Stochastic Schramm-Loewner Evolution (SLE) from Statistical Conformal Field Theory (CFT): An Introduction for (and by) Amateurs

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Abstract

The lectures will be devoted to a somewhat detailed presentation of Stochastic Schramm-Loewner Evolutions (SLE), which are Markov processes describing fractal curves or interfaces in two-dimensional critical systems. A substantial part of the lectures will cover the connection between statistical mechanics and processes which, in the present context, leads to a connection between SLE and conformal field theory (CFT). These lectures aim at filling part of the gap between the mathematical and physics approaches. They are intended to be at an introductory level.

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Monday, March 19: Statistical interfaces and SLE

The lecture will be in three parts:

Monday Statistical interfaces and SLE

Wednesday Relation between SLE and CFT

Friday SLE/CFT Delicatessen

The basic idea is to relate SLE with 2-dimensional conformal field theory. In this relation, SLE martingales, which are similar to physically conserved quantities, will be related to observables in CFT.

Today, what I'm going to describe first are curves, and then the relation between statistical mechanics and curves, and then the definition of SLE.

1 What objects?

Our objects of study are curves in connected and simply connected planar domains. These are geometric objects, and also interfaces.

1.1 Loop erased random walks

To begin, you start with a lattice, with some lattice spacing, filling your planar domain $\mathbb{D}$. Then you take two points, x_0 and x_∞, which I will draw on the boundary but they can be in the interior. Then you take a random walk that connects x_0 and x_∞, and in two dimensions these will go to every point infinitely many times. And "loop-erased" means that you erase all loops. So a *loop erased random walk* is a simple path γ in $\mathbb{D}$ from x_0 to x_∞, and you want to give these a probability measure. We do this by setting a weight $w_\gamma = \sum_{\mathrm{rw}\downarrow\gamma} w_{\mathrm{rw}}$, where $w_{\mathrm{rw}} = x^{\|\mathrm{rw}\|}$; here $\|\mathrm{rw}\|$ is the number of steps in an arbitrary random walk, and $\mathrm{rw} \downarrow \gamma$ means that the random walk becomes γ upon erasing loops.

Since the lattice is finite, there are finitely many simple curves. But we are interested in the limit as the lattice spacing $a \to 0$. This limit is critical at $x = x_c = \frac{1}{4}$. At the critical value, the limit is almost surely a simple curve with fractal dimension $\frac{5}{4}$. The limiting probability measure is called $\mathrm{SLE}_{\kappa=2}$.

Harold: You said there's a critical point at $x = \frac{1}{4}$. What happens away from this value? **Denis:** Away from this, the correlations decay exponentially with the length scale. We want something conformally invariant, and in particular that the correlations are independent of the length scale.

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1.2 Percolation

In this case, you draw a triangular lattice. Now at each vertex I will assign a color with some probability, either yellow with probability p or blue with probability $1-p$. The aim will be to look at the clusters of, say, yellow sites, and the properties of the boundary of such clusters.

To be sure that there is a boundary, we assign boundary conditions appropriately. We work on the upper half plane, and ask that lattice points at $(-n-\frac{1}{2},0)$ are assigned yellow and at $(n+\frac{1}{2},0)$ are assigned blue, for $n \in \mathbb{N}$. Then we start at $(0,0)$, and we see a path disconnecting yellow from blue. The question now is to study properties of this interface as the lattice spacing $a \to 0$. We choose p carefully so that the limit is conformally invariant. Then we see a curve which is not simple: it has infinitely many double points. But it never crosses itself.

In particular, we can look at the curve by an exploration process: we can discover it step-by-step by walking along. Because of a locality property of percolation — you assign the color of each site independently of the colors of some other site. So you start along the curve, and suppose you've decided it so far. Then you have to decide whether to turn left or right. This depends just on the color of the vertex you're about to run into, and so L or R turns happen in probability p or $1-p$. To be critical, $p=\frac{1}{2}$ for the triangular lattice.

This is typical of the SLE process.

I should mention, the curve you get has fractal dimension $\frac{7}{4}$.

1.3 Self-avoiding walk

You take a simple domain, and look at self-avoiding walks, which are random walks where you demand that they do not self-intersect.

2 Statistical mechanics and curves

2.1 Ising model

Again, you take a domain $\mathbb{D}$, and fill it will a lattice. Then there is a degree of freedom at each vertex of the lattice Λ, called a *spin*. For the Ising model, if $i \in \Lambda$, then $\sigma_i = \pm 1$. So the configuration c is a value of σ_i for each i. So there are finitely many configurations $c \in C$. And to each configuration we assign a weight $w_c = e^{-\beta E_c}$, where $\beta E_c = \sum_{i\sim j} J\sigma_i\sigma_j$, where "$i \sim j$" means that i and j are lattice points and neighbors. Then we have the probability $P_c = \frac{1}{Z} w_c$ where $Z = \sum_{c\in C} w_c$. Actually, everything also depends on the domain $\mathbb{D}$.

So, there are some sites $+$ and some sites $-$, and we'd like to understand the curve that is the interface. To control the interface, and in particular to control where it starts and ends, we choose boundary conditions. So we choose two points on the boundary x_0 and x_∞, and put $+$ on all the boundary from x_0 to x_∞ (going counterclockwise) and $-$ on the ones from x_∞ to x_0. In this way again we get a measure on curves in $\mathbb{D}$ connecting x_0 with x_∞. We get a probability measure $P_{\mathbb{D},x_0,x_\infty}[\gamma] = \frac{1}{Z_{\mathbb{D}}} \sum_{c\in C[\gamma]} w_c$, where $C[\gamma]$ are the configurations giving γ, i.e. the sites immediately to the left of γ are $-$ and immediately to the right are $+$.

Question from the audience: Is it important to choose some special β? **Denis:** The model makes sense for any β, but in order to say anything about the measure, we want it to be conformally invariant. **Question from the audience:** So this means that at β_c all the correlations are scale-invariant? **Denis:** At β_c, $\langle \sigma_x \sigma_y \rangle \sim \frac{1}{|x-y|^{\#}}$. At others, it's like $e^{-|x-y|/\xi}$. **Question from the audience:** On both sides of the critical point? **Denis:** Yes, if you look at the connected expectation.

2.2 Induced measure

What we hope is to give a definition of this measure in the limit as the lattice spacing goes to 0. This is very hard to do directly. What SLE does it directly gives a measure on the space of curves, that you can then compare.

And you see that the construction is very general: not just the Ising model, but others like the Potts models or the $O(n)$ models.

2.3 Conditioning → Domain Markov property

I start with a domain $\mathbb{D}$, and two marked points x_∞ and x_0. Then I consider curves γ_{x_0,x_∞}. Depends on four data: $\mathbb{D}, x_0, x_\infty$, and also on a curve $\gamma_{x_0,x}$ going to some point x in the bulk of $\mathbb{D}$. Then with these data we can consider probabilities $P_{\mathbb{D}\smallsetminus\gamma_{x_0,x},x_0,x_\infty}[\,\cdot\,]$. Now, I can think of this as a conditional probability, but I can also think of it as replacing $\mathbb{D}$ with a domain in which I have cut along the curve. If you think about the Ising model, then the new cut domain with its boundary conditions is of the same type as the old one. The *domain Markov property* is that:

$$P_{\mathbb{D};x_0,x_\infty}[\,\cdot\,|\gamma_{x_0,x}] = P_{\mathbb{D}\smallsetminus\gamma_{x_0,x};x,x_\infty}[\,\cdot\,]$$

Let's prove that we have this. Call $\hat\gamma$ the variable curve from x to x_∞. Then we have:

$$P_{\mathbb{D};x_0,x_\infty}[\gamma_{x_0x}\hat\gamma|\gamma_{x_0x}] = \frac{P_{\mathbb{D}}[\gamma_{x_0x}\hat\gamma]}{P_{\mathbb{D}}[\gamma_{xx_0}]}$$

and $P_{\mathbb{D}}[\gamma_{xx_0}] = \frac{1}{Z_{\mathbb{D}}} Z_{\mathbb{D}}[\gamma_{xx_0}]$, and $Z_{\mathbb{D}}[\gamma_{xx_0}] = \sum_{c\in C[\gamma_{xx_0}]} w_c^{\mathbb{D}}$. We sum over all the spins which are away from γ, and we see that $Z_{\mathbb{D};x_0,x_\infty}[\gamma_{x_0x}] = Z_{\mathbb{D}\smallsetminus\gamma_{x_0x};x,x_\infty}$, with one difference: there is a factor $e^{-\beta E_{\gamma_{x_0x}}}$, which is the energy along the curve, but it factors out.

The same argument will be true for the other partition function, and you see that it applies in any local statistical model. **Question from the audience:** Even for next-nearest-neighbor models? **Denis:** You trust the universality of the scaling limit.

We see that the factors $e^{-\beta E_{\gamma_{x_0 x}}}$ cancel out, and

$$\frac{P_{\mathbb{D}}[\gamma_{x_0 x}\hat{\gamma}]}{P_{\mathbb{D}}[\gamma_{x x_0}]} = \frac{Z_{\mathbb{D}}[\gamma_{x_0 x}\hat{\gamma}]}{Z_{\mathbb{D}}[\gamma_{x x_0}]} = \frac{Z_{\mathbb{D}\smallsetminus\gamma_{x_0 x}}[\hat{\gamma}]}{Z_{\mathbb{D}\smallsetminus\gamma_{x_0 x}}} = P_{\mathbb{D}\smallsetminus\gamma_{x_0 x};x,x_\infty}[\hat{\gamma}]$$

3 SLE

Our goal, coming from physics, is to perhaps use lattice models to understand the continuum theory. But the lattice partition function becomes divergent in the continuum limit. So we directly look for a continuous model, and we will lose contact with statistical mechanics, to regain it when we connect to CFT.

We consider γ a random curve in $\mathbb{D}$, connecting x_0 to x_∞. And we consider some open set U. Then we ask for a measure $P_{\mathbb{D};x_0,x_\infty}[\gamma \subset U]$, and we ask for some properties:

1. conformal transport
2. domain Markov property

Then with these two properties, what Schramm understood is that there is a one-parameter family of measures, which are the SLE measures with parameter $\kappa \in \mathbb{R}_+$.

3.1 Basics of conformal transformations

A conformal transformation is a transformation from a planar domain to another planar domain, which locally preserves angles. This means that locally they are holomorphic. So at least locally at z, you have $w = f(z)$, and if you take two points very close, $z_1 = z_0 + \delta z$, this is mapped to w_0 and $w_1 = w_0 + \delta w$, and $\delta w = f'(z_0)\delta z$. This is a dilation and a rotation — it is a complex number $f'(z_0) = |f'(z_0)|e^{i\theta_0}$. These numbers of course can vary from point to point.

The standard example: let $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ upper half plane to $\mathbb{U} = \{w \in \mathbb{C} | |w| < 1\}$. Then we can set $w = f(z) = \frac{z-i}{z+i}$. Here $i \mapsto 0$, $0 \mapsto -1$, and $\infty \mapsto 1$.

The *Riemann mapping theorem* says that any domain in $\mathbb{C}$ which is connected and simply-connected is conformally equivalent to $\mathbb{H}$. This map is not unique — there are conformal transformations of $\mathbb{H}$ to itself, and these form a group which is $\mathrm{SL}(2,\mathbb{R})$, by $\phi(z) = \frac{az+b}{cz+d}$, with real $ad - bc = 1$. So to fix f, you have to fix three parameters.

This is everything I will use about these functions.

3.2 Conformal transport

Now we can state what we mean by *conformal transport.*

We take two planar domains $\mathbb{D}$ and Δ, each with two marked points $x_0, x_\infty \in \partial\mathbb{D}$ and $y_0, y_\infty \in \partial\Delta$. Then we have probabilities $P_{\mathbb{D};x_0,x_\infty}$ and $P_{\Delta;y_0,y_\infty}$. Then these are conformally equivalent, and so we fix two parameters that $f(x_0) = y_0$ and $f(x_\infty) = y_\infty$, and we ask that f relate them. More precisely, we choose such an f, and for each U, we can ask that

$$P_{\mathbb{D};x_0,x_\infty}[\gamma \subset U] = P_{\Delta;y_0,y_\infty}[\hat{\gamma} \subset f(U)]$$

This is tautological, because we're just saying how we transport the measure, in a way similar that if we have a vector field on a manifold, then we know how vector fields transform. But if we knew how to take the scaling limit, then conformal invariance would mean something.

Question from the audience: And the third condition to fix f? **Denis:** It says that the probability is invariant under a one-parameter group. If you work on $\mathbb{H}$ with $x_0 = 0$ and $x_\infty = \infty$, then this group is by dilation.

So since it is a tautological statement. What makes it useful is also to impose the domain Markov property.

3.3 Domain Markov property

We have $\mathbb{D}$, and x_0, x_∞, and x and $\gamma_{x_0 x}$. And we also have $\mathbb{D} \smallsetminus \gamma_{x x_0}$. So we impose that:

$$P_{\mathbb{D};x_0,x_\infty}[\,\cdot\,|\gamma_{x_0 x}] = P_{\mathbb{D}\smallsetminus\gamma_{x_0 x};x,x_\infty}[\,\cdot\,]$$

The cut domain is still connected and simply connected. So we choose a conformal map $f_{x x_0} : \mathbb{D} \smallsetminus \gamma_{x_0 x} \to \mathbb{D}$, taking x to x_0 and x_∞ to x_∞. So our axiom is that:

$$P_{\mathbb{D};x_0,x_\infty}[\,\cdot\,|\gamma_{x_0 x}] = f_{x_0 x} \circ P_{\mathbb{D};x_0,x_\infty}$$

This axiom is a *growth process.* You imaging that you have grown the curve from x_0 to x, and then you want to know what happens in just a little more. By the above axiom, this is the same as just the very beginning step of the curve. It follows that: *to define the measure, it's enough to know the statistics of just the germ of curves.*

This is what Schramm did, and we lose contact (temporarily) with all statistical mechanics.

3.4 Towards the definition of SLE

Now that we understand that it is enough to know the statistics of just the germ of the curve, we need good parameterization in order to understand the formula.

This we do via the Loewner equation. We work on the upper half plane, since this is the same as any other domain by a transformal map. We imagine that we start at 0 and go to ∞: we are studying $P_{\mathbb{H};0,\infty}$.

Now imagine that we have a parameterized curve $\gamma_{[0,\infty)} : t \mapsto \gamma_t$. But we are really interested in the curve as a geometrical object. Anyway, so we have $\gamma_{[0,t]}$, and we choose $g_t : \mathbb{H} \smallsetminus \gamma_{[0,t]} \to \mathbb{H}$. If we fix three conditions, then we have fixed g_t. They are:

$$g_t(z) = z + 0 + \frac{2t}{z} + \frac{\#}{z^2} + \dots$$

This is the Taylor expansion of g near $z = \infty$.

So the first condition says that $\infty \mapsto \infty$, and the second that $g'(\infty) = 1$. Along with the condition that the next coefficient is 0, we define the map uniquely. Here t, as coefficient of $1/z$, is a choice of affine parameter along the curve.

3.5 How g_t varies when the curve groups (Loewner equation)

To know how g_t evolves, it is enough to know how to uniformize a tiny bit of a curve. So we have in $\mathbb{H}$ a tiny curve from ξ_t to $\xi_t + i\delta_t$. The uniformization of this tiny curve is implemented by:

$$g_{t,\delta_t}(z) = \sqrt{(z-\xi_t)^2 + \delta_t^2} + \xi_t$$

with the cut in the square root along the tiny curve.

If δ_t is small, then we can Taylor expand, and we see that:

$$g_{t,\delta_t}(z) \approx z + \frac{\frac{1}{2}\delta_t^2}{z - \xi_t} + \dots$$

This implies that the function g_t transforms by picking up a pole:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t} \tag{L}$$

Let's say this again. We have a map g_t, and $g_{t+\delta_t}$. Then to uniformize just the little bit, we want to use $g_{t+\delta t} \circ g_t^{-1}$, and we just computed $(g_{t+\delta t} \circ g_t^{-1}(z) - \mathrm{id})(g_t(z))$. The factor of 2 is linked to our choice of t-parameterization.

Question from the audience: But then you are demanding that the curve grows perpendicularly to the boundary? **Denis:** No, I did that just for clarity. The better answer is to ask: how do you add on a little to a subdomain? What SLE supposes is that you add matter just at a delta function, but you can of course smooth this out.

Ok, so we said how g_t evolves, depending on this number $\xi_t = g_t(\gamma_t)$. The trick now is to reverse the logic, to understand the curve from the data $t \mapsto \xi_t$.

Remember, given the curve, the function was unique. And conversely. What you can do is integrate (L) with initial data $g_{t=0}(z) = z$, given $t \mapsto \xi_t$, to reconstruct $g_t(z)$ and hence the curve.

3.6 Definition of SLE

To give a measure on curves, we give a measure on maps, which satisfy (L) with initial data. The measure will be: $\xi_t = \sqrt{\kappa} B_t$, with B_t the standard Brownian motion $\mathbb{E}[\xi_t \xi_s] = \kappa \min(t,s)$.

Then $\gamma_t = g^{-1}(\xi_t + i\epsilon)_{\epsilon \to 0^+}$. It is a theorem of Rohde and Schramm that $t \mapsto \gamma_t$ is a curve, for any κ, but with different phases.

If $\kappa = 0$, then ξ_t is not random. If you integrate (L), then you get a square root.

The theorem is that there are three phases. When $\kappa \leq 4$, then you get a simple curve. When $4 < \kappa < 8$, then γ is a curve with infinite number of double points. When $\kappa \geq 8$, then γ is space-filling.

For example, LERW has $\kappa = 2$, Ising $\pm$ is $\kappa = 3$, and SAW should be $\kappa = 8/3$. The percolation model is $\kappa = 6$, and the Ising FK model is $\kappa = 16/3$. I know of no statistical model that gives $\kappa > 8$ or $\kappa < 2$.

3.7 Domain Markov property (again)

Now I will prove that to have domain Markov property, it fixes the statistics of ξ_t to be a Brownian motion.

Look. I have a curve γ_t, and I have the map g_t which is sends the tip to ξ_t, with the normalization at ∞ as above. But this is not quite the same normalization as when I defined the Domain Markov property. So I must translate, and use $f_t(z) = g_t(z) - \xi_t$.

Now the curve continues to evolve to γ_{t+s}, conditional on γ_t. This translates to a small bit of curve $\tilde{\gamma}_{t,s}$ starting at 0. Then there is the full flattening f_{t+s} of γ_{t+s}. But by additivity, we see that $f_{t+s} \circ f_t^{-1}$, which is the flattening of the short curve, must be (by the domain markov property):

$$f_{t+s} \circ f_t^{-1} \cong f_s$$

by which we mean that they should have the same law.

Then you expand at $z = \infty$, and what you learn is that $\xi_{t+s} - \xi_t$ has to have the same law as ξ_s. Because $f_t(z) = z - \xi_t + \frac{2t}{z} + O(1/z^2)$. Now impose continuity, and this imposes that $\xi_t = \sqrt{\kappa} B_t + \alpha t$. And finally, you need invariance by dilation, and this kills that αt factor.

3.8 Some remarks

The first remark is that it has been proved that the dimension of the curve is $d_\kappa = 1 + \frac{\kappa}{8}$ when $\kappa < 8$. And, of course, that it is a curve.

There's been work by Schramm, Lawler, Werner, Rohde. There are also many conjectures by physicists. And I probably forgot many names.

Physicists usually, when they study critical system, are used to using field theory. Field theory is usually defined by actions, integration, quantization. They are determined by local data. So this is why it is difficult for physicists to get information about nonlocal objects. What we learn from SLE is that there is a way to learn from field theory about nonlocal objects.

So, I talked about SLE with data which are two points on the boundary. What I talked about was *chordal SLE*. You can also talk about *radial SLE*, which goes into the bulk. In radial SLE, you fix three numbers, one on the boundary and two in the bulk. There is also *dipolar SLE*, where you fix three points x_0, x_+, and x_-, and just ask that the curve start at x_0 and go somewhere between the other two. You can do this in Ising by putting some area of free boundary. **Question from the audience:** How do you do radial SLE in the Ising model? **Denis:** You can't for $\pm$ model. But for $O(n)$ model you can.

And there are many more, with more data, i.e. more marked points. But then you have only three conditions on conformal map, and so when you do the SLE argument, you get a flow in the moduli space of marked points. That flow is specified by the CFT partition function.

So what I'm going to describe, which is a relation between CFT and SLE, gives you a way to make contact between these. And it will give a hint how to define the more general process. Next time I will give these connections, there are many ways to approach it, and since one way to define CFT is just through algebra, I will do it through algebra and group theory. I will give a group-theoretical description of Loewner equation.

Wednesday, March 21: SLE/CFT correspondence

Today I will talk about the correspondence between SLE and CFT. I will first formulate the relation between statistical mechanics and Martingales for some processes. Then I will tell you what is the statement about the correspondence between CFT and SLE. Then I will describe the objects, and see how we can prove this correspondence. This will be a more algebraic approach.

4 Statistical martingales

4.1 Discrete

So I start again with the model on the lattice that we are to consider. If I am to talk about martingales, then I must talk about stochastic processes. In this case, the process will be the growth of the curve of interfaces.

Again, we have a domain $\mathbb{D}$ which has the topology of a disk, and has a lattice in it. And we have two marked points x_∞ and x_0 on the boundary, and we can look at the interface of fixed length γ_T. So if C is the configuration space, we have $C = \bigcup_{\gamma_T} C[\gamma_T]$. This partition is finer and finer as we increase the lengths.

Since everything is finite, we can from this define a filtration $\mathcal{F}_T \subseteq \mathcal{F}_S$ for $S > T$. Here $\mathcal{F}_T$ and $\mathcal{F}_S$ are sigma algebras. The probability measure P is the one coming from the Boltzman weight that I discussed last time.

That's our process. Now you have to define first what are the observables, and their expectation values, and I want to show that if you look at the expectation values of the observables, then that is a martingale.

What is an observable? We have some observable $\mathcal{O}$ — maybe you measure the values of spins at various locations — and for $c \in C$ its value $\mathcal{O}_c$ (that is an *observable* is a function on C), and we define the *expectation value* to be

$$\langle \mathcal{O} \rangle_{\mathbb{D}}^{\text{stat}} = \frac{1}{Z_{\mathbb{D}}} \sum_{c \in C} w_c \mathcal{O}_c$$

Now we can consider the conditional expectation value:

$$\langle \mathcal{O} \rangle_{\mathbb{D}}^{\text{stat}}|_{\gamma_T} = \frac{1}{Z_{\mathbb{D}}[\gamma_T]} \sum_{c \in C[\gamma_T]} w_c \mathcal{O}_c$$

Then the first is really just an expectation value $\mathbb{E}[\mathcal{O}]$, and the second is the conditional expectation $\mathbb{E}[\mathcal{O}|\gamma_T] = \mathbb{E}[\mathcal{O}|\mathcal{F}_T]$.

Then I explained last time that $Z_{\mathbb{D}}[\gamma_T] = e^{-E_{\gamma_T}} Z_{\mathbb{D} \smallsetminus \gamma_T}$. Here E_{γ_T} is the energy of γ_T. This was because if $c \in C[\gamma_T]$, then $w_c^{\mathbb{D}} = e^{-E_{\gamma_T}} w_c^{\mathbb{D} \smallsetminus \gamma_T}$. So from this relation, we see that

$$\langle \mathcal{O} \rangle_{\mathbb{D}}^{\text{stat}}|_{\gamma_T} = \langle \mathcal{O} \rangle_{\mathbb{D} \smallsetminus \gamma_T}^{\text{stat}}$$

Prop: This is a $\mathcal{F}_T$-martingale.

Proof: More or less, a martingale is something that is conserved in mean. So we want to prove that

$$\mathbb{E}[\langle \mathcal{O} \rangle_{\mathbb{D} \smallsetminus \gamma_T}^{\text{stat}}] = \langle \mathcal{O} \rangle_{\mathbb{D}}^{\text{stat}}$$

If we have a martingale, then it definitely has this property, so this is a hint that it is a martingale.

The LHS is an expectation with respect to the measure on the interface γ_T. And we know that $P[\gamma_T] = Z_{\mathbb{D}}[\gamma_T]/Z_{\mathbb{D}}$. So the LHS is

$$\sum_{\gamma_T} P[\gamma_T] \langle \mathcal{O} \rangle_{\mathbb{D} \smallsetminus \gamma_T}^{\text{stat}} = \sum_{\gamma_T} \frac{Z_{\mathbb{D}}[\gamma_T]}{Z_{\mathbb{D}}} \times \frac{1}{Z_{\mathbb{D}}[\gamma_T]} \sum_{c \in C[\gamma_T} w_c \mathcal{O}_c = \frac{1}{Z_{\mathbb{D}}} \sum_{c \in C} w_c \mathcal{O}_c = \langle \mathcal{O} \rangle_{\mathbb{D}}^{\text{stat}}$$

The double sum on γ_T and $c \in C[\gamma_T]$ reproduces the sum on $c \in C$ because $C = \bigcup_{\gamma_T} C[\gamma_T]$. What have I done? I've rearranged a statistical sum. That's what we learned from SLE. When you have a sum of many things, you can rearrange.

So I have proved that it is conserved in mean. That's not enough to prove that it is a martingale, but there is more, you need to have

$$\mathbb{E}[\langle \mathcal{O} \rangle^{\text{stat}}_{\mathbb{D} \smallsetminus \gamma_T} | \gamma_S] = \langle \mathcal{O} \rangle^{\text{stat}}_{\mathbb{D} \smallsetminus \gamma_S}$$

for $S < T$. What we proved above was the statement for $S = 0$. But you can prove this more generally. You can do it either directly, using a sum as above, or you can use that this is a conditional expectation and $\mathbb{E}\big[\mathbb{E}[\mathcal{O}|\mathcal{F}_T]\big|\mathcal{F}_S\big] = \mathbb{E}[\mathcal{O}|\mathcal{F}_S]$.

The definition of a *martingale* is that you have a filtration $\mathcal{F}_S$ and a map $S \mapsto M_S$ which is $\mathcal{F}_S$-measurable, satisfying $\mathbb{E}[M_T|\mathcal{F}_S]_{S<T} = M_S$. And you see that in our case (of statistical mechanics) this is tautological, more or less, like the Markov property. It is true for any model, even away from criticality, and even more general things, like the renormalization group can be formulated in this way.

4.2 Continuum

In the continuum, we don't have anymore the lattice, but we have a curve $\gamma_{[0,t]}$ that we are parameterizing in some way. What we expect is to give a definition of $\langle \mathcal{O} \rangle^{\text{stat}}_{\mathbb{D} \smallsetminus \gamma_t}$ in the continuum. We have the measure on curves, which I call $\mathbb{E}$, and if both are coming from a lattice model then we want that $t \mapsto \langle \mathcal{O} \rangle^{\text{stat}}_{\mathbb{D} \smallsetminus \gamma_t}$ is a martingale for the $\gamma_{[0,t]}$-process.

This gives information on the relation between $\mathbb{E}$ and the continuum limit of a statistical model, which will be a QFT. Because when you write $\mathbb{E}[\langle \mathcal{O} \rangle_{\mathbb{D} \smallsetminus \gamma_t}] = \langle \mathcal{O} \rangle_{\mathbb{D}}$, then the LHS uses the measure on curves, and the RHS is information for the QFT.

Question from the audience: So you are not assuming that this angle bracket is a scaling limit? **Denis:** I am. Or rather, if you knew how to define $\mathbb{E}$ directly from the lattice, then you would have it, but you can impose the martingale property, when you try to define the continuum theory through some other method. **Question from the audience:** Through SLE? **Denis:** Yes. But you see, there's the parameter κ for $\mathbb{E}$, and the parameters $T, c, \dots$ for the QFT, and you want to find which parameters satisfy that identity.

5 CFT/SLE

5.1 Statement

I will use terms from field theory, and then try to explain them.

I have the domain $\mathbb{D}$, and in the continuum I know that $\mathbb{E} = \mathrm{SLE}_\kappa$. Then the rule is that the QFT expectation value is a ratio of two partition functions — the individual partition functions don't have continuum limit, but their ratio can.

Then since we care about the boundary conditions, we have:

$$\langle \mathcal{O} \rangle_{\mathbb{D}}^{\text{stat}} \longrightarrow \frac{\langle \Psi(x_\infty)\, \mathcal{O}\, \Psi(x_0)\rangle_{\mathbb{D}}^{\text{cft}}}{\langle \Psi(x_\infty)\, \Psi(x_0)\rangle_{\mathbb{D}}^{\text{cft}}}$$

Actually, this type of relation is still true also away from criticality, but then you don't have cft, but some other qft. Why is it useful to study at criticality? Because in a cft, going from $\mathbb{D} \smallsetminus \gamma_t$ to $\mathbb{D}$ is implemented by conformal transformation, and in cft you know how it transforms for such maps. **Question from the audience:** Do you know what to do away from criticality? **Denis:** Yes, in the free theory; no in general. Away from criticality, SLE is not a good approach. In SLE, you grow the curve step-by-step, and you can still do this in QFT, but at the conformal point then you can forget about the past. But if you have a mass term, say, or some density, then when you do the conformal map, if the mass was originally homogeneous, then it is no longer after the conformal map. **Question from the audience:** But if you can control this? **Denis:** Yes, but it is impossible. In Ising model with magnetic field, it may start homogeneous, but afterwards, it has a Jacobian, and it is dilated in a nonuniform way. So to do this away from criticality, you have to control the qft with nonuniform perturbation, and no one knows how to do that, except in the free theory.

At criticality, going from $\mathbb{D} \smallsetminus \gamma_t$ to $\mathbb{D}$ is purely kinematical.

How does it go? You have $\langle \Psi(x_\infty)\, \mathcal{O}\, \Psi(x_0)\rangle_{\mathbb{D}\smallsetminus\gamma_t}^{\text{cft}}$, and you use a conformal map g_t to send $x_\infty \mapsto x_\infty$ and the tip γ_t of the curve to some $\xi_t = g_t(\gamma_t)$. Now how does this transform?

$$\langle \Psi(x_\infty)\, \mathcal{O}\, \Psi(\gamma_t)\rangle_{\mathbb{D}\smallsetminus\gamma_t}^{\text{cft}} = |g_t'(x_\infty)|^h\, |g_t'(\gamma_t)|^h\, \langle \Psi(x_\infty)\, {}^{g_t}\mathcal{O}\, \Psi(\xi_t)\rangle_{\mathbb{D}}^{\text{cft}} \qquad \text{(CFT property)}$$

where h is the scaling dimension of the operator Ψ. But in the ratio, this part will cancel out.

Question from the audience: And there is a Jacobian from $\mathcal{O}$? **Denis:** Yes, it is hidden in ${}^{g_t}\mathcal{O}$. If $\mathcal{O} = \prod_k \Phi_k(z_k, \bar{z}_k)$, with Φ_k of weight Δ_k, then ${}^{g_t}\mathcal{O} = \prod_k |g_t'(z_k)|^{\Delta_k} \Phi_k(g_t(z_k))$.

So it is now easy to insert, and we have

$$\langle \mathcal{O} \rangle_{\mathbb{D}\smallsetminus\gamma_t}^{\text{stat}} \longrightarrow \frac{\langle \Psi(x_\infty)\, {}^{g_t}\mathcal{O}\, \Psi(\xi_t)\rangle_{\mathbb{D}}^{\text{cft}}}{\langle \Psi(x_\infty)\, \Psi(\xi_t)\rangle_{\mathbb{D}}^{\text{cft}}} =: Z_{\mathbb{D}}\big(z_k^t, \dots; \xi_t\big)$$

And we want to make this a martingale.

Claim: $Z_{\mathbb{D}}\big(z_k^t, \dots; \xi_t\big)$ are SLE_κ martingales if the CFT has central charge $c = (6-\kappa)(3\kappa-8)/2\kappa$ and $h = (6-\kappa)/2\kappa$. And the field Ψ has to have a null vector at level 2, but I'll discuss that later.

So now we know how to match things up. For example, when $c = 1$, then $\kappa = 4$, and this is the free Gaussian field theory. You consider a domain $\mathbb{D}$ with a free boson with action $S = \frac{1}{2}\int (\partial\phi)^2$, and then you impose boundary conditions that on one side of the boundary you have $\phi = \lambda_*$, and on the other $\phi = 0$. And then for any configuration you have a discontinuity, and this curve is SLE(4).

For other values of c, there are two values of κ for the same c. For example, for $c = \frac{1}{2}$, then you have $\kappa = 3$ corresponding to the Ising $\pm$ model, and you have $\kappa = 16/3$ which is FK (high temp). For $c = 0$, you have $\kappa = 6$ percolation, and $\kappa = 8/3$ which is SAW. Etc. **Question from the audience:** c can also be negative? **Denis:** Yes. **Question from the audience:** But c does not uniquely fix the theory — you need the whole operator content.

Denis: What you see is that $c < 1$. But when you want to study SLE, you see that you need to work with many more operators than usual. You can't just work with local operators, because in SLE we look at non-local objects.

5.2 Hint for the proof

We have the Loewner equation $\mathrm{d}g_t = \frac{2}{g_t - \xi_t}\mathrm{d}t$, and we have $Z_{\mathbb{D}}(z_k^t, \dots; \xi_t)$. We have:

$$\mathrm{d}Z_{\mathbb{D}}(g_t(z_\bullet), \xi_t) = \mathrm{d}t \underbrace{\left(\frac{\kappa}{2}\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2}\right)}_{\text{diff op of 2nd order}} Z_{\mathbb{H}} + \mathrm{d}\xi_t \left(\mathcal{L}_{-1} Z_{\mathbb{H}}\right)$$

In order to be a martingale, it should be conserved, and we have to ask that the factor of $\mathrm{d}t$ is 0 for $Z_{\mathbb{H}}$. And this has an interpretation in terms of representation theory. What we have to do is look for a representation of the Virasoro algebra such that $\left(\frac{\kappa}{2}\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2}\right)|\mathrm{hw}\rangle = 0$. This is what gives the above claim.

This is called the *null vector equation*. I'm sure you've seen the representation theory of SL(2). In particular, the spin-$\frac{1}{2}$-representation, which is 2-dimensional, how do you construct it? It is defined as the representation such that if you act on the highest-weight vector twice with the lowering operator J^-, you get 0. Put another way, you take some Verma module, and then you demand that when you act on the highest weight vector twice, you get the highest weight vector of a sub-module. What we will do is analogous.

6 Details

This is for Loewner equation in $\mathbb{H}$, and chordal.

6.1 Basics of Ito Calculus

We start with Brownian motion B_t. This is some very irregular curve: it is almost surely continuous and nowhere differentiable. We have Gaussian measure with $\mathbb{E}[B_t B_s] = \min(t, s)$, and $\mathbb{E}[B_t] = 0$.

This has iid increments, where conditioned on B_s we have that $B_{t+s} - B_s$ is distributed like B_t. And this is independent of $B_{s'}$ for $s' \leq s$.

Why is this? Because we have a random walk, and if you post time at some moment, then resume your random walk, your choices are independent of your history.

Ito calculus is an attempt to define a continuum limit with these properties. Specifically, it is the goal to define $\int_0^T f(B_t)\,\mathrm{d}B_t$. You have to be careful. If you look at the $\mathbb{E}[(B_t - B_s)^2] = |t-s|$, and so $(\mathrm{d}B_t)^2 \sim \mathrm{d}t$. This is why almost surely B_t is nowhere differentiable.

So how to define the integral? I will do it by example. Let's consider $f(B) = B$. Then we want $\int_0^T f(B_t)\,\mathrm{d}B_t$, and we do it by a Riemann sum. So we divide the interval $0 = t_0 < t_1 < \cdots < t_N = T$. Then we would have:

$$\int_0^T f(B_t)\,\mathrm{d}B_t = \sum_i f(B_{t_i})\,(B_{t_{i+1}} - B_{t_i})$$

This is the *Ito convention*: you do the sampling of f before the increment, so as to have independence.

In our example, this is:

$$= -\frac{1}{2}\sum_i (B_{t_{i+1}} - B_{t_i})^2 + \frac{1}{2}\sum_i (B_{t_{i+1}}^2 - B_{t_i}^2)$$

using $B_{t_i} = \frac{1}{2}(B_{t_{i+1}} + B_{t_i}) - \frac{1}{2}(B_{t_{i+1}} - B_{t_i})$.

Well, the second sum is telescopic, and so is $\frac{1}{2}(B_T^2 - B_0^2)$. And the first is a sum of random variables that are about $|t_{i+1} - t_i|$, and by the law of large numbers in the limit the sum is a non-random variable with value T. So what we find is:

$$\int_0^T B\,\mathrm{d}B = -\frac{1}{2}T + \frac{1}{2}\left(B_T^2 - B_0^2\right)$$

We proved the following for $F(B) = B^2/2$, but in general the fact of Ito calculus is:

$$F(B_T) - F(B_0) = \int_0^T F'(B_t)\,\mathrm{d}B_t + \frac{1}{2}\int_0^T F''(B_t)\,\mathrm{d}t$$

In infinitesimal form, this is

$$\mathrm{d}F(B_t) = F'(B_t)\,\mathrm{d}B_t + \frac{1}{2}F''(B_t)\,\mathrm{d}t$$

which is Taylor expansion with $(\mathrm{d}B_t)^2 = \mathrm{d}t$. You must use the Ito convention to understand the first term. Then

$$\mathrm{d}\mathbb{E}[F(B_t)] = 0 + \frac{1}{2}\mathbb{E}[F''(B_t)]\,\mathrm{d}t.$$

6.2 Example: Loewner equation

Recall we have $\partial_t g_t = \frac{2}{g_t - \xi_t}$, with $g_{t=0}(z) = z$. Then it is convenient to use $f_t = g_t - \xi_t$, with $\xi_t = \sqrt{\kappa}\, B_t$, and so $\mathrm{d}f_t(z) = \mathrm{d}g_t(z) - \mathrm{d}\xi_t = \frac{2\mathrm{d}t}{f_t} - \mathrm{d}\xi_t$. Now we consider $F(f_t)$. When, expanding Taylor to second order as we must above, we have:

$$\mathrm{d}F(f_t) = \mathrm{d}t\, \frac{2}{f}\, F'(f) - \mathrm{d}\xi_t\, F'(t) + \frac{\kappa}{2}\, F''(f)\, \mathrm{d}t = \left(\frac{2}{f}\, F'(f) + \frac{\kappa}{2}\, F''(f)\right) \mathrm{d}t - F'(f)\, \mathrm{d}\xi_t$$

Let $\ell_n = -z^{n+1}\partial_z$. Then we have:

$$\mathrm{d}F(f_t) = \left(-2\ell_{-2} + \frac{\kappa}{2}\ell_{-1}^2\right) F(f_t)\, \mathrm{d}t + (\ell_{-1}F)(f)\, \mathrm{d}\xi_t$$

So you can see that the hinted proof above is simply an application of the Ito calculus.

Then from what I said, in order to be a martingale you have to kill the term in front of $\mathrm{d}t$. Because the other term is zero in mean, so you just have to kill the drift.

So now there are two approaches. You either take the equation (and its multi-point generalization), and just try to solve it, and classify the solutions. Or you can recognize that the equations you need to solve already show up in field theory, and are related to representation theory.

6.3 Group tautology

In order to make contact with Virasoro algebra, I need to make contact with algebra.

The functions f, g are all of the same kind. They are conformal transformations, and in particular Laurent series. Specifically, we consider N_- to be the set

$$N_- = \{f(z) = z + \sum_{n\geq 0} a_n z^{-n}\} \qquad \text{(finite radius of convergence)}$$

So N_- is a group under composition. Given $f \in N$, we denote the group element by Γ_f, so that $\Gamma_f \Gamma_g = \Gamma_{f\circ g}$. Of course N_- acts on itself.

Note, for SLE $f_t \in N_-$, with $a_0 = -\xi_t$ and $a_1 = \int \xi \mathrm{d}\xi_t$ or so, and so on.

For SLE, we set Γ_t to be the group element corresponding to f_t. Then the Loewner equation reads:

$$\Gamma_t^{-1}\, \mathrm{d}\Gamma_t = \left(-2\ell_{-2} + \frac{\kappa}{2}\ell_{-1}^2\right) \mathrm{d}t - \ell_{-1}\, \mathrm{d}\xi_t$$

Now, usually when you write a differential equation for a group, the RHS should be in the Lie algebra. But here, because of Ito calculus there is on the RHS a term which is not.

In any case, the goal now is to promote this equation in CFT data, so that $\left(-2\ell_{-2} + \frac{\kappa}{2}\ell_{-1}^2\right)$ becomes some operator acting on the CFT data, and we look for some object which kills this operator.

6.4 Basics of CFT

There are three data in a cft. There is the Virasoso algebra, its representations, and fields. Oh, and a 1000-page book by Di Francesco et al. But the main data are the first three.

We have the differential $\ell_n = -z^{n+1}\partial_z$, and they form an algebra $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$. The *Virasoro algebra* is the unique central extension of this algebra. Namely, it is:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\,n(n^2-1)\,\delta_{n+m,0}$$

Here c is the *central charge*: it is central, and so commutes with *everything*. Why do you get a central charge? Because in quantum mechanics, you may have projective representations.

Now we want a qft. The Hilbert space for the qft will be a sum of representations of Virasoro, all with the same central charge, so that the value of c is a characteristic of the qft. But we want specific representations, namely the *highest weight vector representations*. These are $\mathcal{H}$, with $|\omega\rangle \in \mathcal{H}$ unique such that $L_n|\omega\rangle = 0$ for all $n > 0$. And it is an eigenvector, so that the representation is characterized by the eigenvalue h with $L_0|\omega\rangle = h|\omega\rangle$. Since $L_0 \approx z\partial_z$, we see that h is the scaling dimension.

Question from the audience: So one qft is one representation? **Denis:** A qft involves a sum of representations. There is a rule how these representations interact, but I will not discuss it.

Now I give an example of how to construct such representations. One class are the *Verma modules*. What you do, you set $\mathfrak{n}_- = \{L_n, n \leq -1\}$. Then the Verma module is $V_{h,c} = \mathcal{U}(\mathfrak{n}_-)|\omega\rangle$. So you have a graded representation. In grading 0, you have $|\omega\rangle$. Then $L_{-1}|\omega\rangle$. In grading -2, you have two basis vectors $L_{-1}^2|\omega\rangle$ and $L_{-2}|\omega\rangle$. And so on.

But this is nor irreducible. In general, you can let N be any submodule of $V_{h,c}$, and consider $M = V_{h,c}/N$. This is still highest-weight. These quotients are the class of representations that we will usually use in qft. Asking that $N \to 0$ — these are the *null vectors*.

6.5 CFT/SLE

Now we are almost done, which is good because it is almost 2 o'clock. We consider $\overline{\mathcal{U}(\mathfrak{n}_-)}$, and for each $f_t \in N_-$ we assign to $\hat{\Gamma}_t \in \overline{\mathcal{U}(\mathfrak{n}_-)}$. By construction, this object satisfies the same relations as the earlier one:

$$\hat{\Gamma}_t^{-1}\,\mathrm{d}\hat{\Gamma}_t = \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)\mathrm{d}t + L_{-1}\,\mathrm{d}\xi_t$$

Then we look for highest-weight representations with $|\mathrm{hw}\rangle = |\omega\rangle$ such that $\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)|\omega\rangle = 0$. If this is OK, then $\hat{\Gamma}_t|\omega\rangle$ is a generating function of martingales.

Why is it a martingale? Because you apply the above, and then the drift part is killed. Why is it a generating function? Because it is something that is valued in the representation, so each coordinate is a martingale.

So this is the null vector equation.

And now the question is: In $V_{h,c}$, when is $|\Theta\rangle = \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)|\omega\rangle$ a highest-weight vector? There are two equations to check. We need $L_n|\Theta\rangle = 0$ for all $n > 0$, but it is enough to check for L_1 and L_2, because L_3 is in the commutator of L_1 and L_2, and so on. So there are two equations to check. You check $L_1|\Theta\rangle = 0$, and this gives one equation for c, h, κ. You check $L_2|\Theta\rangle = 0$, and this gives a second equation. Then you solve, and you get h_κ and c_κ.

So you have found what is the vector, and you know that in the representation which is the quotient by the submodule, the vector is zero, and so that's the generating function.

That's an algebraic way of doing it. What I have done algebraically is I use L_n as a formal way to represent the differential operators from the beginning.

6.6 Relation with correlation functions

In sum, that's the correspondence between SLE and CFT. But now we have more information. In particular, a remark. We defined Virasoro, and gave the representation. And the *fields* are the operators or intertwiners acting on Virasoro modules.

So now consider

$$\langle\omega| \prod_k \Phi_k(z_k)\,\hat{\Gamma}_t\,|\omega\rangle.$$

This is just a number, but by the construction, if I commute $\hat{\Gamma}_t$ through the operator, it implements the conformal transformation, so it is

$$= \langle\omega| \prod_k |f_t'(z_k)|^{\Delta_k}\Phi_k(f_t(z_k))\,|\omega\rangle,$$

which are the CFT correlations in $\mathbb{H} \smallsetminus \gamma_t$; i.e. they are what we called $Z_{\mathbb{H}}(z_k^t, \dots; \xi_t)$.

Next time will be less formal, and we will give applications. For example, there's a recent proof using SLE of the Mandelbrot conjecture, that if you take a random walk in two dimensions, well it comes back to itself many times, and it fills in some region, and if you take the boundary of the region, then you get a curve with dimension $\frac{4}{3}$.

Friday, March 23: SLE/CFT delicatessen

The lecture today will be lighter than last time. I'm going to talk about applications of the link between SLE and CFT. First I will say a few words on applications. Then I will talk about how to generalize SLE, using input from CFT. The goal is to try to show that the CFT partition function encodes dynamics of the SLE. Finally, I will talk about something completely different, which is the restriction measure in SLE. This won't use much CFT, but will work towards the proof of the Mandelbrot conjecture I said last time.

7 Application of CFT/SLE

Last time we said that ratios of CFT partition functions are SLE martingales. In particular, if we think about chordal SLE, then we had a process of the form

$$t \mapsto \frac{\langle \Psi(x_\infty) \cdots \Psi(\gamma_t) \rangle_{\mathbb{H} \smallsetminus \gamma_t}}{\langle \Psi(x_\infty) \Psi(\gamma_t) \rangle_{\mathbb{H} \smallsetminus \gamma_t}}$$

and it is a martingale for any operator "$\cdots$". Here the Ψ are of dimension $h_\kappa = \frac{6-\kappa}{2\kappa}$ and the central charge for the CFT is $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$.

Why do you want a martingale? You want to know about the probability that the SLE curve does this or that. These probabilities can often be formulated in terms of stopping time. And they are going to be given in terms of a ratio of correlation functions, with the operator "$\cdots$" depending on the question you want to ask. If physicists were sufficiently clever, then they could have calculated directly all of these probabilities directly from statistical mechanics: if comes just from the Boltzman weight definition that all probability are ratios of partition functions, and this fact survives going to the continuum. So a posteriori we could have solved (or guessed the answers to) all of these questions without going to SLE. But History is different.

Question from the audience: Is it going to be local the operator? **Denis:** It depends on the question that you ask. If you ask "does the curve visit within ϵ of some point?" then that's local. But normally the operators are very complicated.

So we started with statistical mechanics, and then wrote down SLE, and now we're back to statistical mechanics, which is good, because we closed the loop.

And one thing you can do is ask: how to learn about CFT statistical questions from SLE? But there are more degrees of freedom in CFT, and so you have to invent generalization of SLE to "CLE".

8 Generalization of SLE

8.1 More marked points

We are in $\mathbb{H}$, and we have the curve starting at 0 and going to ∞. But we also add some more marked points, perhaps in the bulk of $\mathbb{H}$, that maybe impose some spin conditions. Since we still have the curve, then we can still parameterize γ_t via the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}.$$

This means that we have normalized at ∞, and the "2" fixes the time parameterization.

Then we can ask still about the measure/statistics on $t \mapsto \xi_t$. We should still impose conformal invariance.

Question from the audience: But now you do not have domain markov property? Because before, we said that if we have domain markov property, then $t \mapsto \xi_t$ must be Brownian motion. **Denis:** We still have domain markov property. I'm imagining just the Ising model, but with insertion of operators. Before, I argued that the law for the evolution of the curve must be the same after unfolding the cut. But now I have the function g_t depending on extra parameters w_i for the locations of the marked points. And these points must move with g_t.

Question from the audience: What does it mean to include marked points, and not forget them? **Denis:** You get to ask questions of the curve. This is the same as imposing conditions on the curve, and from that point of view it is to change the statistics.

So the short way to do this is to impose that the correlation functions $\langle\rangle^{\text{stat}}$ are martingales.

Prop: Let $Z^0_{\mathbb{H}}(w^t_k, \xi_t)$ be a chordal SLE$_\kappa$ martingale. (So we pick one, and it's the one associated to the marked points w_k.) And take another one $Z_{\mathbb{H}}(w^t_k, {}^t\mathcal{O}, \xi_t)$. Then the ratio

$$\frac{Z_{\mathbb{H}}(w^t_k; {}^t\mathcal{O}, \xi_t)}{Z^0_{\mathbb{H}}(w^t_k; \xi_t)}$$

is a martingale for SLE process defined by

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad w^t_k = g_t(w_k), \quad \mathrm{d}\xi_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial\xi_t \log Z^0_{\mathbb{H}}(w^t_k, \xi_t)\,\mathrm{d}t.$$

So ratio of martingales is a martingale for the theory with a drift term. Why? I said that everything is ratio of partition functions. If we have marked points, then statistical observable

$$\langle \mathcal{O} \rangle^{\text{stat}}_{\mathbb{D}} = \frac{\langle \psi \prod_k \Phi_k(w_k)\,\mathcal{O}\,\psi\rangle_{\mathbb{D}}}{\langle \psi \prod_k \Phi_k(w_k)\,\psi\rangle_{\mathbb{D}}}$$

and this becomes the ratio of $Z_{\mathbb{H}}$ above. We want this ratio to be a martingale. You know that each is itself a martingale for original SLE. And you impose that the ratio is a martingale for some new SLE, and it's the SLE with drift term.

Proof: Option 1: Explicit computation (with Ito calculus). Option 2: Girsanov's theorem. Once you have a martingale, you can use it to change the measure and this adds drift terms to the stochastic equations.

In summary: if you condition SLE, you get a new SLE with a drift term depending on the conditioning.

Question from the audience: But then you should be able to get radial SLE, but that's not Brownian? **Denis:** You do get radial motion, by inserting an operator that forces the curve to go through some point in the bulk. But notice that always I use the same normalization at ∞, and from that perspective it is still locally Brownian with a drift term.

8.2 Example: chordal SLE from 0 to a

This you can just get from our other theory by conformal transformation, but I impose the normalization at ∞ that $g(\infty) = \infty$, $g'(\infty) = 1$, and the second Taylor coefficient is 0.

So how do you do this? You insert an operator to create the curve at $x_0 = 0$, and an operator at a. So you have:

$$Z^0_{\mathbb{H}}(a, x_0) = \langle \Psi(a)\Psi(x_0)\rangle_{\mathbb{H}} = \frac{1}{(a - x_0)^{2h_\kappa}}$$

by scaling dimensions. Let's call this $\mathrm{SLE}_\kappa(0 \to a, \mathbb{H})$. Then using $2h_\kappa = \frac{6-\kappa}{\kappa}$, we can write down the theory: it is

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad \mathrm{d}\xi_t = \sqrt{\kappa}\,\mathrm{d}B_t - (6-\kappa)\,\frac{\mathrm{d}t}{a_t - \xi_t}, \quad \mathrm{d}a_t = \frac{2}{a_t - \xi_t}\mathrm{d}t$$

This is chordal SLE, but with a normalization that is not at the final boundary point.

Remark: when $\kappa = 6$, we have $\mathrm{d}\xi_t = \sqrt{\kappa}\,\mathrm{d}B_t$, and thus we have locality of $\mathrm{SLE}(\kappa = 6)$. This is related to the locality property of percolation. By "locality" we mean "independent of the target point." And that's the only SLE which has this property.

8.3 Example: 3 marked points on $\partial\mathbb{D}$, giving $\mathrm{SLE}(\kappa, \rho)$, or *dipolar SLE*

We are still in $\mathbb{H}$. We have three marked points $0, x_+, x_-$, and we put in three observables: Ψ at 0 with dimension h_κ, and $\Phi_\pm$ at $x_\pm$ with scaling dimensions $\delta_\pm$. I don't put anything to absorb the curve — it can go anywhere.

So we have $Z^0_{\mathbb{H}}(x_0, x_-, x_+) = \langle \Phi_-(x_-)\,\Phi_+(x_+)\,\Psi(x_0)\rangle_{\mathbb{H}}$. By Möbius invariance, this is completely determined; it is

$$\frac{\text{constant}}{(x_0 - x_+)^{h+\delta_+-\delta_-}\,(x_0 - x_-)^{h+\delta_--\delta_+}\,(x_+ - x_-)^{\delta_++\delta_--h}}$$

But now we know something from CFT, which is that $Z^0_{\mathbb{H}}$ satisfies a second-order differential equation. So this imposes one condition on $(\delta_+, \delta_-, h_\kappa)$. This imposes the "fusion rules" from CFT. I mean, the constant vanishes if the equation is not satisfied. So given κ, there is one parameter, usually called ρ.

Then I get $\mathrm{SLE}(\kappa, \rho)$, and it still satisfies $\partial_t g_t(z) = \frac{2}{(g_t - \xi_t)}$. And now the point moves, as always: $\mathrm{d}x^t_\pm = \frac{2}{x^t_\pm - \xi_t}\mathrm{d}t$. And the rest of the dynamics is the Brownian motion plus drift for ξ_t:

$$\mathrm{d}\xi_t = \sqrt{\kappa}\,\mathrm{d}B_t - \left(\frac{\rho}{\xi_t - x^t_+} - \frac{\rho + \kappa - 6}{\xi_t - x^t_-}\right)\mathrm{d}t$$

The *dipolar* case is when these two numbers are the same: $\rho = \frac{\kappa-6}{2}$. And more generally, the curve tends to end up between $x_\pm$, and in the dipolar case then it tends symmetrically between x_+, x_-, and for other ρs the curve will tend towards x_+ or perhaps towards x_-.

Question from the audience: How do you see that the partition function must be exactly as you said? **Denis:** It is just the Möbius invariance. I will explain it for the 2-point function in the earlier example.

We have the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$, with $ad - bc = 1$. Then this is an action of $\mathrm{SL}(2,\mathbb{R})$. The Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ is spanned by L_0 and $L_{\pm 1}$ with commutators $[L_0, L_{\pm 1}] = \pm L_{\pm 1}$ and $[L_+, L_-] = 2L_0$. So this includes into the Virasoro algebra. A representation is $\ell_n = -z^n\bigl[(n+1)\delta + z\partial_z\bigr]$. When $\delta = 0$, this is acting on functions, and when $\delta = -1$ it is action on vectors, and when $\delta = 1$ it is acting on 1-forms, and and when δ is arbitrary it is action on "δ-forms."

Then we have $Z_{\mathbb{H}}(z_1, \dots, z_N) = \langle \Phi_{\delta_1}(z_1) \cdots \Phi_{\delta_N}(z_N) \rangle$. For the $\mathrm{SL}(2,\mathbb{R})$ invariance, we impose that

$$\sum_{j=1}^{N} \ell_k^{(j)} Z_{\mathbb{H}}(z_1, \dots, z_N) = 0 \text{ for } k = 0, \pm 1$$

For the 2- and 3-point functions, this completely determines the function.

Question from the audience: Does it matter whether the points are in the bulk or in the boundary? **Denis:** It is important that they are on the boundary. If the points are in the bulk, then you have two operators, for z and $\bar{z}$. If you have two points in the bulk, then you have four operators, and so it is not fixed. But for example, in radial SLE, you have one point on the boundary and one in the bulk, so it's still preserved.

But I think it's less important the explicit formulas, and more important what we've learned: when you condition on other operators, then you keep the same SLE structure, just with a drift depending on the conditioning.

Question from the audience: So how do you continue if you want to study these more complicated theories? **Denis:** We have N variables, and 3 equations, which isn't enough and so you stop. If one of the operators is Ψ, then you also get the null-vector equations, this is another 2nd-order equation and so for four points you can still do something. In general, you have to be specific to the problem you're looking at. In certain cases, you can work with the minimal model, which has enough structure that you can proceed.

8.4 N-SLE

We pick N points on $\partial\mathbb{D}$, and try to grow N curves. For example, in Ising, perhaps we color the boundary alternatingly $+$ and $-$. The curves grow and do not cross, and so with some probabilities they meet up in various ways. We want to understand the statistics of the model.

We have N points $x_1, \dots, x_N$, and we work with $\mathbb{D} = \mathbb{H}$, so that x_j are real. You grow all the curves, and then uniformize with some fixed parameterizations at ∞. Then you get N new locations ξ_j^t. You can play with the speeds, but I will fix all of them, so that we have;

$$\partial_t g_t(z) = \sum_{j=1}^{N} \frac{2}{g_t(z) - \xi_j^t}$$

Then what happens? You have N Brownian motions, but with some interaction terms as well:

$$d\xi_j^t = \sqrt{\kappa}\, dB_j^t + \sum_{k\neq j} \frac{2dt}{\xi_k^t - \xi_j^t} + dt\, \kappa\, \partial_{\xi_j} \log Z_{\mathbb{H}}(\xi_1^t, \dots, \xi_N^t) \qquad (*)$$

The first two terms comprise the original definition, which is wrong. What you need to include is an extra drift, like we had above.

With the third term, it is a good definition. Why? First, we think in terms of statistical physics. The correlation functions should have a description from field theory, so that we have $Z_{\mathbb{H}}(x_1, \dots, x_N) = \langle \Psi(x_1) \cdots \Psi(x_N)\rangle_{\mathbb{H}}^{\text{cft}}$, where the Ψs are boundary-changing operators, with dimension $h_\kappa = \frac{6\kappa}{2-\kappa}$.

Second, with $(*)$, all expectation values $\langle \mathcal{O}\rangle_{\mathbb{H}}^{\text{stat}}$ given by ratios of CFT correlation functions are martingales. This request completely fixes the drift term.

Finally, let me address what freedom we have in the choice of $Z_{\mathbb{H}} = \langle \Psi(x_1) \cdots \Psi(x_N)\rangle_{\mathbb{H}}$? Last time we learned that each time we insert Ψ, we get a 2nd-order differential equation. So $Z_{\mathbb{H}}$ satisfies N differential equations, and there is a finite number of solutions and you can take linear combinations of them.

We didn't prove everything, but you can find some $Z_{\mathbb{H}}^{(\alpha)}$ of them (how many depending on the Catalan numbers) which have certain positivity conditions to be partition functions, where α is indexing over a discrete set. Then we are interested in $Z_{\mathbb{H}} = \sum_\alpha p_\alpha\, Z_{\mathbb{H}}^{(\alpha)}$. When $N = 4$, there are two of them: $Z_{\mathbb{H}}^{(1)}$ and $Z_{\mathbb{H}}^{(2)}$. These correspond to the two *arch configurations*. If you choose $Z^{(1)}$, then with probability 1 the final curves connect in a certain way, and for $Z^{(2)}$ the final curves almost surely connect the other way. The conjecture (coming from cft) in general is that with N marked points, the $Z^{(\alpha)}$s are in one-to-one correspondence to arch configurations. So for the $Z_{\mathbb{H}}$ process, the probability to have arch configuration α is $\frac{p_\alpha Z_{\mathbb{H}}^{(\alpha)}}{\sum_\beta p_\beta Z_{\mathbb{H}}^{(\beta)}}$.

This lets you get explicit formulas for percolation or for Ising.

In summary, these objects $Z_{\mathbb{H}}^{(\alpha)}$ come just from statistical mechanics, and they must give the correct answer. But SLE is a way to prove it.

9 Restriction measures

Consider the domain $\mathbb{D}$, with two marked points x_0, x_∞, and some variable region K connecting them, which might be a curve but might be thicker. We care about probability $P_{\mathbb{D}}$ which is the probability for K to enter some region. For example, let's take some region A in $\mathbb{D}$ so that $\mathbb{D} \setminus A$ is still the topology of $\mathbb{D}$, and let's ask about $P_{\mathbb{D}}[K \cap A = \emptyset]$. By taking sufficiently complicated A, this should determine the statisticts of K. We again ask for two properties:

1. Conformal transport (so far a tautology)

2. Restriction: $P_{\mathbb{D}\setminus A}[\,\cdot\,] = P_{\mathbb{D}}[\,\cdot\,|K \cap A = \emptyset]$.

Theorem (LSW): There is a one-parameter family of such measures. Moreover, you can define it explicitly:

We are in $\mathbb{H}$. We have the region K going from 0 to ∞, and we have the region A. Since $\mathbb{H}\setminus A$ has the topology of A, then we have the map $\Phi_A : \mathbb{H}\setminus A \to \mathbb{H}$, and it flattens out A to some interval. The map Φ is uniquely determined by asking $\Phi_A(0) = 0$, $\Phi_A(\infty) = \infty$, and $\Phi'_A(\infty) = 1$. Then the only measures are:

$$P_{\mathbb{D}}[K \cap A = \emptyset] = [\Phi'_A(0)]^\alpha, \quad \text{for some } \alpha \geq \frac{5}{8}$$

It is easy to check that this measure effectively satisfies the restriction property. As you increase α, it becomes a thicker and thicker domain.

9.1 Example: $\alpha = \frac{5}{8}$.

This corresponds to SLE($\kappa = \frac{8}{3}$), which is cft with $c = 0$ and $h = \frac{6-\kappa}{2\kappa} = \frac{5}{8}$. Conjecture: this is the scaling limit of Self-Avoiding Walk.

If SAW is conformally invariant, then it has to be SLE with $\kappa = \frac{8}{3}$, because it does satisfy the restriction property.

A consequence: if you take SLE in $\mathbb{H}$ with a chord connecting 0 to ∞, and ask that it not enter A, then

$$P_{\mathrm{SLE}(\frac{8}{3})}[\gamma_{\mathrm{SLE}} \cap A = \emptyset] = \Phi'_A(0)^{5/8}$$

SLE(8/3) is the only SLE satisfying the restriction property.

9.2 Example: Brownian excursion

You take Brownian random walk in the upper half plane $\mathbb{H}$, and ask that the path starts at 0. Well, this doesn't exist, but you can ask to start at $i\epsilon$, and condition that you escape by $i\Lambda + \mathbb{R}$ before escaping by $\mathbb{R}$. Then you take the limits as $\epsilon \to 0$ and then as $\Lambda \to \infty$. Then this satisfies the restriction property with $\alpha = 1$. Let's prove this.

You see, this is really a one-dimensional problem, because you are just asking the probability to cross the strip from ϵ to Λ. This is a standard exercise, and occurs with probability ϵ/Λ.

Now you take the domain A, and ask the probability that the Brownian excursion B doesn't intersect A. But this is

$$\frac{P[\mathrm{B} \cap A = \emptyset \text{ and B escapes } \Lambda]}{P[\mathrm{B} \text{ escapes } \Lambda]}$$

So after you flatten, then the roof is still essentially flat, and so you are still doing a probability that you escape the roof before you escape the floor. But now you start not at $i\epsilon$, but at $i\epsilon\Phi'_A(0)$.

So you get the probability

$$= \frac{\epsilon \Phi'_A(0)/\Lambda}{\epsilon/\Lambda} = \Phi'_A(0)$$

hence $\alpha = 1$.

9.3 Mandelbrot conjecture

This is a physics proof. We have seen that 8 independent SLE(8/3) satisfies the same law as 5 independent Brownian excursions. Now the boundary of the hull of 5 independent Brownian excursions is locally given by the exterior perimeter of one of the Brownian excursions. By the above argument, this has to have the identical dimension as the boundary of the hull of 8 independent SLE(8/3). The latter is locally one of the SLE(8/3) with dimension 4/3. Thus the exterior perimeter of a Brownian excursion has dimension 4/3.

A reference (with more references therein) is: “2D grown processes: SLE and Loewner chains”, M. Bauer and D. Bernard, `arXiv:math-ph/0602048`, Phys. Rept. **432** (2006) pp. 115–221.

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