla_z^a-2\nabla_z^a\log \Pi(\Gamma;g)\right)\nabla_z^a\right]$$

$$\times\Pi(\Gamma;g). \tag{5.25}$$

Here we have introduced a new differential operator called ∇_z^a. It is defined by

$$\nabla_z^a=\mathscr{L}_z^a(z|g)-\Xi(z|g)^{ad}\rho(t^d), \tag{5.26}$$

when it acts on automormorphic forms $\Psi(z)$ satisfying $(\gamma^*\Psi)(z)=\rho(g(\gamma))\Psi(z)$, with ρ a representation of the Lie group G. Under the pull-back by an element γ of Γ, the differential operator ∇_z^a acts (almost) covariantly

$$\gamma^*\left(\nabla_z^a\Psi(z)\right)=\rho(g(\gamma))\mathrm{Ad}\, g(\gamma)^{ad}\left[\nabla_z^d+\frac{t^d}{z-\gamma^{-1}(\infty)}\right]\Psi(z). \tag{5.27}$$

Eq. (5.27) is the analogue of the transformation of the derivative of $\Psi(z)$, $\partial_z\Psi(z)$, by the pull-back by the Mobius transformation γ.

To argue that both the denominator $\Pi(\Gamma;g)$ and the numerator $Z(\Gamma;g)\Pi(\Gamma;g)$ satisfy a heat equation on the moduli space, observe that, up to now, we never have supposed that the theory was unitary. For the non-unitary representation whose highest weight is a scalar for the global gauge group, but whose central charge is not integral, there is no null-vector in the Verma module (except those associated with the global-gauge group) [18]. Therefore, the partition function of this theory is the inverse of the infinite product (5.24), $Z^{\mathrm{qu.}}(\Gamma;g)=1/\Pi(\Gamma;g)$. In this case, eq. (5.25) implies

$$\left[\sum_{\alpha=1}^{3h-3} h_{zz}^{\alpha}(z)\frac{\partial}{\partial m^{\alpha}}+\frac{1}{2h^{*}}\left(\nabla_z^a-2\nabla_z^a\log \Pi(\Gamma;g)\right)\nabla_z^a\right]\Pi(\Gamma;g)=0. \tag{5.28}$$

In the case of the torus, this remark proves Fegan's result [20]; namely, the infinite product (5.24) is solution of the heat equation on the group manifold. It thus completes the proof of the Weyl–Kac character formula presented in ref. [10]. Hence, it gives a new proof of MacDonald's identities [21]. We understand that, in genus $h \geq 2$, our argument does not constitute a mathematical proof of eq. (5.28). For mathematicians, eq. (5.28) has to be understood as a conjecture. A mathematical proof of eq. (5.28) could arise from a perturbative study ($K \to \infty$) of the WZW models.

From eqs. (5.25) and (5.28), it follows that the numerator of the partition function is also a solution of this kind of heat equation

$$\left[\sum_{\alpha=1}^{3h-3} h_{zz}^{\alpha}(z)\frac{\partial}{\partial m^{\alpha}} + \frac{1}{2(K+h^*)}\left(\nabla_z^a - 2\nabla_z^a \log \Pi(\Gamma; g)\right)\nabla_z^a\right] Z(\Gamma; g)\Pi(\Gamma; g) = 0. \tag{5.29}$$

Unfortunately, we are not able (at present) to integrate this differential equation.

We end this section by writing the mixed Ward identities for the correlation functions. The mixed Ward identities for the expectation values $\langle \cdots \rangle_g$ are obtained by mixing the Ward identities of the current algebras with the Sugawara construction [9]. The property (5.28) satisfied by the infinite product $\Pi(\Gamma; g)$ allows us to factorize these identities

$$\begin{aligned}\sum_{n=1}^{N}&\left[\Delta_n \partial_{\xi_n}\mathbb{G}_{zz}(z,\xi_n) + \mathbb{G}_{zz}(z,\xi_n)\partial_{\xi_n} + \sum_{\alpha=1}^{3h-3} h_{zz}^{\alpha}(z)\frac{\partial}{\partial m^{\alpha}}\right]\langle\langle \Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle\rangle \\ &= -\frac{1}{2(K+h^*)}\left[\nabla_z^a - 2\nabla_z^a \log\Pi(\Gamma; g) + \sum_{n=1}^{N}\Theta_z(z,\xi_n|g)^{ad}t_{(n)}^d\right] \\ &\quad\times\left[\nabla_z^a + \sum_{n=1}^{N}\Theta_z(z,\xi_n|g)^{ab}t_{(n)}^b\right]\langle\langle \Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle\rangle,\end{aligned} \tag{5.30}$$

where the double bracket means

$$\langle\langle \Phi_1(\xi_1)\dots\Phi_N(\xi_n)\rangle\rangle = Z(\Gamma; g)\Pi(\Gamma; g)\langle \Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g, \tag{5.31}$$

It would be interesting to have a geometrical interpretation (i.e. via a path-integral analysis?) of this factorization.

As on the sphere, the double pole in eq. (5.30) at $z = \xi_m$ determines the conformal weight of the affine primary field $\Phi_m(\xi_m)$: $\Delta_m = \mathrm{Casimir}(\rho_{(m)})/2(K+h^*)$. The simple pole gives rise to the fundamental Knizhnik–Zamolodchikov [9] differential equations

$$\left[\frac{\partial}{\partial\xi_m}+\frac{1}{K+h^*}\sum_{n\neq m}\Theta_{\xi_m}(\xi_m,\xi_n|g)^{ad}t^a_{(m)}t^d_{(n)}+\frac{1}{2(K+h^*)}\left\{t^a_{(m)},\nabla^a_z\right\}_+\right]$$

$$\times Z(\Gamma;g)\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g=0. \tag{5.32}$$

As a check, one may verify that, thanks to the property (5.27), eq. (5.32) is covariant under the pull-back by an element γ of Γ. Notice that eq. (5.32) remains valid even if eq. (5.17) is not true. Eq. (5.32) together with the unitary constraint (4.25) are supposed to fully determine the correlation functions.

6. Conclusion

We have succeeded in formulating the Wess–Zumino–Witten models at higher genus such that the Ward identities of the current algebra become complete. Thus, besides reducing the theory to the analysis of the correlation functions between the affine primary fields only, the Ward identities provide linear differential equations which should fully characterize the correlation functions.

This formulation requires the definition of a twisted version of the models. The twists belong to the Lie groups on which the models are defined. They give rise to new Lie-group-valued moduli. The action of the zero modes of the affine currents on the correlation functions are defined by Lie derivatives acting in these moduli. It is the very analogue of the definition of the zero modes of the stress tensor as derivatives on the moduli space of the Riemann surfaces. The differential equations satisfied by the partition function and by the correlation functions involve derivations with respect to all of these moduli parameters.

To illustrate the method, we have identified the denominator of the chiral partition function. We also have argued that both the denominator and the numerator of the chiral partition function satisfy a kind of heat equation on the moduli space.

It evidently remains to integrate these linear differential equations.

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APPENDIX

In this appendix we point out few details concerning the Virasoro Ward identities.

Let us define an automorphic 2-form in z by

$$Q_{zz}(z,w)=\sum_{\gamma\in\Gamma}\frac{[\gamma'(z)]^2}{\gamma(z)-w},\quad \gamma_z^*Q_{zz}(z,w)=Q_{zz}(z,w). \tag{A.1}$$

If w is in the fundamental domain F, $Q_{zz}(z,w)$ has only a simple pole in F located at $z=w$. But if $w=\gamma^{-1}(\infty)$ for some γ in Γ, $Q_{zz}(z,\gamma^{-1}(\infty))$ is a holomorphic quadratic differential defined over $\Sigma=\mathbb{C}/\Gamma$. As it has been proved in ref. [22], a basis of holomorphic quadratic differentials can be made up of automorphic 2-forms $Q_{zz}(z,\gamma^{-1}(\infty))$; i.e., if one chooses judiciously $(3h-3)$ elements γ_α in Γ, then the automorphic 2-forms $h^\alpha_{zz}(z)=Q_{zz}(z)=Q_{zz}(z,\gamma_\alpha^{-1}(\infty))$ span the space of holomorphic quadratic differentials.

The Poincaré series $Q_{zz}(z,w)$ fails to be a vector field in w. But for any γ in Γ, the difference between $Q_{zz}(z,\gamma(w))[\gamma'(w)]^{-1}$ and $Q_{zz}(z,w)$ is a quadratic differential in z which has no pole in the fundamental domain F. As such it decomposes on the holomorphic quadratic differentials $h^\alpha_{zz}(z)$

$$Q_{zz}(z,\gamma(w))[\gamma'(w)]^{-1}-Q_{zz}(z,w)=-\sum_{\alpha}^{3h-3}h^\alpha_{zz}(z)\Upsilon_\alpha(w|\gamma). \tag{A.2}$$

The function $\Upsilon_\alpha(w|\gamma)$ are elements of $\Pi_{(2)}$; the set of polynomials of degree less than or equal to two.

For deriving Ward identities for one insertion of the stress tensor by the method of the operator valued differentials, we have to introduce the following 1-form

$$\hat{\Omega}_x(x)=Q_{zz}(z,x)\langle T(x)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g. \tag{A.3}$$

From the OPE of the Virasoro algebra and the residue theorem we deduce that

$$\begin{aligned}&\langle T(x)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g\\ &\quad=+\sum_{n=1}^{N}\left[\Delta_n\partial_{\xi_n}Q_{zz}(z,\xi_n)+Q_{zz}(z,\xi_n)\partial_{\xi_n}\right]\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g\\ &\qquad+\sum_{\alpha=1}^{3h-3}h^\alpha_{zz}(z)\left[\frac{1}{2i\pi}\sum_{j=1}^{h}\oint_{a_j}\mathrm{d}x\,\Upsilon_\alpha(x|\gamma_j)\langle T(x)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g\right]\end{aligned} \tag{A.4}$$

We would like to interpret the last term in eq. (A4) as a derivation of the correlation functions with respect to the moduli parameters. But to do that we have first to study the polynomials $\Upsilon_\alpha(w|\gamma)$.

First, we show that they are cocycles for the Eichler cohomology [23]. In concrete terms, it means that the maps Υ_α from Γ to $\Pi_{(2)}$ defined by Υ_α: $\gamma \to \Upsilon_\alpha(w|\gamma)$ satisfy the following relation

$$\Upsilon_\alpha(w|\gamma\mu) = \Upsilon_\alpha(\mu(w)|\gamma)[\mu'(w)]^{-1} + \Upsilon_\alpha(w|\mu), \tag{A.5}$$

for any γ and μ in Γ. Eq. (A.5) is proved by applying the relation (A.2) twice. It also means that the maps Υ_α are completely specified by the data of the h functions $\Upsilon_\alpha(w|\gamma_j)$.

Kra introduces Eichler coboundaries [23]: a map Υ from Γ to $\Pi_{(2)}$ is called a coboundary if there exists a polynomial $p(w)$ in $\Pi_{(2)}$ such that

$$\Upsilon(w|\gamma) = p(\gamma(w))[\gamma'(w)]^{-1} - p(w). \tag{A.6}$$

Obviously, Eichler coboundaries are Eichler cocycles.

Second, we show that they are dual to the holomorphic quadratic differentials $h^\alpha_{zz}(z)$. To give a meaning to this duality, we define a scalar product between the holomorphic quadratic differentials $h_{zz}(z)$ and the Eichler cocycles Υ by

$$(\Upsilon; h_{zz}) = \sum_{j=1}^{h} \oint_{a_j} \mathrm{d}x\, \Upsilon(x|\gamma_j) h_{xx}(x). \tag{A.7}$$

Here we have supposed that the Eichler cocycle Υ was specified by the data of the functions $\Upsilon(x|\gamma_j)$. This scalar product is actually defined for the Eichler cocycles modulo the Eichler coboundaries; i.e.: it is defined on the equivalence classes of Eichler cocycles.

The duality between the holomorphic quadratic differentials $h^\alpha_{zz}(z)$ and the maps Υ_α is

$$\left(\Upsilon_\alpha; h^\beta_{zz}\right) = 2i\pi\delta^\beta_\alpha, \tag{A.8}$$

To prove this duality it is sufficient to apply the residue theorem to the following 1-form: $\Omega_x(x) = Q_{zz}(z, x) h^\beta_{xx}(x)$. The dimension of the space of Eichler cocycles modulo the Eichler coboundaries is $(3h-3)$ [23]. Therefore, an equivalence class of Eichler cocycles is specified by its scalar products with the holomorphic quadratic differentials.

Third, we remark that there is an one-to-one map between the equivalence classes of Eichler cocycles and the equivalence classes of Beltrami differentials. Beltrami differentials are (1-1)-forms; the equivalence relation is defined by identifying two Beltrami differentials which differ by the $\bar\partial$ of a vector field. The space of equivalence classes of Beltrami differentials is dual to the space of holomorphic quadratic differentials. The scalar product is defined by

$$[\mu; h_{zz}] = \int_{\Sigma} \mu_{\bar{x}}^{x}(x, \bar{x}) h_{zz}(x). \tag{A.9}$$

An equivalence class of Beltrami differentials is completely specified by its scalar product with the holomorphic quadratic differentials. Therefore, the duality relation (A.8) induces a one-to-one map, $\Upsilon \to \mu$, between the equivalence classes of Beltrami differentials and of Eichler cocycles such that $(\Upsilon; h_{zz}) = [\mu; h_{zz}]$.

We now are able to identify the last term in eq. (A.3) as a derivation on the moduli space. The expectation value of the stress tensor is a holomorphic quadratic differential. Therefore, from the one-to-one correspondence between the Eichler cocycles and the Beltrami differentials, and from the duality property (A.8), we have

$$2i\pi \frac{\partial}{\partial m^{\alpha}} \log Z(\Gamma; g) = \left(\Upsilon_{\alpha}; \langle T(z) \rangle_{g}\right) = \sum_{j=1}^{h} \oint_{a_j} \mathrm{d}x\, \Upsilon_{\alpha}(x|\gamma_j) \langle T(x) \rangle_{g}. \tag{A.10}$$

This equation is equivalent to eq. (5.5). (Note the analogy with eq. (4.6).)

A similiar relation cannot be used for the correlation functions. Indeed, in contrast to eq. (A.10) in which the scalar product is independent of the representative Υ_{α} of the Eichler cohomology class, the contour integral in eq. (A.3) depends on the representative Υ_{α}. Therefore, this contour integral cannot be directly identified with the derivative of the correlation functions. But as explained in sect. 5, to each deformation of the kleinian group there is associated canonically a unique Eichler cocycle (or a unique Beltrami coefficient). These Eichler cocycles are defined by equations similar to eq. (5.13). In other words, to each modular parameter m^{α} there is associated canonically a Eichler cocycle that we denote by χ_{α}. Therefore, we have the following identification

$$\begin{aligned} &2i\pi \frac{\partial}{\partial m^{\alpha}} \left[\langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g Z(\Gamma; g)\right] \\ &\quad = \sum_{j=1}^{h} \oint_{a_j} \mathrm{d}x \left(\chi_{\alpha}(x|\gamma_j) \langle T(x) \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g Z(\Gamma; g)\right). \end{aligned} \tag{A.11}$$

There are no ambiguities in this last equation. On the contrary, eq. (11) of ref. [4] may suffer from some ambiguities due to the presence of poles in the correlation functions but also due to the ambiguities inherent to the definition of the Beltrami differentials.

Finally, χ_α and Υ_α being in the same Eichler cohomology class, there exists an element p_α of $\Pi_{(2)}$ such that

$$\Upsilon_\alpha(w|\gamma) = \chi_\alpha(w|\gamma) + p_\alpha(\gamma(w))[\gamma'(w)]^{-1} - p_\alpha(w), \tag{A.12}$$

for all elements γ of Γ.

Therefore, if we define the Green function $\mathbb{G}_{zz}(z, w)$ by

$$\mathbb{G}_{zz}(z, w) = Q_{zz}(z, w) - \sum_{\alpha}^{3h-3} h_{zz}^{\alpha}(z) p_\alpha(w), \tag{A.13}$$

the identities (A.3) become the Ward identities (5.4) discussed in sect. 5.

References

[1] Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. B241 (1984) 333
[2] D. Friedan and S. Shenker, Phys. Lett. B174 (1986) 287; Nucl. Phys. B281 (1987) 505
[3] E. Witten, Commun. Math. Phys. 113 (1988) 529
[4] T. Eguchi and H. Ooguri, Nucl. Phys. B282 (1987) 308
[5] N. Ishibashi, Y. Matsuo and H. Ooguri, Tokyo preprint UT-489-87;
L. Alverez-Gaume, C. Gomez and C. Reina, Phys. Lett. B190 (1987) 55;
C. Vafa, Phys. Lett B190 (1987) 47; HUTP-87/A066
[6] T. Eguchi and H. Ooguri, Phys. Lett. B187 (1987) 127;
L. Alverez-Gaume, J.-B. Bost, G. Moore, P. Nelson and C. Vafa, Commun. Math. Phys. 112 (1987) 503;
R. Dijkgraaf, E. Verlinde and H. Verlinde, preprint THU-87/17;
M. Dugan and H. Sonoda, Nucl. Phys. B289 (1987) 227;
J. Atick, L. Dixon, D. Griffin and D. Nemeschansky, Nucl. Phys. B298 (1988) 1;
P. DiFrancesco, H. Saleur and J.-B. Zuber, Nucl. Phys. B290 [FS20] (1987) 527;
D. Bernard, Meudon preprint 1987, Nucl. Phys. B302 (1988) 251–279
[7] E. Witten, Commun. Math. Phys. 92 (1984) 455
[8] P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88
[9] V. Knizhnik and A. Zamolodchikov, Nucl. Phys. B247 (1984) 83
[10] D. Bernard, Nucl. Phys. B303 (1988)
[11] J. Fay, Theta functions on Riemann surfaces, Lecture notes in mathematics, vol. 352, (Springer, Berlin, 1973);
G. Springer, Introduction to Riemann surface (Chelsea, 1957)
[12] P. Martin and S. Schwinger, Phys. Rev. 115 (1959) 1342;
R. Kubo, J. Phys. Soc. Jpn., 12 (1957) 570;
R. Haag, N. Hugenholtz and M. Winnink, Commun. Math. Phys. 5 (1967) 215;
R. Haag, D. Kastler and E. Trych-Pohlmeyer, Commun. Math. Phys. 38 (1974) 173
[13] L.R. Ford, Automorphic functions (Chelsea, 1951);
W. Burnside, Proc. Lond. Math. Soc. 23 (1891) 49
[14] R. Gunning, Lectures on Riemann surfaces (Princeton University Press, Princeton, NJ, 1966)
[15] L. Ahlfors, Lectures on quasi-conformal mapping (Van Nostrand, New York, 1966);
L. Bers, Bull. Lond. Math. Soc. 4 (1972) 257

[16] C. Lovelace, Phys. Lett. B32 (1970) 703;
V. Alessandrini, Nuovo Cim. 2A (1971) 321;
V. Alessandrini and D. Amati, Nuovo Cim. 4A (1971) 793;
C. Montonen, Nuovo Cim. A19 (1974) 69;
S. Mandelstam, Proc. Santa Barbara Workshop on Unified string theories, eds. M. Green, D. Gros (World Scientific, Singapore, 1986)
[17] E. Martinec, Nucl. Phys. B281 (1987) 157
[18] V.G. Kac, Infinite dimensional Lie algebras (Cambridge University Press, Cambridge, 1985)
[19] T. Eguchi and H. Ooguri, Tokyo Univ. Preprint UT-518
[20] H. Fegan, Trans. Am. Math. Soc. 248 (1978) 339; J. Diff. Geo. 13 (1978) 589
[21] I. MacDonald, Inv. Math. 15 (1972) 91;
B. Kostant, Ad. Math. 20 (1976) 179; see also ref. [17]
[22] D. Heijhal, C. R. Acad. Sc. Paris, 279A (1979) 713; 280A (1979) 341;
I. Kra and B. Maskit, Commun. Math. Helv. 57 (1982) 603
[23] I. Kra, Acta Math. 153 (1984) 47