The functional renormalization group approach to disordered systems

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References: PRL 86 (2001) 1785: 2loop
PRL 89 (2002) 125702: large $N$
cond-mat/0205116: pedagogical

http://theory.itp.ucsb.edu/~wiese/
Statistical Mechanics

PURE SYSTEMS: pretty well understood

\[ \text{add disorder} \rightarrow \]\n
STRONGLY DISORDERED SYSTEMS

???

- disorder dominates over entropy
- what is the ground state?
- metastability
- very slow dynamics

Examples

- glasses: spin-glass, vortex-glass, electron-glass, structural glass
- random field magnet
- elastic systems in disorder
Current understanding of disordered systems

Still many puzzles despite 30 years of research...

- simulations
- very few exact solutions
- phenomenological models (droplet-picture)
- mean-field approximation
  unclear of whether that applies to any real physical system

Recent advances

- for elastic manifolds in random media
- advantage: approachable by other (analytical) methods, while containing all ingredients of strongly disordered systems
Physical Realizations

Domain-walls in magnets

Contact line of liquid Helium/water
Vortex-lattice/Bragg glass

Charge Density wave

Cracks, earthquakes, directed polymer (KPZ),...
Model and Observables

Displacement field

\[ x \in \mathbb{R}^d \quad \rightarrow \quad u(x) \in \mathbb{R}^N \]

coordinate displacement-field

Elastic energy

\[ \mathcal{H}_{\text{el}} = \int d^d x \frac{1}{2} [\nabla u(x)]^2 \]

Disorder energy

\[ \mathcal{H}_{\text{DO}} = \int d^d x V(u(x), x) \]

with correlations \((N = 1)\)

\[ \frac{V(u, x)V(u', x')}{V(u, x)V(u', x')} = \delta^d(x - x')R(u - u') \]

Observables

roughness \(\zeta\)

\[ [u(x) - u(x')]^2 \sim |x - x'|^{2\zeta} \]

full probability-distribution function
How to treat disorder? The replica-trick

Example: free energy averaged over disorder

\[ \overline{\mathcal{F}} = \overline{\ln \mathcal{Z}} \]

But how to calculate?

\[ \ln \mathcal{Z} = \lim_{n \to 0} \frac{1}{n} \left( e^{n \ln \mathcal{Z}} - 1 \right) = \lim_{n \to 0} \frac{1}{n} (\mathcal{Z}^n - 1) \]

\( n \) times replicated system

“Replica Hamiltonian”

\[ \mathcal{H}[u] = \frac{1}{T} \sum_{a=1}^{n} \int d^d x \frac{1}{2} [\nabla u_a(x)]^2 \]

\[ - \frac{1}{2T^2} \sum_{a,b=1}^{n} \int d^d x R(u_a(x) - u_b(x)) \]
The problem in the treatment of disorder: dimensional reduction

“Theorem”: (Efetov, Larkin 1977) A $d$-dimensional disordered system at zero temperature ($T = 0$) is equivalent to all orders in perturbation theory to a pure system in $d - 2$ dimensions at finite temperature. (“Holds” under quite general assumptions.)

Example: Elastic manifolds in disorder

The thermal 2-point function becomes

$$\langle [u(x) - u(0)]^2 \rangle \sim |x|^{2-d} \quad \rightarrow \quad [u(x) - u(0)]^2 \sim x^{4-d}$$

roughness exponent

$$\zeta = \frac{4 - d}{2}$$

Counter-example:

3d disordered Ising-model at $T = 0$ is ordered; in contrast to the 1d Ising-model without disorder at $T > 0$. 
The Larkin-length (and some difference to standard critical phenomena)

Be the disorder force $F_x$ gaussian, with correlation length $r$. Typical disorder force on segment

$$F_{DO} = \bar{f} \left( \frac{L}{r} \right)^{d/2}$$

Elastic force

$$F_{el} = c L^{d-2}$$

$F_{DO} = F_{el}$ at $L = L_c$ (Larkin-length)

$$L_c = \left( \frac{c^2}{\bar{f}^2 r^d} \right)^{\frac{1}{4-d}}$$

$d < 4$: Membrane pinned by disorder on scales larger than the Larkin-length
Functional Renormalization, WHY?

Old idea: Wegner, Houghton (1973)
For disordered systems: D. Fisher (1985)

Larkin’s argument:
\[ d = 4 \text{ is critical dimension} \]
Make an \( \epsilon = 4 - d \) expansion

Dimensional reduction says:
\[ \zeta = \frac{4 - d}{2} \]

Even though wrong for \( d < 4 \), it correctly says: field is marginal in \( d = 4 \).

NEED FOR A FUNCTIONAL RENORMALIZATION GROUP!
Functional renormalization group (FRG) (D. Fisher 1986)

\[ \mathcal{H}[u] = \int x \frac{1}{2T} \sum_{a=1}^{n} [\nabla u_a(x)]^2 - \frac{1}{2T^2} \sum_{a,b=1}^{n} R(u_a(x) - u_b(x)) \]

Functional renormalization group equation (FRG) for the disorder correlator \( R(u) \):

\[ \partial_\ell R(u) = (\epsilon - 4\zeta)R(u) + \zeta u R'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0) \]

Solution for force-force correlator \(-R''(u)\):

Cusp: \( R''''(0) = \infty \) appears after finite RG-time (at Larkin-length); dimensional reduction invalid

\[ R''_{L>L_c}(0) \neq \text{dim.red.} \]

\[ \partial_\ell R''(0) = (\epsilon - 2\zeta)R''(0) \]

\( \equiv \text{dim.red.} \)

Renormalization of whole function overcomes dimensional reduction
Why is a cusp necessary?
Consider simple model with one mode

\[ \mathcal{H}[u] = \frac{1}{2} q^2 u^2 + \sqrt{\varepsilon} V(u) \]

Physics beyond the Larkin-length \( L_c \): multiple minima

This implies that for all \( \varepsilon \) and some \( u \)

\[ \frac{d^2}{du^2} \mathcal{H}[u] = q^2 + \sqrt{\varepsilon} V''(u) < 0 \]

Thus

\[ R''''(0) = \left. \frac{V''(u)V''(u')}{u = u'} \right|_{u = u'} = \infty \]
Beyond leading order (1 loop) ???

As a consistent theory, it should

• allow for systematic corrections beyond 1 loop
• be renormalizable
• and thus make universal predictions.

A puzzle since 1986 . . .

Next order involves $R'''(0) = ?$

$$\lim_{u \to 0^+} R'''(u) = - \lim_{u \to 0^-} R'''(u)$$

Solution of the puzzle

• 2-loop statics: PRL 76 (2001) 1785
• 2-loop driven dynamics: PRL 76 (2001) 1785, cond-mat/0205108
• large $N$: cond-mat/0109204
2 loop statics

\[ \partial_\ell R(u) = (\varepsilon - 4\zeta)R(u) + \zeta uR'(u) + \frac{1}{2} R''(u)^2 - R''(u)R''(0) \]
\[ + \frac{1}{2} [R''(u) - R''(0)] R'''(u)^2 - \frac{1}{2} R'''(0^+)R''(u) \]

Result of sloop-algorithm, recursive construction, non-analytic field theory. Only result consistent with:

- renormalizability
- potentiality (forces are gradient of a potential)

Solution for the fixed point

- periodic case: \( A_d = \frac{\varepsilon}{18} + \frac{7\varepsilon^2}{108} \) (universal amplitude)
- random field \( \zeta = \frac{\varepsilon}{3} \) (exact)
- random bond \( \zeta = 0.20829804\varepsilon + 0.006858\varepsilon^2 \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>( \varepsilon )</th>
<th>( \varepsilon^2 )</th>
<th>simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.208</td>
<td>0.215 ± 0.003</td>
<td>0.22 ± 0.01</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>0.417</td>
<td>0.444 ± 0.007</td>
<td>0.41 ± 0.01</td>
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<tr>
<td>1</td>
<td>0.625</td>
<td>0.687 ± 0.02</td>
<td>2/3</td>
</tr>
</tbody>
</table>
Solution at large $N$

$\vec{u}(x) \in \mathbb{R}^N$, e.g. polymer in $N$ dimensions

Calculate free energy in presence of an external field; do Legendre-transform; obtain self-consistent equation for effective action (exact)

$$\tilde{R}'(u^2) = R'(u^2 + 2TI_1 + 4I_2(\tilde{R}'(u^2) - \tilde{R}'(0)))$$

$R(\ldots)$ bare disorder

$\tilde{R}(\ldots)$ effective (renormalized) disorder

$T =$ temperature

$I_n = \int \frac{d^dk}{(k^2 + m^2)^n}$

Functional renormalization group equation (FRG)

$$- m\frac{\partial}{\partial m} \tilde{R}(x) = (\epsilon - 4\zeta)\tilde{R}(x) + 2\zeta x\tilde{R}'(x)$$

$$+ \frac{1}{2}\tilde{R}'(x)^2 - \tilde{R}'(x)\tilde{R}'(0) + T\tilde{R}'(x) \left[ 1 + \tilde{R}''(0)/\epsilon \right]^{-1}$$

complicated non-linear partial differential equation: solved analytically; cusp under analytical control.
Replica Symmetry Breaking (RSB)

No symmetry-breaking field (Mézard, Parisi 1992).

Gaussian variational ansatz exact at $N = \infty$:

$$R \left( (u_a - u_b)^2 \right) = \sigma_{ab} u_a u_b$$

RS: $\sigma_{ab} = \sigma \forall a \neq b$: dimensional reduction

RSB:

\[
\sigma_{ab} = \begin{pmatrix}
\sigma & & & \\
& \sigma & & \\
& & \ddots & \\
& & & \sigma
\end{pmatrix}
\]

Infinit-step RSB: $\sigma_{ab} \to [\sigma](\sigma), \quad z \in [0, 1]$

$z = \text{overlap}, \quad \begin{cases} 
z = 0 & \text{distant states} \\
z = 1 & \text{nearby states} \end{cases}$

Observables are constructed out of $[\sigma](z)$

\[
\left\langle u_k u_{-k} \right\rangle = \frac{1}{k^2} \left( 1 + \int_0^1 \frac{dz}{z^2} \left[ \sigma(z) + m^2 \right] + \left[ \sigma(z) \right] + m^2 \right)
\]
RSB and FRG, the relation

\[ [\sigma](z) + m^2 \]

from UV–cutoff

IR–cutoff

\[ m^2 \]

0 \hspace{1cm} z_m \hspace{1cm} z_c \hspace{1cm} 1

• FRG gives the contribution of the RSB-states with minimal overlap
• RSB: spontaneous symmetry breaking
• FRG: explicit symmetry breaking by applied field
• green part is RG-invariant: \( m \frac{d}{dm} ([\sigma](z) + m^2) = 0 \)
• RSB-curve can be scanned by varying \( m^2 \)
• RSB-reconstruction-formula (out of FRG-objects)

\[
\langle u_a u_b \rangle = \frac{\tilde{R}'_m(0)}{m^4} + \int_m^{m_c} \frac{d\tilde{R}'_\mu(0)}{\mu^4} + \frac{1}{m_c^2} - \frac{1}{m^2}
\]

• no hierarchic matrix was ever inverted!
Driven dynamics

- domain-walls in magnets
- contact line in liquid Helium or water
- crack-propagation

For the last two systems, the effective elasticity is mediated through the bulk, leading to “long-range” elasticity:

\[ \mathcal{H}[u] = \int \frac{d^d k}{(2\pi)^d} |\tilde{u}(k)|^2 |k| \]

instead of \( k^2 \).
Observables

- **roughness**
  \[ [u(x) - u(0)]^2 \sim |x|^{2\xi} \]

- **velocity-force-characteristics, pinning**
  \[ v \sim |f - f_c|^\beta \]

- **dynamic exponent** $z$
  \[ t \sim x^z \]

- **correlation length** $\xi$
  \[ \xi \sim |f - f_c|^{-\nu} \]

- **exponent relations**
  \[ \beta = \nu(z - \xi) \quad \nu = \frac{1}{2 - \xi} \]
RG-treatment, 1loop

Nattermann et al., Narayan & Fisher, 1992

- same RG-equation as in the statics, even though physics should be different
- claim: $\zeta = \frac{\varepsilon}{3}$ is exact to all orders; in contradiction with experiments and simulations

2 loop

PRL 86 (2002) 1785, cond-mat/0205108

- Membrane only jumps ahead ($T = 0$):
  \[ t > t' \implies u(x, t) \geq u(x, t') \]
- renders perturbation theory unique

\[
\partial_\ell R(u) = (\varepsilon - 4\zeta)R(u) + \zeta uR'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0)
\]
\[
+ \frac{1}{2} \left[ R''(u) - R''(0) \right] R'''(u)^2 + \frac{1}{2} R'''(0^+)^2 R''(u)
\]

- results for RF:
  \[
  \zeta = \frac{\varepsilon}{3} \left( 1 + 0.14331\varepsilon + \ldots \right)
  \]
  \[
  z = 2 - \frac{2}{9} \varepsilon - 0.04321\varepsilon^2 + \ldots
  \]
Some comparison with numerics

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\varepsilon$</th>
<th>$\varepsilon^2$</th>
<th>estimate</th>
<th>simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.33</td>
<td>0.38</td>
<td>$0.38 \pm 0.02$</td>
<td>$0.34 \pm 0.01$</td>
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<tr>
<td>$\zeta$</td>
<td>0.67</td>
<td>0.86</td>
<td>$0.82 \pm 0.1$</td>
<td>$0.75 \pm 0.02$</td>
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<td>1.00</td>
<td>1.43</td>
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<tr>
<td>3</td>
<td>0.89</td>
<td>0.85</td>
<td>$0.84 \pm 0.01$</td>
<td>$0.84 \pm 0.02$</td>
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<tr>
<td>$\beta$</td>
<td>0.78</td>
<td>0.62</td>
<td>$0.53 \pm 0.15$</td>
<td>$0.64 \pm 0.02$</td>
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<tr>
<td>1</td>
<td>0.67</td>
<td>0.31</td>
<td>$0.2 \pm 0.2$</td>
<td>$0.25 \ldots 0.4$</td>
</tr>
<tr>
<td>3</td>
<td>0.58</td>
<td>0.61</td>
<td>$0.62 \pm 0.01$</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.67</td>
<td>0.77</td>
<td>$0.85 \pm 0.1$</td>
<td>$0.77 \pm 0.04$</td>
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<tr>
<td>1</td>
<td>0.75</td>
<td>0.98</td>
<td>$1.25 \pm 0.3$</td>
<td>$1 \pm 0.05$</td>
</tr>
</tbody>
</table>

Depinning, long-range elasticity ($d = 1$)

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$\varepsilon$</th>
<th>$\varepsilon^2$</th>
<th>estimate</th>
<th>simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>0.47</td>
<td>$0.47 \pm 0.1$</td>
<td>$0.39 \pm 0.002$</td>
<td></td>
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<tr>
<td>$\beta$</td>
<td>0.78</td>
<td>0.59</td>
<td>$0.6 \pm 0.2$</td>
<td>$0.68 \pm 0.06$</td>
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<tr>
<td>$\zeta$</td>
<td>0.78</td>
<td>0.66</td>
<td>$0.7 \pm 0.1$</td>
<td>$0.74 \pm 0.03$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>1.33</td>
<td>1.58</td>
<td>$2 \pm 0.4$</td>
<td>$1.52 \pm 0.02$</td>
</tr>
</tbody>
</table>
Perspectives

- higher order calculations: very cumbersome, but under control
- exact solution of the large-$N$ limit
- cusp analytically under control
- precise relation to RSB
- $1/N$-expansion should be possible

$$
\delta \tilde{R}(\bar{u}^2) = \frac{1}{N} \left[ + \left( \begin{array}{c}
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\end{array} \right) + \left( \begin{array}{c}
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\end{array} \right)

+ T \left( \begin{array}{c}
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\end{array} \right)
\end{array} \right)

+ T^2 \left( \begin{array}{c}
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\end{array} \right) + \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right) + \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right)
\end{array} \right) \right]

\begin{array}{c}
\begin{array}{c}
\end{array} \right) = R''(\chi_{ab}) (1 - 4I_2(p)R''(\chi_{ab}))^{-1}, \quad . = R(\chi_{ab})
\end{array} \right]

- most promising method to obtain strong-coupling behaviour of strongly disordered systems as e.g. KPZ beyond mean-field.
- random field, anisotropic depinning . . .