

# Hyperuniformity in the Manna Model, Conserved Directed Percolation and Depinning

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We use an exact mapping of the Manna model, or equivalently conserved directed percolation, onto disordered elastic manifolds at depinning to show that particle-density fluctuations in these two models are hyperuniform. The structure factor of the particle density behaves for small  $q$  as  $S(q) \sim |q|^\sigma$  with  $\sigma = 4 - d - 2\zeta$ , where  $\zeta$  is the roughness exponent at depinning. In dimension  $d = 1$ ,  $\sigma = 1/2$ , while for all dimensions  $0.6 > \sigma \geq 0$ . Our results fit well known simulations in the literature, except in  $d = 1$ , where we perform our own simulations to confirm our findings.

*Hyperuniformity.* Consider a particle system of size  $L$ , where the total number  $N_{\text{tot}}$  of particles is conserved. We ask how many particles  $N_R$  are in a part of the system of size  $R \ll L$ . If the system is translationally invariant, then

$$\langle N_R \rangle = \frac{N_{\text{tot}}}{L^D} R^d. \quad (1)$$

How does  $N_R$  fluctuate? We expect that

$$\text{var}(N_R) = \langle N_R^2 \rangle - \langle N_R \rangle^2 \sim R^\kappa. \quad (2)$$

One can show [1] that (except for fine-tuned models [2])

$$d - 1 \leq \kappa \leq d. \quad (3)$$

A Poisson process has  $\kappa = d$ , a regular lattice  $\kappa = d - 1$ . When  $\kappa < d$  the system is said to be *hyperuniform*. This terminology was introduced in [3] for  $\kappa = d - 1$ , but is now used for any  $\kappa < d$  [4]. Alternatively, one can consider the structure factor of the Fourier transform  $n_q$  of the density  $n(x)$ . Its small- $q$  behavior is

$$S(q) = \langle n_q n_{-q} \rangle \sim q^\sigma, \quad \kappa + \sigma = d. \quad (4)$$

We are interested in class-III hyperuniform systems [5], which correspond to  $0 < \sigma \leq 1$ . Larger values of  $\sigma$  are possible [5, 6],  $\kappa$  then freezes at its lower bound  $\kappa = d - 1$ .

*The Manna sandpile and conserved directed percolation.* For sandpile models, hyperuniformity was first observed in [7], and later in [4, 8–11]. It renders simulations better convergent, allowing larger sizes for the Manna or Oslo models than for the depinning of disordered elastic manifolds. Here we evaluate  $\sigma$  analytically for the Manna model (MM) and conserved directed percolation (CDP).

There was a long debate whether the Manna model, or the corresponding CDP theory, are in the same universality class as disordered elastic manifolds or whether they instead belong to a different universality class, the directed-percolation (DP) class. This question was finally settled by showing that the Manna model is equivalent to the CDP equations [12], and that an exact mapping exists between CDP and disordered elastic manifolds [13]. Let us recall this mapping: CDP can be written in terms of two fields, a density of particles  $n(x, t)$ ,

and an activity  $\rho(x, t)$ ,

$$\begin{aligned} \partial_t \rho(x, t) &= \nabla^2 \rho(x, t) + [2n(x, t) - 1] \rho(x, t) - 2\rho(x, t)^2 \\ &\quad + \sqrt{2\rho(x, t)} \xi(x, t), \end{aligned} \quad (5)$$

$$\partial_t n(x, t) = \nabla^2 \rho(x, t). \quad (6)$$

Here  $\xi(x, t)$  is a standard white noise

$$\langle \xi(x, t) \xi(x', t') \rangle = \delta^d(x - x') \delta(t - t'). \quad (7)$$

Consider now the Manna model [14]. If we denote the fraction of  $i$  times occupied sites as  $a_i$ , then (for each site  $x$  and time  $t$ )  $\sum_{i=0}^\infty a_i = 1$ , the number of particles is  $\sum_{i=1}^\infty i a_i = n$ , and the activity  $\sum_{i=2}^\infty (i - 1) a_i = \rho$ . The last definition, introduced in [12], gives a higher toppling rate to triple and higher occupied sites than the standard definition of the Manna model. Since we are interested in densities close to the transition, this does not matter [12]. The benefit of this definition is the existence of the exact sum rule

$$n - \rho + e = 1, \quad (8)$$

where  $e := a_0$  is the fraction of empty sites.

Instead of writing coupled equations for  $n(x, t)$  and  $\rho(x, t)$ , with the help of the sum rule (8) we can write coupled equations for  $\rho(x, t)$  and  $e(x, t)$ ,

$$\partial_t e(x, t) = [1 - 2e(x, t)] \rho(x, t) + \sqrt{2\rho(x, t)} \xi(x, t), \quad (9)$$

$$\partial_t \rho(x, t) = \nabla^2 \rho(x, t) + \partial_t e(x, t). \quad (10)$$

In this formulation, it is easy to show the equivalence to disordered elastic manifolds [13, 15]. To this aim define

$$\rho(x, t) = \partial_t u(x, t) \quad (\text{the velocity of the interface}), \quad (11)$$

$$e(x, t) = \mathcal{F}(x, t) \quad (\text{the force acting on it}). \quad (12)$$

The second equation (10) is the time derivative of the equation of motion of an interface, subject to a random force  $\mathcal{F}(x, t)$ ,

$$\partial_t u(x, t) = \nabla^2 u(x, t) + \mathcal{F}(x, t). \quad (13)$$

Since  $\rho(x, t)$  is positive for each  $x$ ,  $u(x, t)$  is monotonously increasing. Instead of parameterizing  $\mathcal{F}(x, t)$  by space  $x$  and time  $t$ , it can be written as a function of space  $x$  and *interface*

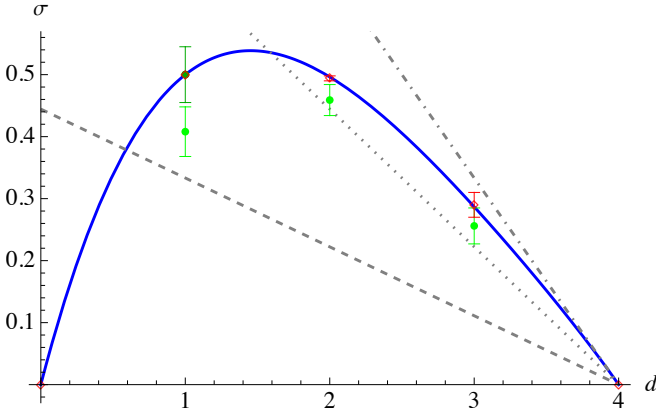


FIG. 1. The exponent  $\sigma$  of the structure factor  $S(q) \sim |q|^\sigma$  as a function of dimension  $d$  for Manna model. Solid line from the  $\epsilon$ -expansion of [17], red dots (with error bars) from [18, 19]. Numerical simulations in green are from [20] as cited in [8]. The Dark green data point is from Fig. 2. In gray are the different  $\epsilon$ -expansion results,  $\sigma = \epsilon/9$  (dashed) [8],  $\sigma = 2\epsilon/9$  (dotted)[21] and  $\sigma = \epsilon/3$  (dot-dashed) (leading term of Eq. (22)).

position  $u(x, t)$ . Setting  $\mathcal{F}(x, t) \rightarrow F(x, u(x, t))$ , the first equation (9) becomes

$$\begin{aligned} \partial_t \mathcal{F}(x, t) &\rightarrow \partial_t F(x, u(x, t)) \\ &= \partial_u F(x, u(x, t)) \partial_t u(x, t) \\ &= \left[ 1 - 2F(x, u(x, t)) \right] \partial_t u(x, t) \\ &\quad + \sqrt{2\partial_t u(x, t)} \xi(x, t). \end{aligned} \quad (14)$$

For each  $x$ , this equation is equivalent to an Ornstein-Uhlenbeck [16] process  $F(x, u)$ , defined by

$$\partial_u F(x, u) = 1 - 2F(x, u) + \sqrt{2} \xi(x, u), \quad (15)$$

$$\langle \xi(x, u) \xi(x', u') \rangle = \delta^d(x - x') \delta(u - u'). \quad (16)$$

It is a Gaussian Markovian process with mean  $\langle F(x, u) \rangle = 1/2$ , and variance in the steady state of

$$\langle [F(x, u) - \tfrac{1}{2}] [F(x', u') - \tfrac{1}{2}] \rangle = \frac{1}{2} \delta^d(x - x') e^{-2|u - u'|}. \quad (17)$$

Writing the equation of motion (13) as

$$\partial_t u(x, t) = \nabla^2 u(x, t) + F(x, u(x, t)), \quad (18)$$

it is the equation of motion of an interface with position  $u(x, t)$ , subject to a disorder force  $F(x, u(x, t))$ . The latter is  $\delta$ -correlated in the  $x$ -direction, and short-ranged correlated in the  $u$ -direction. In other words, this is a disordered elastic manifold subject to Random-Field (RF) disorder. As a consequence, results for disordered elastic manifolds can immediately be used for CDP and the Manna model.

*Hyperuniformity in the Manna model.* After these reminders, let us calculate the particle-density correlation function. We first have to identify  $n(x, t)$  with the appropriate

random-manifold field. Using Eq. (6), and the identification (11), we find  $\partial_t n(x, t) = \nabla^2 \partial_t u(x, t)$ , or after integration over time

$$n(x, t) = \nabla^2 u(x, t) + n_0. \quad (19)$$

Here  $n_0$  is the conserved mean density of particles, i.e. the conserved total number of particles divided by the volume. (We used  $\int_x \nabla^2 u(x, t) = 0$ .) As a consequence, for equal time  $t$ ,

$$\langle n(x) n(y) \rangle^c \sim |x - y|^{2\zeta - 4}. \quad (20)$$

In Fourier space this reads

$$S_q := \langle n_q n_{-q} \rangle \sim |q|^\sigma, \quad \sigma = 4 - d - 2\zeta. \quad (21)$$

Denoting  $\epsilon = 4 - d$ ,  $\sigma$  has  $\epsilon$  expansion, see [22, 23] (2-loop) and [17] (3-loop)

$$\sigma = \epsilon - 2\zeta = \frac{\epsilon}{3} - 2\zeta_2 \epsilon^2 - 2\zeta_3 \epsilon^3 + \mathcal{O}(\epsilon^4) \quad (22)$$

$$\zeta_2 = 0.0477709715468230578... \quad (23)$$

$$\zeta_3 = -0.0683544(2). \quad (24)$$

In dimension  $d = 1$  we know that  $\zeta = 5/4$  [4, 19], s.t.

$$\sigma(d = 1) = 1/2. \quad (25)$$

The exponent  $\sigma$  vanishes in dimensions  $d = 4$  ( $\epsilon = 0$ ) and  $d = 0$ : up to logarithmic corrections  $\zeta = 2$  there. Using  $\zeta(d = 2) = 0.753 \pm 0.002$  [18] and  $\zeta(d = 3) = 0.355 \pm 0.01$  [18], we find for the remaining dimensions

$$\sigma(d = 2) = 0.494(4), \quad \sigma(d = 3) = 0.29(2). \quad (26)$$

Thus  $0 \leq \sigma < 1$  in all dimensions, the signature given in Eqs. (3)-(4) for a class-III hyperuniform system. This is summarized in Fig. 1 (red dots with error bars). Also shown is the prediction (21) using the 3loop-result of [17] (blue line), and numerical simulations for the Manna model [20].

*Numerical checks.* There is some tension between [20] and our exact result  $\sigma(d = 1) = 1/2$ . For this reason we performed numerical simulations for systems of sizes up to  $L = 10^4$ . The results of the latter compensated for the predicted behavior are shown on the left of Fig. 2. There are strong finite-size corrections which make understandable the relatively small value given in [20]. However, in the relevant limit of small  $q$ , the data are consistent with  $\sigma = 0.5$  (red dashed line), while the cyan (bright) lines for  $\sigma = 0.45$  and  $\sigma = 0.55$  are the confidence interval reported on Fig. 1. We also show the results for a generalized Laplace-transform  $\mathcal{L}_\beta \circ C(t) := \sum_q e^{-|q|^\beta t} C_q$ , with  $\beta = 1$  the standard Laplace-transform, introduced to reduce the statistical noise.  $\beta = 1$  was used e.g. in [24],  $\beta = 2$  is now popular under the name “diffusion spreadability” [25]. Our data analysis shows  $\beta = 1, 2$  or  $4$  to be equivalent for all practical purposes. As Fig. 2 for  $\beta = 2$  shows, the noise is indeed reduced, but it is

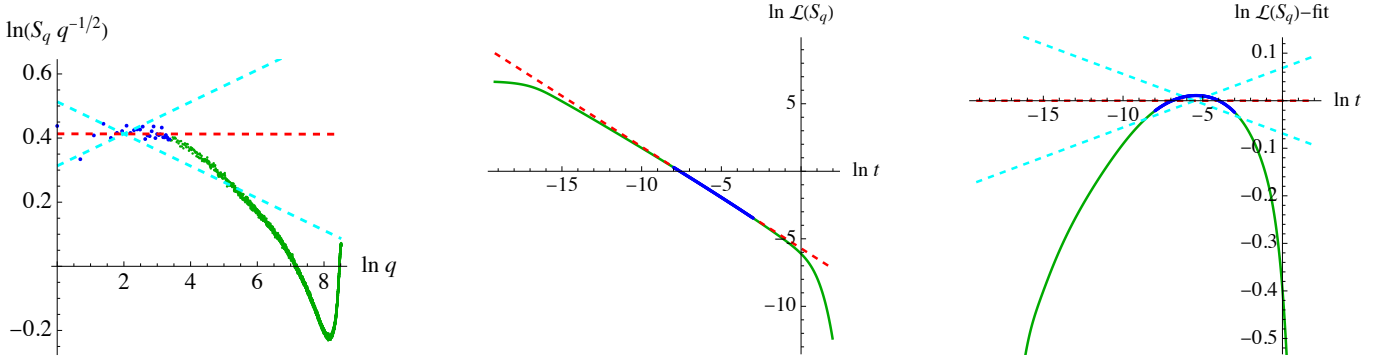


FIG. 2. Left: the compensated structure factor  $S_q q^{-1/2}$  in a  $\ln\text{-}\ln$  plot for a periodic system of size  $L = 10^5$ , with  $8 \times 10^7$  samples. The red dashed line with slope 0 indicates the behavior  $S_q \sim \sqrt{q}$ , the cyan curves power laws with an exponent deviating by  $\pm 0.05$ , indicating our interval of confidence.

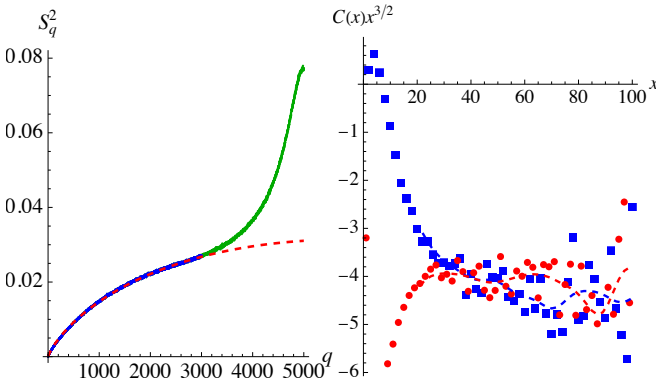


FIG. 3. Left: fit (red dashed) of  $S_q^2$  (solid blue used for fit, green not used) for  $L = 10^5$  with  $S_q^2 \simeq 5.31 \cdot 10^{-6} q \ln(16111/q)$ . Right: The compensated correlation function  $C(x)x^{3/2}$  for even (blue squares) and odd (red discs) distances  $x$ . In dashed weakly filtered data as guide for the eye. One sees strong even/odd lattice effects, which start to disappear at  $x \approx 30$ .

more difficult to choose the proper domain to fit too. All these fits indicate that  $\sigma = 0.5 \pm 0.05$ .

The reader may wonder where this problem in such a large system comes from, and whether there might be systematic corrections. While there is no proper theoretical motivation, on a phenomenological level the deviations from a pure power law are well fitted with a logarithm, as Fig. 3 attests. To proceed, it is instructive to plot the density correlations as a function of distance. For short even distances, we find positive correlations, due to events where one grain is moved to the right, and one to the left. These positive correlations become negative for  $x \geq 8$ , but one has to wait to  $x \approx 30$  until even and odd correlations become comparable. This indicates that  $\ell = 30$  is the minimal coarse graining size, taking out one and a half decades from the range to which one can fit  $S_1$ , certainly one reason for its slow convergence. One may also wonder whether this is related to the saturation of the apparent roughness exponent at depinning  $\zeta_{\text{app}}^{\text{dep}}(d=1) \approx 1$  [26, 27].

A hyperuniformity exponent of  $1/2$  is also observed in the

related Oslo model [4].

*Relation to the literature.* Our results contradict two works from the literature:  $\sigma = \epsilon/9$  [8] and  $\sigma = 2\epsilon/9$  [21]. None of these works use functional RG, which seems to be crucial to account for the non-trivial structure present at 2-loop and [22, 23] 3-loop order [17] at depinning. Ref. [21] does this calculation in terms of active and passive particles. The density of the latter is a linear combination of fields used here,  $n_p = a_1 \approx n - 2\rho$ ; since  $n - n_0 = \nabla^2 u$  and  $\rho = \partial_t u$ , the scaling dimensions of the two terms differs by  $z - 2 = \mathcal{O}(\epsilon)$ . As a result,  $n_p$  is not a proper scaling field of the RG, a problem known in other contexts [28]. Since the two fields are degenerate at  $\epsilon = 0$ , their respective  $\mathcal{O}(\epsilon)$  corrections are easily attributed to the  $\mathcal{O}(\epsilon)$  correction of their linear combination  $n_p$ . Another problem in [21] is the introduction of an additional noise in Eqs. (4) and (5) which does not vanish in the absorbed state  $\rho = 0$ . It would be possible to add  $\sqrt{\rho(x,t)}\nabla\tilde{\eta}(x,t)$ , with  $\langle\eta^i(x,t)\eta^j(x',t')\rangle = \delta^d(x-x')\delta(t-t')\delta^{ij}$ , or  $\nabla[\sqrt{\rho(x,t)}\tilde{\eta}(x,t)]$ , but comparing  $\nabla^2\rho \sim L^{\zeta-z-2}$  to  $\sqrt{\rho(x,t)}\nabla\tilde{\eta}(x,t) \sim L^{\frac{\zeta-z}{2}-1-\frac{d+z}{2}}$  we conclude that the latter is irrelevant as long as  $d + \zeta > 2$ , which is satisfied for all  $d > 0$ .

*Conclusions.* In this letter, we have shown how hyperuniformity in the Manna model is related to depinning. This equivalence yields precise theoretical predictions for the hyperuniformity exponent in all dimensions.

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