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ON THE WESS-ZUMINO-WITTEN MODELS ON THE TORUS*

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We discuss the Ward identities of the Wess-Zumino-Witten models on Riemann surfaces and point out some ambiguities in the description of the zero modes of the currents. In the case of the torus, we show how to describe them and we write the Ward identities in such a way that they become complete. We examine in detail how the Ward identities are related to the Kubo-Martin-Schwinger condition. As an illustration of this formulation, we present a new proof of the Weyl-Kac character formula. The proof essentially relies on the mixed Virasoro × Kac-Moody Ward identities and explains the relation of the heat equation on the group manifold to the Weyl-Kac character formula.

1. Introduction

Both in the study of the two-dimensional critical phenomena and in the string theory one of the more important purposes is to find a complete description of all conformal field theories [1]. Moreover, as emphasized by Polyakov [2] and Cardy [3], we have to look at these theories on the torus and on higher genus. There is now a clear understanding - in the functional approach [4-8] as well as in the operator formalism [9, 10] - of the free conformal field theories on Riemann surfaces. Bosonic and fermionic free field theories have been extensively discussed in the recent literature, especially in connection with the abelian bosonization - with [10-13] or without [4-7] twisted fields. But, little is known concerning the interacting conformal field theories, in particular for those which have a central charge greater than one. Therefore, to describe the Wess-Zumino-Witten models [14] on higher genus is a concrete approach to this problem. It is also the first step in a complete description of the non-abelian bosonization [14] on Riemann surfaces. Stringy speaking, it also corresponds to describe the loop expansions of strings compactified on group manifolds [15].

In this article, we point out that to have complete Ward identities for the current algebras on Riemann surfaces requires to consider "character valued expectation values"; i.e. expectation values with an insertion of an element of the Lie group. Here, complete Ward identities mean that the Ward identities reduce the study of

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the theory to the evaluation of the correlation functions involving the primary fields only. We discuss in details the case of the torus, and present our formulation of the Ward identities for the current algebras. We also describe how the Ward identities are related to the Kubo-Martin-Schwinger condition [16].

As an illustration of this formulation of the Ward identities, we give a new proof of the Weyl-Kac character formula [17]. The proof essentially relies on the mixed Virasoro × Kac-Moody Ward identities, which are no more than the cyclicity property of the trace together with the Sugawara's construction. This proof explains the "mysterious relation of the heat equation on the Lie group to the representations of the loop group" (ref. [18], p. 286). An application of this relation can be found in the work of Frenkel [19] concerning the orbital theory for the affine Lie algebras. In his study, he extensively uses the relation of the denominator of the Weyl-Kac character formula to the heat equation, see also ref. [20]. In addition, in ref. [18], p. 283, it was noted – without explanation – that the Weyl-Kac character formula possesses "the striking property that the power of $q=\exp(i2\pi\tau)$ which accompanies each finite dimensional character $\chi_\lambda(g)$ does not depend on the affine representation (except on its level), and is proportional to the value of the Casimir of the finite dimensional representation". This property turns out to be a simple consequence of the mixed Ward identities.

2. General remarks

One of the more powerful properties of the conformal field theories resides in the following fact [1]: because of the Ward identities associated to the Virasoro algebra the correlation functions with insertions of the stress-tensor are expressed in terms of the correlation functions without insertion. It is therefore sufficient to only determine the correlation functions involving the primary fields. In addition to the conformal symmetry, the WZW models also possess chiral conserved currents, $J^a(z)$ and $\bar{J}^a(\bar{z})$, which represent two commuting affine Kac-Moody algebras $\mathscr{G}^{(1)}$ and $\bar{\mathscr{G}}^{(1)}$. As described by Knizhnik and Zamolodchikov [21] for the WZW models on the sphere, the Ward identities for the current algebras reduce the study of these models to the computation of the correlation functions between the affine primary fields only; the correlation functions with currents inserted in them are deduced from the correlation functions without insertion.

What about on higher genus? The Ward identities for the Virasoro and for the current algebras have been described by Eguchi and Ooguri [22]. For the Virasoro algebra, the situation is similar: if one knows the correlation functions between the primary fields one also knows the correlation functions between the primary fields and the stress-tensor. The relation is evidently more complicated. In particular it involves derivatives with respect to the moduli parameters of the Riemann surfaces. But for the affine algebras the situation is rather different. Let us recall how the Ward identities of the current algebras look like on a Riemann surface of genus g.

Consider a collection of N primary fields of the current algebra, $\Phi_1(\xi_1), \dots, \Phi_N(\xi_N)$. They belong to some representations $\rho_{(j)}$ of the finite semi-simple Lie algebra $\mathscr{G}$. $\mathscr{G}$ is the Lie algebra of the Lie group G on which the WZW model is defined. The Ward identities for one insertion of the current $J^a(z)$ are [22]:

$$\langle J_z^a \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle = \sum_{k=1}^{N} \partial_z \log E(z, \xi_k) t_{(k)}^a \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle + \sum_{j=1}^{g} \omega_j(z) \langle J_{0;\,j}^a \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle . \tag{2.1}$$

Here, $t_{(k)}^a = \rho_{(k)}(t^a)$ is the representation of the Lie algebra $\mathscr{G}$ which only acts on the field $\Phi_k(\xi_k)$. The matrices t^a satisfy $[t^a, t^b] = f_c^{ab} t^c$ where f_c^{ab} are structure constants of $\mathscr{G}$. To write eq. (2.1), we have chosen a basis of canonical cycles on the Riemann surface, (a_j, b_j), $j = 1, \dots, g$. $E(z, \xi)$ is the prime form on the Riemann surface; it depends on the choice of the canonical cycles. $\omega_j(z)$, $j = 1, \dots, g$, are the holomorphic differential forms dual to the canonical cycles (a_j, b_j). The operators $J_{0;\,j}^a$ are the zero modes of the currents:

$$J_{0;\,j}^a = \oint_{a_j} J_z^a . \tag{2.2}$$

These Ward identities have been demonstrated in [22] by using (non-analytic) Green functions. They also can be derived by analytic considerations. The contraction of the current J_z^a with a primary field $\Phi(\xi)$ must produce a (multivalued) meromorphic 1-form with only a simple pole located at $z = \xi$. Therefore, this 1-form is the derivative of the logarithm of the prim form: this is the first term of the r.h.s. in (2.1). To determine the second term note that it must be holomorphic in z. As such it is completely defined by its a-periods. Evaluation of the a-periods of both sides in (2.1) gives the result since $\partial_z \log E(z, \xi)$ has vanishing a-periods. These Ward identities can also be derived from the method of the operator valued differential forms described by Witten [10].

In the same way, one derives the Ward identities for two insertions:

$$\begin{aligned} \langle J_z^a J_w^b \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle = & -K \delta^{ab} \partial_z \partial_w \log E(z, w) \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle \\ & + \left\{ f_c^{ab} \partial_z \log E(z, w) + \delta_c^b \sum_{k=1}^{N} t_{(k)}^a \partial_z \log E(z, \xi_k) \right\} \\ & \times \langle J_w^c \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle \\ & + \sum_{j=1}^{g} \omega_j(z) \langle J_{0;\,j}^a J_w^b \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle . \end{aligned} \tag{2.3}$$

In order to illustrate why and how the zero modes carry almost all the non-trivial information concerning the WZW model, let us deduce the implication of eq. (2.3) for the partition function. The variation of the logarithm of the partition function with respect to the moduli parameters, m^k, $k=1,\dots,3g-3$, is the expectation value of the stress-tensor [22,23]:

$$\langle T_{zz}\rangle = \sum_{k=1}^{3g-3} h^k_{zz}\frac{\partial}{\partial m^k}\log Z(m,\bar{m}), \tag{2.4}$$

where the h^k_{zz} form a basis of quadratic abelian differentials.

In the WZW models, we postulate that the stress-tensor is given by Sugawara's construction. Namely, inside the correlation functions we have:

$$T^{\mathrm{Sug.}}_{zz} = \lim_{w\to z}\left\{-\frac{1}{2(K+h^*)}\left[J^a_z J^a_w + \frac{K\dim \mathrm{G}}{(z-w)^2}\right]\right\}, \tag{2.5}$$

where h^* is the dual Coxeter number of $\mathcal{G}$ and K the level of the affine representation. Therefore, from the Ward identities (2.3), we have:

$$\langle T^{\mathrm{Sug.}}_{zz}\rangle = \frac{K\dim \mathrm{G}}{K+h^*}\frac{S(z)}{12} - \sum_{j,k=0}^{g}\omega_j(z)\left\langle\frac{J^a_{0;j}J^a_{0;k}}{2(K+h^*)}\right\rangle\omega_k(z). \tag{2.6}$$

Here $S(z)$ is the holomorphic projective connection, see ref. [24]. Under a change of holomorphic coordinate, $z\to w(z)$, it transforms inhomogeneously:

$$S(w)\,\mathrm{d}w^2 = S(z)\,\mathrm{d}z^2 + \{z;w\}\,\mathrm{d}w^2, \tag{2.7}$$

where $\{z;w\}$ is the schwarzian derivative. Therefore the Virasoro central charge is $C_{\mathrm{G}} = K\dim\mathrm{G}/(K+h^*)$.

In addition, the variation of the partition function of a free boson under a change of the complex structure is the projective connection. Thus, we have the following identity:

$$\sum_{k=1}^{3g-3} h^k_{zz}\frac{\partial}{\partial m^k}\log\left[Z(m,\bar{m})\left\{\frac{\det'(-\Delta)}{\int\sqrt{g}\,\det(\mathrm{Im}\,\tau)}\right\}^{C_{\mathrm{G}}/2}\right]$$

$$= -\sum_{j,k=1}^{g}\omega_j(z)\left\langle\frac{J^a_{0;j}J^a_{0;k}}{2(K+h^*)}\right\rangle\omega_k(z). \tag{2.8}$$

Note that eq. (2.8) looks like a multi-time heat equation on the group manifold; the time variables being the moduli parameters. This remark will be made more precise in sect. 4. In the case of the torus, eq. (2.8) gives rise to "monstrous" identities between theta functions and string functions of the affine representations. Sometimes, they reduce to Jacobi identities.

Eq. (2.8) perfectly illustrates that almost all the special features of a given WZW model are encoded inside its zero modes. The zero modes correspond to the solitonic sector of the Lie algebra valued fields. The quantum part is simply described by free field quantities. Therefore, to have a complete description of the model, and a powerful formulation of the Ward identities, we need to define the action of the zero modes inside the correlation functions.

One way to define these zero modes is obviously to describe them as a derivation. To do that, we have to make precise which kind of derivation it is. We also must consider "character valued expectation values"; i.e. expectation values with an insertion of an element of the Lie group G. For concreteness, let us restrict ourself to the case of the torus. For each element g of the Lie group G with $g=\exp(\gamma)$ where γ is an element of the Lie algebra $\mathscr{G}$, we define character valued expectation values and denoted them $\langle \cdots \rangle_g$ or $\langle \cdots \rangle_\gamma$. Under a conjugation of g by a constant element g_0 of G, $g \to g_0 g g_0^{-1}$, the expectation values transform as follows:

$$\langle \Phi_1 \dots \Phi_N \rangle_{g_0 g g_0^{-1}} = \rho_{(1)}(g_0) \dots \rho_{(N)}(g_0) \langle \Phi_1 \dots \Phi_N \rangle_g \tag{2.9}$$

if the field Φ_j belongs to the representation $\rho_{(j)}$ of $\mathscr{G}$. In particular setting $g_0 = g$ in the above equation, we find:

$$\langle \Phi_1 \dots \Phi_N \rangle_g = \rho_{(1)}(g) \dots \rho_{(N)}(g) \langle \Phi_1 \dots \Phi_N \rangle_g . \tag{2.10}$$

This equation expresses the invariance of the correlation functions under global gauge transformation. Lie derivation of this equation produces other conservation laws.

The correlation functions $\langle \cdots \rangle_g$, but also the partition function, noted $Z(\tau; g)$ or $Z(\tau; \gamma)$, now depend on all the moduli parameters. They depend on τ but also on the element g of the Lie group G. $Z(\tau; g)$ depends only on the conjugation class of g; i.e.: $Z(\tau; g_0 g g_0^{-1}) = Z(\tau; g)$.

We define the action of the zero modes J_0^a by:

$$\langle J_0^a \rangle_g = \mathscr{L}_a \log Z(\tau; g) , \tag{2.11}$$

$$\langle J_0^a \Phi_1 \dots \Phi_N \rangle_g - \langle J_0^a \rangle_g \langle \Phi_1 \dots \Phi_N \rangle_g = \mathscr{L}_a \langle \Phi_1 \dots \Phi_N \rangle_g , \tag{2.12}$$

where $\mathscr{L}_a$ denotes the Lie derivitive on the group manifold G along the left-invariant Killing vector e_{L}^a. Similarly, the zero modes $\bar{J}_0^a$ could be described by Lie derivative along the right invariant Killing vector e_{R}^a. This definition ensures that the currents $J^a(z)$ and $\bar{J}^a(\bar{z})$ commute.

The formulae (2.9) to (2.12) can be derived from the definition of the correlation functions in the operator formalism,

$$\langle \Phi_1 \dots \Phi_N \rangle_g = Z^{-1}(\tau; g) \mathrm{Tr}\left(g \Phi_1 \dots \Phi_N q^{L_0} \right) ,$$

and the following well-known formulae:

$$e^{-\gamma} d e^{\gamma} = \frac{1 - e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma} \cdot d\gamma \qquad \text{and} \qquad g \cdot \Phi_j(\xi) \cdot g^{-1} = \rho_{(j)}(g) \cdot \Phi_j(\xi). \quad (2.13)$$

Here ad γ denotes the adjoint representation of γ.

Now, contrary to eqs. (2.1) and (2.3), we have a precise definition of the zero modes. But, we do not know what is the contraction function; i.e. we do not know what is the generalization of the derivative of the logarithm of the prime form. A priori, we imagine that it depends on g so that the Ward identities for one insertion of a current will take the following form:

$$\langle J_z^a \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g - \langle J_z^a \rangle_g \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g$$
$$= \sum_{k=1}^{N} \omega_z(z, \xi_k | g)^a_b t^b_{(k)} \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g + \mathscr{L}_a \langle \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g. \quad (2.14)$$

Since we do not know the 1-form $\omega_z(z, \xi | g)$, it could seem that we have made no progress: we only have moved the problem from the zero modes to the contraction function $\omega_z(z, \xi | g)$. But, in the following we will describe how we can specify this function. In this way the Ward identities will become complete. Evidently, they will involve derivatives with respect to all the moduli parameters τ and g.

3. The Kubo-Martin-Schwinger condition and the Ward identities

In this section we show that the Kubo-Martin-Schwinger (KMS) condition [16] yields to the Ward identities on the torus and we show how it allows us to determine the contraction function $\omega(z, \xi | g)$. The KMS condition is a relation satisfied by an equilibrium state at finite inverse temperature β. It relates the expectation values of observable at time t and at time $(t + i\beta)$. To be precise, for any observables A and $B(t)$, we must have:

$$\langle A B(t) \rangle_\beta = \langle B(t + i\beta) A \rangle_\beta . \quad (3.1)$$

The condition (3.1) is almost equivalent to the cyclic property of the trace. In simple field theories, like free fermions, it uniquely determines the vacuum. A beautiful application of this condition to the Hawking radiation of the black holes has been described by Haag and co-workers in ref. [25].

For the character valued expectation values introduced above the KMS condition reads:

$$\langle J^a(z) J^b(w) \dots \Phi_1(\xi_1) \dots \Phi_N(\xi_N) \rangle_g$$
$$= q (e^{\operatorname{ad}\gamma})^a_d \langle J^b(w) \dots \Phi_1(\xi_1) \dots \Phi_N(\xi_N) J^d(zq) \rangle_g \quad (3.2)$$

where $q = \exp(i2\pi\tau)$, with $\mathrm{Im}(\tau) > 0$. The fields are supposed to be ordered in such a way that $|z| > |w| > \cdots > |\xi_1| > \cdots > |\xi_N| > |zq|$. A simple derivation of eq. (3.2) uses the cyclicity property of the trace and the fact that $J^a(z)$ is a Virasoro primary field with conformal weight one. Note that from the l.h.s. to the r.h.s. of (3.2) we have moved the current J^a from the left to the right, but on the r.h.s. its argument is zq instead of z.

Let us illustrate on the two-point function of the currents, $\langle J^a(z)J^b(w)\rangle_g$, how we use the KMS condition (3.2) to derive the Ward identities. Consider a chiral affine current $J^a(z)$,

$$J^a(z) = \sum_n J^a_n z^{-n-1}, \tag{3.3}$$

whose modes J^a_n satisfy the commutation relations of an untwisted affine Kac-Moody algebra:

$$[J^a_n, J^b_m] = f^{ab}_c J^c_{n+m} - nK\delta^{ab}\delta_{n+m,0}. \tag{3.4}$$

Here, f^{ab}_c are normalized by $f^{ab}_c f^{dc}_b = -2h^*\delta^{ad}$. (In eq. (3.4), there is a minus sign in front of the central charge K because the Killing form is negative on $\mathscr{G}$. It evidently disappears if we change the normalization of the structure constants by a factor i: $f^{ab}_c \to i f^{ab}_c$). We suppose that the currents $J^a(z)$ have only integer modes, so that we describe the affine algebra in one of its homogeneous gradations.

In terms of the modes J^a_n, the KMS condition becomes:

$$q^n\langle J^a_n J^b_m\rangle_g = (\mathrm{e}^{-\mathrm{ad}\,\gamma})^a_d\langle J^b_m J^d_n\rangle_g. \tag{3.5}$$

Eq. (3.5) allows us to express the expectation value $\langle J^a_n J^b_m\rangle_g$ in terms of the commutator $[J^a_n, J^b_m]$. Namely, for $n \neq 0$, we have:

$$\begin{aligned}\langle J^a_n J^b_m\rangle_g &= \left(\frac{1}{1-q^n\mathrm{e}^{\mathrm{ad}\,\gamma}}\right)^a_d\langle [J^d_n, J^b_m]\rangle_g \\ &= \left(\frac{1}{1-q^n\mathrm{e}^{\mathrm{ad}\,\gamma}}\right)^a_d\left[f^{db}_c\langle J^c_0\rangle_g - nK\delta^{db}\right]\delta_{n+m,0}.\end{aligned} \tag{3.6}$$

Note that eq. (3.6) does not determine the expectation values with an insertion of a zero mode since the matrix $(1-\mathrm{e}^{-\mathrm{ad}\,\gamma})$ is not invertible. Therefore, there is no contradiction between the KMS condition and the definition (2.11) of the zero modes.

To find the two-point function $\langle J^a(z)J^b(w)\rangle_g$ we just have to re-sum the expression (3.6). We find:

$$zw\langle J^a(z)J^b(w)\rangle_g = -Kw\partial_w\omega(z,w|g)^{ab} + \omega(z,w|g)^a_d f^{db}_c\langle J^c_0\rangle_g + \langle J^a_0 J^b_0\rangle_g, \tag{3.7}$$

where the matrix $\omega(z, w|g)$ is:

$$\omega(z, w|g) = \frac{w}{z-w} + \sum_{n=1}^{\infty}\left(\frac{q^n}{e^{-\mathrm{ad}\,\gamma} - q^n}\left(\frac{w}{z}\right)^n - \frac{q^n}{e^{\mathrm{ad}\,\gamma} - q^n}\left(\frac{z}{w}\right)^n\right). \tag{3.8}$$

The expectation values of the zero modes are defined in eq. (2.11). The factor zw in front of the two-point function is due to our choice of coordinate z instead of the standard choice $\exp(i2\pi z)$.

Notice that in eq. (3.8), we have defined an analytic continuation of the correlation function $\langle J^a(z)J^b(w)\rangle_g$, initially defined for $|z| > |w| > |zq|$, in the region $|w/q| > |z| > |qw|$, - or $|z/q| > |w| > |qz|$. Actually, with a little more work, we can define an analytic continuation of $\omega(z, w|g)$ on any compact domain of the complex plane. But later on, the formulae will be more transparent if we keep the expression (3.8) for $\omega(z, w|g)$. We just mention that this analytic continuation can be used to show that the Ward identities (3.7), or (3.10) and (3.11) below, reduce to those described in eqs. (2.1) or (2.3) in the borderline case where there is no insertion of an element of the Lie group. Indeed, when $g = e$ with e the unity of the Lie group, the function $\omega(z, w) = \omega(z, w|g = e)$ evaluated in the standard coordinate $e(z) = \exp(i2\pi z)$ is (almost) the derivative of the logarithm of the Θ_1 function:

$$2i\pi\,\omega(e(z), e(w)|g = e) = \partial_z \log \Theta_1(z - w|\tau) - i\pi. \tag{3.9}$$

Θ_1 is the theta function with characteristic $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. The factor $(-i\pi)$ cancels out in the Ward identities since at $g = e$ the correlation functions are singlet under the Lie group G.

In a similar way, one derives the Ward identities in presence of primary fields. We have for one insertion:

$$z\langle J^a(z)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g - z\langle J^a(z)\rangle_g\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g$$
$$= \sum_{k=1}^{N}\omega(z, \xi_k|g)^a_d t^d_{(k)}\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g + \mathscr{L}_a\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g \tag{3.10}$$

and for two insertions:

$$z\langle J^a(z)J^b(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g - z\langle J^a(z)\rangle_g\langle J^b(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g$$
$$= -K\,\partial_w\omega(z, w|g)^{ab}\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g$$
$$+\left\{\omega(z, w|g)^a_d f^{db}_c + \delta^b_c\sum_{k=1}^{N}\omega(z, \xi_k|g)^a_d t^d_{(k)}\right\}\langle J^c(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g$$
$$+\mathscr{L}_a\langle J^a(w)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g. \tag{3.11}$$

Here $\omega(z, w|g)$ is the same function as in eq. (3.8). The Ward identities (3.10) and (3.11) effectively have the structure suspected in (2.14). They are complete in the sense that it is now sufficient to know the correlation functions between the primary fields to know all the correlation functions. But, the price to pay for having this completeness is to evaluate the correlation functions for any element g of the Lie group.

In the WZW models, there are three kinds of Ward identities. There are the Ward identities associated to the Virasoro algebra, those associated to the affine Kac-Moody algebras, and the mixed Ward identities [21]. These latter arise when we impose the Sugawara's expression for the stress-tensor, equation (2.5). In that case, the correlation functions $\langle T(z)\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g$ can be obtained in two different ways: either directly from the Virasoro algebra or by taking the limit $w \to z$ in the Ward identities of the current algebra. Comparison of the two results gives the mixed Ward identities. These identities are powerful. They determine the conformal weight of the primary fields: $\Delta_{(k)} = \mathrm{Casimir}(\rho_{(k)})/2(K+h^*)$; they give rise to differential equations which allow us to solve the model [21;26]; etc. ...

On the torus, the mixed Ward identities look like:

$$\begin{aligned} &z^2\langle T_{\mathrm{Sug.}}(z)\Phi(\xi)\rangle_g - z^2\langle T_{\mathrm{Sug.}}(z)\rangle_g\langle\Phi(\xi)\rangle_g \\ &\quad = \left[\Delta\,\xi\partial_\xi\omega(z,\xi) + \omega(z,\xi)(\xi\partial_\xi + \Delta)\right]\langle\Phi(\xi)\rangle_g \\ &\quad\quad + q\partial_q\langle\Phi(\xi)\rangle_g . \end{aligned} \tag{3.12}$$

Here $\omega(z, \xi) = \omega(z, \xi|g = e)$ is the function defined in eqs. (3.8) or (3.9).

To obtain the fundamental differential equations satisfied by the primary fields you have to extract the residue as $z \to \xi$ in eq. (3.12). Without going into the details, we just report the result which can be synthesized as follows:

$$\begin{aligned} &(\xi_j\partial_{\xi_j} + 2\Delta_{(j)})\langle\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle\rangle \\ &\quad + \frac{1}{K+h^*}\sum_{k\neq j}\omega(\xi_j,\xi_k|g)_{ab}t^a_{(j)}t^b_{(k)}\langle\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle\rangle \\ &\quad + \frac{1}{K+h^*}t^a_{(j)}\mathscr{L}_a\langle\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle\rangle = 0 . \end{aligned} \tag{3.13}$$

The double bracket means:

$$\langle\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle\rangle = Z(\tau; g)\Pi(\tau; g)\langle\Phi_1(\xi_1)\dots\Phi_N(\xi_N)\rangle_g , \tag{3.14}$$

where $Z(\tau; g)$ is the partition function and $\Pi(\tau; g)$ an infinite product defined below in eq. (4.3)

Eq. (3.13) will allow us to determine the correlation functions of the primary fields. In the following we will not continue in going into the depths of the WZW models on the torus or on higher genus; this discussion will be developed elsewhere [27]. But, we will describe how we derive the Weyl-Kac character formula by using the method described above.

4. A new proof of the Weyl-Kac character formula

In the following we will use the mixed Ward identities to give a new proof of the Weyl-Kac character formula [17]. The proof goes in two steps: We first exhibit the link between the Sugawara's construction, the heat equation on the Lie group and the Weyl-Kac character formula. In this way, we will be able to determine the general structure of the Weyl-Kac character formula. In a second step, we show how the affine Weyl group determines the coefficients left undetermined by the first analysis.

Recall that on the torus the stress-tensor expectation value gives the variation of the partition function:

$$q\partial_q \log Z(\tau; g) = z^2 \langle T_{\mathrm{Sug.}}(z) \rangle_g . \tag{4.1}$$

The limit of the Ward identities (3.7) as $w \to z$ yields to the following expectation value of $T_{\mathrm{Sug.}}(z)$:

$$\begin{aligned} z^2 \langle T_{\mathrm{Sug.}}(z) \rangle_g = &+ \frac{K}{K+h^*} \sum_{n=1}^{\infty} \mathrm{tr}\left(\frac{nq^n}{\mathrm{e}^{-\mathrm{ad}\,\gamma} - q^n} \right) \\ &+ \frac{1}{K+h^*} \sum_{n=1}^{\infty} \mathrm{tr}\left(\frac{q^n}{\mathrm{e}^{-\mathrm{ad}\,\gamma} - q^n} \mathrm{ad}\, t_a \right) \langle J_0^a \rangle_g \\ &- \frac{1}{2(K+h^*)} \langle J_0^a J_0^a \rangle_g , \end{aligned} \tag{4.2}$$

where the traces are taken in the adjoint representation of $\mathscr{G}$.

Let us introduce the following infinite product:

$$\Pi(\tau; g) = \prod_{n=1}^{\infty} \det(1 - q^n \mathrm{e}^{\mathrm{ad}\,\gamma}) . \tag{4.3}$$

The determinant is defined in the adjoint representation. $\Pi(\tau; g)$ is related to the denominator of the Weyl-Kac character formula. It is invariant under a conjugation by any element g_0 of G: $\Pi(\tau; g_0^{-1} g g_0) = \Pi(\tau; g)$. It satisfies the following properties:

$$q\partial_q \log \Pi(\tau; g) = - \sum_{n=1}^{\infty} \mathrm{tr}\left(\frac{nq^n}{\mathrm{e}^{-\mathrm{ad}\,\gamma} - q^n} \right), \tag{4.4}$$

$$\mathscr{L}_a \log \Pi(\tau; g) = - \sum_{n=1}^{\infty} \mathrm{tr}\left(\frac{q^n}{\mathrm{e}^{-\mathrm{ad}\,\gamma} - q^n} \mathrm{ad}\, t_a \right). \tag{4.5}$$

On the other hand, from the relations (2.11) and (2.12) between the zero modes and the partition function, we know that the expectation value of the Casimir of the zero modes is:

$$\langle J_0^a J_0^a \rangle_g = \frac{1}{Z(\tau; g)} \mathscr{L}_a \mathscr{L}_a Z(\tau; g). \tag{4.6}$$

Using all these properties and the expression (2.11) of the expectation value of the zero modes, we write equation (4.2) as follows:

$$\begin{aligned} q\partial_q \log[Z(\tau; g)\Pi(\tau; g)] = &+ \frac{h^*}{K+h^*} q\partial_q \log \Pi(\tau; g) \\ &- \frac{1}{K+h^*} \mathscr{L}_a \log Z(\tau; g) \mathscr{L}_a \log \Pi(\tau; g) \\ &- \frac{1}{2(K+h^*)} \frac{1}{Z(\tau; g)} \mathscr{L}_a \mathscr{L}_a Z(\tau; g). \end{aligned} \tag{4.7}$$

This is an example of the mixed Virasoro × Kac-Moody Ward identities. They yield to differential equations on the moduli space. To prove the Weyl-Kac character formula we just have to integrate this differential equation.

The infinite product $\Pi(\tau; g)$ possesses the surprising property to be a solution of the heat equation on the group manifold [20]:

$$q\partial_q \Pi(\tau; g) = -\frac{1}{2h^*} \Delta_G \Pi(\tau; g), \tag{4.8}$$

where Δ_G is the laplacian on the Lie group. We have chosen the normalization such that:

$$\Delta_G \chi_\lambda(g) = -C(\lambda)\chi_\lambda(g), \tag{4.9}$$

if $\chi_\lambda(g)$ is the character of a finite dimensional representation of $\mathscr{G}$ with highest weight λ and $C(\lambda)$ the Casimir of this representation. The property (4.8) has been proved by Fegan [20]. His proof gives a demonstration of the Macdonald's identities [28] which does not use the Weyl-Kac character formula. Thus, there is no illogicality in our demonstration of the Weyl-Kac formula. For completeness we recall Macdonald's identities:

$$\Pi(\tau; g) = \sum_{\lambda \in P_+} \varepsilon_\lambda \chi_\lambda(g) q^{C(\lambda)/2h^*}, \tag{4.10}$$

where $\varepsilon_\lambda = 0, \pm 1$ and where the sum is over all the finite dimensional representation of $\mathscr{G}$. P_+ denotes the set of the highest weights of the representations of $\mathscr{G}$. The precise definition of ε_λ is $\varepsilon_\lambda = \varepsilon(\omega)$ if $\lambda \equiv \rho - \omega(\rho)$ mod$[h^* Q^\vee]$ for some Weyl transformation ω. ρ is the Weyl vector and $Q^\vee$ the co-root lattice of $\mathscr{G}$. Using identities (4.10) it is easy to check eq. (4.8).*

Therefore, since $\Delta_G = \mathscr{L}_a \mathscr{L}_a$, we have:

$$q\partial_q[Z(\tau; g)\Pi(\tau; g)] = -\frac{1}{2(K+h^*)}\Delta_G[Z(\tau; g)\Pi(\tau; g)]. \qquad (4.11)$$

This is the striking property of the Weyl-Kac character formula: the numerator satisfies the heat equation on the group manifold. Note that the demonstration only uses the mixed Ward identities which are no more than the cyclic property of the trace (the KMS condition) together with Sugawara's construction.

Knowing the general form of the solutions of the heat equation we deduce that:

$$Z(\tau; g) = \frac{1}{\Pi(\tau; g)} \sum_{\lambda \in P_-} N_\lambda \chi_\lambda(g) q^{C(\lambda)/2(K+h^*)} \qquad (4.12)$$

for any representation of the affine algebra of level K. As above $\chi_\lambda(g)$ is the character of the finite dimensional representation of $\mathscr{G}$ whose highest weight is λ and $C(\lambda)$ is its Casimir. The sum is over all the finite dimensional representations of $\mathscr{G}$.

The coefficients N_λ are integers which depend on the affine representation. They evidently cannot be determined by the Ward identities.

Therefore, from now let us concentrate on an irreducible highest weight representation of the affine algebra $\mathscr{G}^{(1)}$. Denote by Λ its highest weight, by $Z^\Lambda(\tau; g)$ its character – or partition function – and by $N_\lambda^{(\Lambda)}$ the associated integers. We have to specify the conditions which define $Z^\Lambda(\tau; g)$ as the partition function for the affine representation Λ.

The affine representations are in one to one correspondence with the weights of the dilated affine chamber KC_{af}. Let us explain this statement. A unitary affine representation of a given level K is uniquely determined by its highest finite dimensional representation of the finite Lie algebra $\mathscr{G}$. Here "highest" representation of $\mathscr{G}$ means the representation of $\mathscr{G}$ which has the "lowest" conformal weight, i.e. the representation which has the lowest Virasoro eigenvalue. Furthermore, the unitarity of the representation requires that the highest weight of this highest representation be inside the affine dilated chamber KC_{af} if the level of the representation is K [17]. If we denote by Φ the highest root of $\mathscr{G}$, the affine dilated chamber is $KC_{af} = \{\lambda \in P_+ / \langle\lambda, \Phi\rangle \leqslant K\}$. Since the correspondence is one to one, we must have:

$$N_\lambda^{(\Lambda)} = \delta_{\Lambda;\lambda} \qquad \text{if } \lambda \in KC_{af}. \qquad (4.13)$$

* Fegan's result, eq. (4.8), can also be proved by using the mixed Virasoro × Kac-Moody Ward identities, eq. (4.7). This new proof of eq. (4.8) is presented in the note added in proof.

We will now prove that this initial condition together with the action of the affine Weyl group on the partition function completely determine all the coefficients $N_\lambda^{(\Lambda)}$. We first quote the result and then describe the proof. For a representation Λ of level K, the integers $N_\lambda^{(\Lambda)}$ are:

$$N_\lambda^{(\Lambda)} = \begin{cases} \varepsilon(\omega) & \text{if } \lambda + \rho \equiv \omega(\Lambda + \rho) \bmod[(K + h^*)Q^\vee] \\ 0 & \text{otherwise,} \end{cases} \tag{4.14}$$

where ω is any element of the Weyl group W of the finite Lie algebra $\mathscr{G}$. $\varepsilon(\omega) = \pm 1$ is the determinant of the Weyl transformation ω.

The demonstration uses more standard algebraic methods and is a little more technical. Before describing the action of the affine Weyl group $\mathrm{W_{af}}$ we write the partition function in a different way. First note that the partition function $Z^\Lambda(\tau; g)$ is $\mathrm{Tr}(gq^{L_0})$ without the factor $\exp(-i2\pi\tau C_G/24)$. This is due to our choice of coordinate z instead of $\exp(i2\pi z)$. To restore this factor we define:

$$Z^\Lambda_{\text{torus}}(\tau; g; t) = \mathrm{e}^{i2\pi K t} q^{-C_G/24} Z^\Lambda(\tau; g) = \mathrm{Tr}\left(g\, \mathrm{e}^{i2\pi K t} q^{L_0 - C_G/24}\right), \tag{4.15}$$

where we have introduced the central charge K for later convenience. We now choose γ in the Cartan subalgebra: $g = \exp(\gamma) = \exp(i2\pi\nu \cdot h)$, and we remember ourself the Weyl formula for the characters of the finite dimensional representations of $\mathscr{G}$:

$$\chi_\lambda(\exp(i2\pi\nu \cdot h)) = \frac{H(\lambda + \rho | \nu)}{H(\rho | \nu)} \tag{4.16}$$

with

$$H(x|\nu) = \sum_{\omega \in \mathrm{W}} \varepsilon(\omega) \exp(i2\pi\langle\omega(x), \nu\rangle). \tag{4.17}$$

Then, using the value of the Casimir, $C(\lambda) = |\lambda + \rho|^2 - |\rho|^2$, the strange formula of Freudenthal-de Vries, $|\rho|^2/2h^* = \dim G/24$, and the formula (4.16), we re-express the partition function $Z^\Lambda_{\text{torus}}(\tau; \nu; t)$ as:

$$Z^\Lambda_{\text{torus}}(\tau; \nu; t) = \mathscr{D}^{-1}(\tau; \nu; t) \mathrm{e}^{i2\pi(K+h^*)t} \sum_{\lambda \in P_+} N_\lambda^{(\Lambda)} H(\lambda + \rho | \nu) q^{|\lambda + \rho|^2/2(K+h^*)}, \tag{4.18}$$

where $\mathscr{D}(\tau; \nu; t)$ is the Weyl-Kac denominator:

$$\mathscr{D}(\tau; \nu; t) = \mathrm{e}^{i2\pi h^* t} q^{|\rho|^2/2h^*} H(\rho|\nu) \Pi(\tau; \nu). \tag{4.19}$$

The affine Weyl group acts on the weights of the affine algebra. By duality, it acts on the triplet $(\tau; \nu; t)$. The affine Weyl group is the semi-direct product $\mathrm{W_{af}} = \mathrm{W} \times \mathrm{T}$

where W is the finite Weyl group of $\mathcal{G}$ and T a group isomorphic to the co-root lattice $Q^\vee$ [17]. The action of W_{af} on the triplet $(\tau; \nu; t)$ is the following:

$$\omega(\tau;\nu;t) = (\tau;\omega(\nu);t) \qquad \text{if } \omega \in \mathrm{W},$$

$$T_\alpha(\tau;\nu;t) = \left(\tau;\nu+\tau\alpha;t+\langle\nu,\alpha\rangle+\tfrac{1}{2}\tau\langle\alpha,\alpha\rangle\right) \qquad \text{if } \alpha\in Q^\vee. \tag{4.20}$$

Since the multiplicities of two affine weights which are conjugated by W_{af} are equal, the partition function $Z^{\Lambda}_{\text{torus}}(\tau; \nu; t)$ is invariant under the transformations (4.20). By using the second form of the Macdonald's identities:

$$\mathcal{D}(\tau;\nu;t) = e^{i2\pi h^* t}\sum_{\alpha\in h^* Q^\vee} H(\alpha|\nu)\,q^{|\alpha|^2/2h^*} \tag{4.21}$$

or directly from the product expression of the denominator $\mathcal{D}(\tau; \nu; t)$, eq. (4.19), it is easy to check that $\mathcal{D}(\tau; \nu; t)$ is anti-invariant under the action of W_{af}. Therefore the numerator $\mathcal{N}^{\Lambda}(\tau; \nu; t)$,

$$\mathcal{N}^{\Lambda}(\tau;\nu;t) = e^{i2\pi(K+h^*)t}\sum_{\lambda\in P_+} N^{(\Lambda)}_{\lambda} H(\lambda+\rho|\nu)\,q^{|\lambda+\rho|^2/2(K+h^*)} \tag{4.22}$$

is also anti-invariant. Namely,

$$\omega_{af}[\mathcal{N}^{\Lambda}](\tau;\nu;t) = \varepsilon(\omega_{af})\mathcal{N}^{\Lambda}(\tau;\nu;t) \tag{4.23}$$

for any elements ω_{af} of the affine Weyl group.

To analyse the anti-invariance of $\mathcal{N}^{\Lambda}(\tau; \nu; t)$, we write it as a sum over all the weight lattice P. Indeed since the Weyl orbit of the positive weights covers the weight lattice, $P = \mathrm{W}[P_+]$, we have:

$$\mathcal{N}^{\Lambda}(\tau;\nu;t) = e^{i2\pi(K+h^*)t}\sum_{y\in P} M^{(\Lambda)}_{y} e^{i2\pi\langle y,\nu\rangle} q^{|y|^2/2(K+h^*)} \tag{4.24}$$

with

$$M^{(\Lambda)}_{y} = \varepsilon(\omega)N^{(\Lambda)}_{\lambda} \qquad \text{if } y\equiv\omega(\lambda+\rho)\ \mathrm{mod}[(K+h^*)Q^\vee]$$

and $M^{(\Lambda)}_{y} = 0$ otherwise.

The anti-invariance of the numerator, eq. (4.23), implies that:

$$M^{(\Lambda)}_{x} = \varepsilon(\omega)M^{(\Lambda)}_{y} \qquad \text{if } x\equiv\omega(y)\ \mathrm{mod}[(K+h^*)Q^\vee]. \tag{4.25}$$

In particular, eq. (4.25) implies that $M^{(\Lambda)}_{y}$ vanishes if y lies on the boundary of the affine dilated chamber $(K+h^*)C_{af}$. Indeed, if $y\in\partial(K+h^*)C_{af}$, either there exists a simple root α_i such that $\langle\alpha_i, y\rangle = 0$, or $\langle\Phi, y\rangle = K+h^*$. In the first case, y is invariant under the Weyl reflection ω_i: $\omega_i(y) = y - \langle\alpha_i, y\rangle\alpha_i^\vee = y$, whereas in the latter case, y is invariant under the Weyl reflection ω_Φ modulo $(K+h^*)Q^\vee$, $\omega_\Phi(y) = y - \langle y, \Phi\rangle\Phi = y - (K+h^*)\Phi$. In both cases equation (4.25) implies that $M^{(\Lambda)}_{y}$ vanishes.

The vanishing of $M_y^{(\Lambda)}$ on $\partial(K+h^*)C_{af}$ and the boundary conditions (4.13) for $N_\lambda^{(\Lambda)}$ induce the following initial conditions for $M_y^{(\Lambda)}$:

$$M_y^{(\Lambda)} = \delta_{y;\,\Lambda+\rho} \qquad \text{if } y \in (K+h^*)C_{af} \tag{4.26}$$

On the other hand, the quotient $P/(K+h^*)Q^\vee$ is isomorphic to the Weyl orbit of the dilated affine chamber $(K+h^*)C_{af}$, $P/(K+h^*)Q^\vee = \mathrm{W}[(K+h^*)C_{af}]$. Therefore, the initial conditions (4.26) together with the relation (4.25) completely determine the coefficients $M_y^{(\Lambda)}$:

$$M_y^{(\Lambda)} = \begin{cases} \varepsilon(\omega) & \text{if } y = \omega(\Lambda+\rho) \bmod[(K+h^*)Q^\vee] \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

This proves the Weyl-Kac character formula. The definition (4.27) of the coefficients $M_y^{(\Lambda)}$ is equivalent to the expression (4.14) of the coefficients $N_\lambda^{(\Lambda)}$.

To characterize the Weyl-Kac formula via the coefficients $M_y^{(\Lambda)}$ corresponds to writing it in a more familiar form; namely in terms of theta functions:

$$Z^{\Lambda}_{\text{torus}}(\tau;\nu;t) = \mathscr{D}^{-1}(\tau;\nu;t)\,\mathrm{e}^{i2\pi(K+h^*)t} \sum_{\omega\in \mathrm{W}} \varepsilon(\omega)\,\Theta^{K+h^*}_{\omega(\Lambda+\rho)}(\tau;\nu), \tag{4.28}$$

where the Θ's are the theta functions with characteristics defined over the co-root lattice $Q^\vee$.

5. Conclusion

In summary, we have been able to write the Ward identities for the Wess-Zumino-Witten models defined on the torus in such a way that they become complete. In this formulation, the Ward identities entirely determine the correlation functions with any insertions of currents in terms of the correlation functions between the primary fields of the affine algebras. This construction demands the introduction of "character valued expectation values"; i.e. correlation functions with an insertion of an element of the Lie group. The Ward identities now involve derivatives with respect to all the moduli parameters: the moduli parameters of the Riemann surface but also those associated to the Lie group. The completeness of the Ward identities requires the evaluation of the correlation functions for all values of these moduli parameters.

In this article, we have restricted ourself to the description of the Ward identities without going into the details of the WZW models on Riemann surfaces. This will be developed in a separate publication [27]. But, we have made explicit the relation between the Ward identities on the torus and the Kubo-Martin-Schwinger condition. The methods described above seem to be closely related to those used in the dual models in the seventies [29].

Using the Ward identities in the form described above, we have presented a new proof of the Weyl-Kac character formula. Let us recall how the proof works. First, it turns out that the mixed Virasoro $\times$ Kac-Moody Ward identities imply that the numerator of the Weyl-Kac formula is a solution of the heat equation on the group manifold. This property completely determines the structure of the Weyl-Kac character formula. During the proof, beside the Ward identities, we only have used the known fact that the denominator of the Weyl-Kac formula is a solution of the heat equation (but see the note added in proof below). In a second step, we show that the invariance of the partition function under the action of the affine Weyl group completely determines the coefficients left undetermined by the Ward identities. This part of the demonstration is technical but quite standard. It would be interesting to have a different demonstration which does not use the affine Weyl group but rather the unitarity of the conformal field theory.

This "physical" proof of the Weyl-Kac character formula shows once more the predominant role played by Sugawara's construction in the theory of the affine Kac-Moody algebra.

There still remains a challenge to prove the (non-chiral) Weyl-Kac character formula by path-integrating the Wess-Zumino-Witten action.

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Note added in proof

For integrating the Ward identity (4.7), and thus for proving the Weyl-Kac character formula, we have used Fegan's result:

$$q\partial_q \Pi(\tau;g) = -\frac{1}{2h^*}\Delta_G \Pi(\tau;g)$$

In this little note, we point out that this result can be proved using the Virasoro × Kac-Moody Ward identities for *non-unitary representation* of the affine albegra $\mathscr{G}^{(1)}$.

After some algebra, eq. (4.7) can be written as follows:

$$\left[q\partial_q + \frac{1}{2(K+h^*)}\Delta_G\right] Z(\tau;g)\Pi(\tau;g) = \frac{h^*}{K+h^*} Z(\tau;g)\left[q\partial_q + \frac{1}{2h^*}\Delta_G\right]\Pi(\tau;g).$$

For proving this relation we only have used the commutation relations of the affine currents together with the cyclic property of the trace. Therefore, this relation is valid whether the representation is unitary or not.

Now consider a non-unitary representation whose highest weight is a scalar for the finite Lie algebra $\mathscr{G}$ but whose central charge K is not an integer. In this case, there is no null vector in the Verma module except those associated to the Lie algebra $\mathscr{G}$. Therefore, the character of this non-unitary representation is:

$$Z^{\text{non unit.}}(\tau;g) = \frac{1}{\Pi(\tau;g)}$$

Insertion of this relation in the Ward identity (4.7) proves Fegan's result.

Thus, with this little improvement, the proof of the Weyl-Kac character formula is now only based on the Virasoro × Kac-Moody Ward identities of the WZW models. Therefore, it gives a completely new proof of the MacDonald's identities.

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