The Toric Code Model in a Magnetic Field

Julien Vidal
Laboratoire de Physique Théorique de la Matière Condensée
(Paris)

in collaboration with

Sébastien Dusuel (Paris)
Kai Phillip Schmidt (Dortmund)
Ronny Thomale (Karlsruhe)

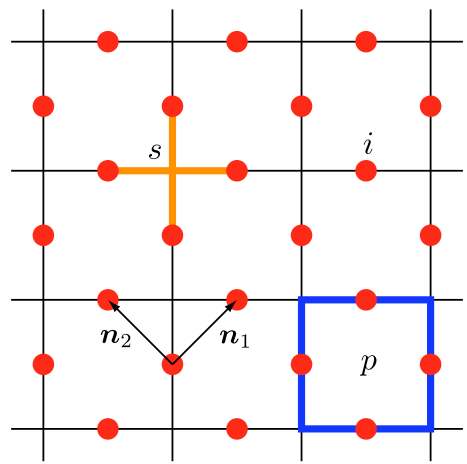
Outline

1. The Toric Code Model
2. The Parallel Case
3. The Transverse Case
4. The General Case ?

The Toric Code Model*

$$H = -J \sum_s A_s - J \sum_p B_p$$

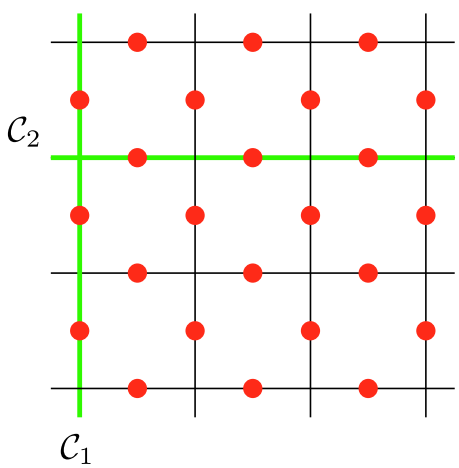
with $A_s = \prod_{i \in s} \sigma_i^x$, and $B_p = \prod_{i \in p} \sigma_i^z$



* A. Y. Kitaev Ann. Phys. **303**, 2 (2003)

Conserved quantities

- $[H, A_s] = [H, B_p] = [A_s, B_p] = 0$,
- ${A_s}^2 = {B_p}^2 = \mathbb{I}$
- $\prod_s A_s = \prod_p B_p = \mathbb{I}$
- $N_s + N_p = N$
- 2 $\mathbb{Z}_2$ operators conserved, *e. g.*, $Z_1 = \prod_{i \in \mathcal{C}_1} \sigma_i^z$ and $Z_2 = \prod_{i \in \mathcal{C}_2} \sigma_i^z$



$\Rightarrow$ Exactly solvable AND trivially solved !

Spectrum and Eigenstates

- Equidistant energy spectrum : $-NJ, -NJ + 4J, -NJ + 8J, \ldots$

- 4-fold degenerate ground-state :

$$|\psi_0, \pm 1, \pm 1\rangle = \mathcal{N}\left(\frac{\mathbb{I} \pm Z_1}{2}\right)\left(\frac{\mathbb{I} \pm Z_2}{2}\right)\prod_s \left(\frac{\mathbb{I} + A_s}{2}\right)\prod_p \left(\frac{\mathbb{I} + B_p}{2}\right)|\text{Ref}\rangle$$

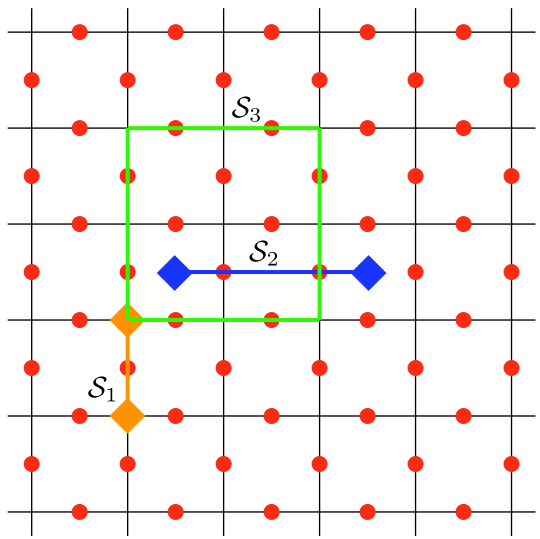
- Elementary excitations :

- "One pair of charges" : $|A_i, A_j, \pm 1, \pm 1\rangle = \sigma^z_{k \in (i,j)}|\psi_0, \pm 1, \pm 1\rangle$

- "One pair of fluxes" : $|B_i, B_j, \pm 1, \pm 1\rangle = \sigma^x_{k \in (i,j)}|\psi_0, \pm 1, \pm 1\rangle$

Quantum Statistics of the Excitations

A's and B's are individually hard-core bosons but...



$$|\psi_1\rangle = \prod_{i\in\mathcal{S}_1} \sigma_i^z \prod_{i\in\mathcal{S}_2} \sigma_i^x |\psi_0\rangle$$

$$|\psi_2\rangle = \prod_{i\in\mathcal{S}_3} \sigma_i^z |\psi_1\rangle = -|\psi_1\rangle$$

(since $\quad \sigma_i^z \sigma_i^x = -\sigma_i^x \sigma_i^z$)

Charge going around a π flux = double exchange and $e^{i\pi} = -1$

Excitations are Abelian anyons (semions)

Topological Degeneracy and its Robustness

- Ground-state degeneracy= 4^g on a genus g surface

- Topological order "destroyed" by thermal fluctuations (vanishing topological entropy**, vanishing expectation values of cycle operators***, ...)

- Topological degeneracy robust against local perturbation but :

"Of course, the perturbation should be small enough, or else a phase transition may occur."*

Breakdown of the topological phase at $T = 0$?

* A. Y. Kitaev Ann. Phys. **303**, 2 (2003)
** C. Castelnovo and C. Chamon, Phys. Rev. B **76**, 184442 (2007)
***Z. Nussinov and G. Ortiz, Phys. Rev. B **77**, 064302 (2008),
R. Alicki, M. Fannes, and M. Horodecki, J. Phys. A **42**, 065303 (2009)

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1. The Toric Code Model
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TCM in a parallel field : the single-component case*

$$H = -J \sum_s A_s - J \sum_p B_p - h_z \sum_i \sigma_i^z$$

- B_p's are still conserved quantities but A_s's can move

- Low-energy effective theory ($B_p = +1, \forall p$) $\Leftrightarrow$ 2D transverse field Ising model

$$\sigma^z_{k\in(i,j)} = \mu_i^x \mu_j^x,\ A_i = \mu_i^z \Rightarrow H_{\text{eff}} = -J \sum_i \mu_i^z - h_z \sum_{\langle i,j \rangle} \mu_i^x \mu_j^x - JN_p$$

- Weak (strong) field in TCM $\Leftrightarrow$ Strong (weak) field in Ising model

Second-order phase transition at $h_z/(2J) = 0.1642(2)$**

$\Rightarrow$ Topological degeneracy lifted for $h_z/(2J) > 0.1642(2)$

* S. Trebst, P. Werner, M. Troyer, K. Shtengel, and C. Nayak, Phys. Rev. Lett. **98**, 070602 (2007)
** H.-X. He, C. J. Hamer, and J. Oitmaa, J. Phys. A **23**, 1775 (1990)

TCM in a parallel field : the two-component case

$$H = -J \sum_s A_s - J \sum_p B_p - h_x \sum_i \sigma_i^x - h_z \sum_i \sigma_i^z$$

- A_s's and B_p's are no more conserved. Anyons can move !

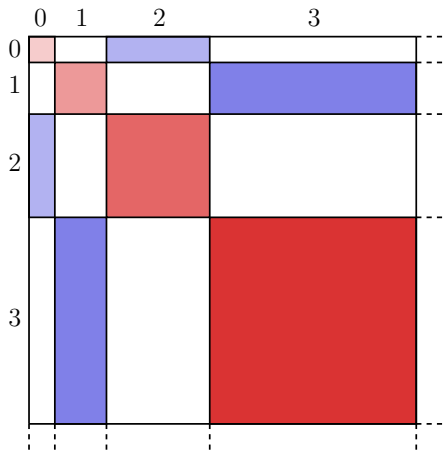
- Mapping onto a classical 3D $\mathbb{Z}_2$ gauge Higgs model + Monte-Carlo simulations*

- Perturbative analysis (weak-field and strong-field expansions)** using

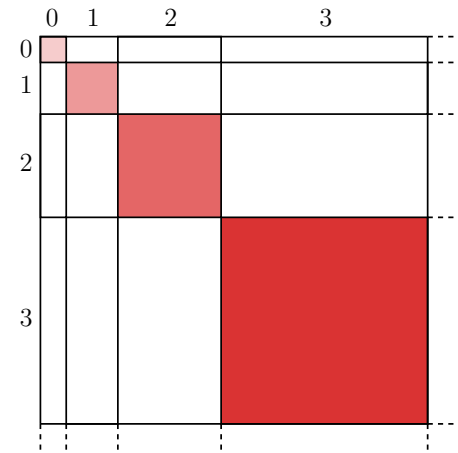
  Continuous Unitary Transformations

* I. Tupitsyn, A. Kitaev, N. V. Prokof'ev and P. C. E. Stamp, arXiv:0804.3175
** J. Vidal, S. Dusuel, and K. P. Schmidt, Phys. Rev. B **79**, 033109 (2009)

Perturbative Continuous Unitary Transformations*



$\Rightarrow$



$$H = -\frac{N}{2} + Q + T_0 + T_{+2} + T_{-2} \quad \Rightarrow \quad H_{\text{eff}} = -\frac{N}{2} + Q + T_0 + \frac{1}{2}\left(T_{+2}T_{-2} - T_{-2}T_{+2}\right) + \ldots$$

$H_{\text{eff}} = U^{\dagger} H U$ such that $[H_{\text{eff}}, Q] = 0$

Landau-like Quasiparticle description

* F. Wegner Ann. Phys. (Leipzig), **3**, 77 (1994), A. Mielke Eur. Phys. J. B **5**, 605 (1998), C. Knetter and G. S.Uhrig, Eur. Phys. J. B **13**, 209 (2000).

Weak-field expansion ($J = 1/2$)

- 0-QP sector (ground-state energy per spin) :

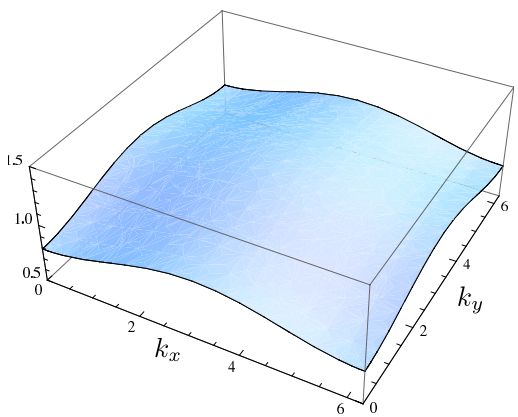
$$
\begin{aligned}
e_0 \;=\; & -\frac{1}{2} - \frac{1}{2}(h_z^2 + h_x^2) - \frac{15}{8}(h_z^4 + h_x^4) + \frac{h_x^2 h_z^2}{4} - \frac{147}{8}(h_z^6 + h_x^6) + \frac{113}{32}(h_x^2 h_z^4 + h_x^4 h_z^2) \\
& + \frac{6685}{128}(h_x^2 h_z^6 + h_x^6 h_z^2) + \frac{20869}{384} h_x^4 h_z^4
\end{aligned}
$$

- 1-QP sector (gap of a "dressed charge" or a 'dressed flux") :

$$
\begin{aligned}
\Delta_{\text{charge}} \;=\; & 1 - 4h_z - 4h_z^2 - 12h_z^3 + 2h_x^2 h_z - 36h_z^4 + 3h_x^2 h_z^2 + 5h_x^4 - 176h_z^5 + \frac{83}{4} h_x^2 h_z^3 \\
& + \frac{27}{2} h_x^4 h_z - \frac{2625}{4} h_z^6 + 63 h_x^2 h_z^4 + 71 h_x^4 h_z^2 + 92 h_x^6 - \frac{14771}{4} h_z^7 + \frac{28633}{64} h_x^2 h_z^5 \\
& + \frac{925}{4} h_x^4 h_z^3 + \frac{495}{2} h_x^6 h_z - \frac{940739}{64} h_z^8 + \frac{118029}{64} h_x^2 h_z^6 + \frac{19263}{16} h_x^4 h_z^4 \\
& + \frac{80999}{96} h_x^6 h_z^2 + \frac{495}{2} h_x^6 h_z^2 + \frac{35649}{16} h_x^8
\end{aligned}
$$

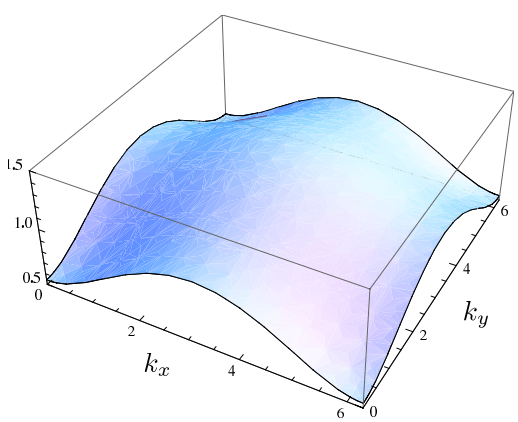
Single Quasiparticle Dispersion

($h_z = 0.1$ and $h_x = 0.05$)



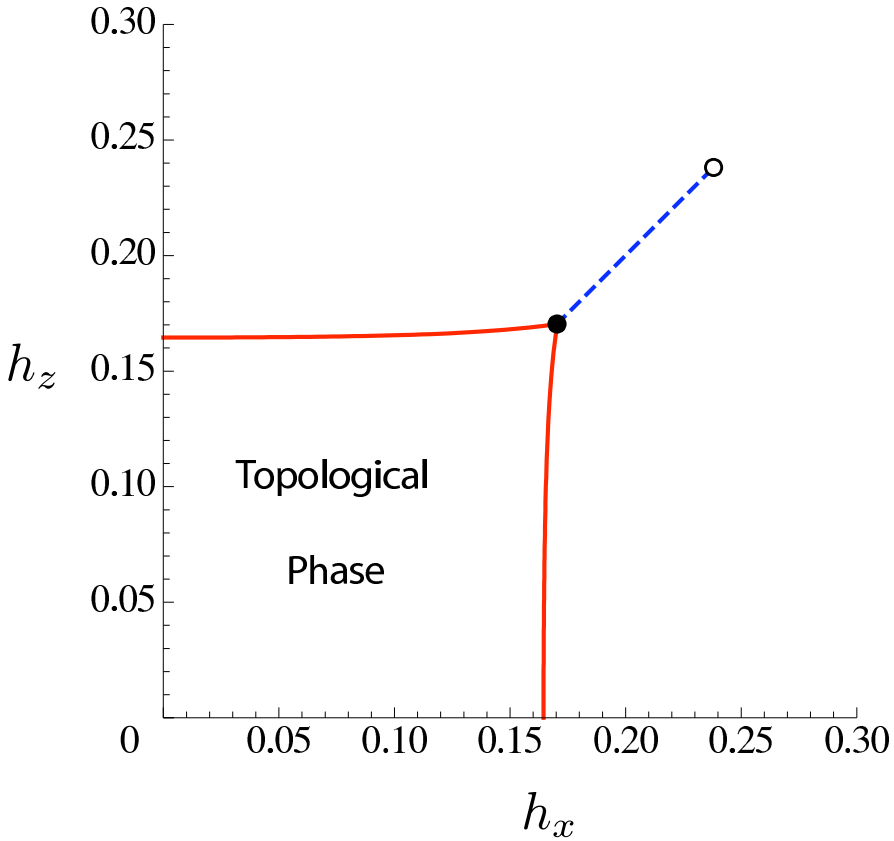
Single-flux dispersion

$$\Delta = \min(\Delta_{\text{charge}}, \Delta_{\text{flux}})$$



Single-charge dispersion

Phase diagram of the parallel case



—— 2nd order transition line

(Ising Universality Class ?)

– – 1st order transition line

• Multicritical point at

$h_x = h_z = 0.1703(2)$

$(\nu_{\text{mult.}} > \nu_{\text{Ising}})$

○ Ising-like critical point at

$h_x = h_z = 0.24(1)$

No local order parameter !

Transition $\Leftrightarrow$ $\Delta = 0$

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TCM in a transverse field*

$$H = -J \sum_s A_s - J \sum_p B_p - h_y \sum_i \sigma_i^y$$

- A_s's and B_p's are no more conserved

- Parity of the number of spins $\uparrow_y$ per row and per column is conserved

- Spectrum invariant under $J \leftrightarrow h_y$: Self-Dual Model

- Perturbative analysis using PCUT + Exact Diagonalizations

* J. Vidal, R. Thomale, K. P. Schmidt, and S. Dusuel, arXiv:0902.3547

Self-Duality and Related Problems

$$H = -J\sum_s A_s - J\sum_p B_p - h_y\sum_i \sigma_i^y \qquad \left(\widetilde{\sigma}^z_{j_s} = A_s, \widetilde{\sigma}^z_{j_p} = B_p, \text{and} \quad \widetilde{\sigma}^x_j = \prod_{j>i}\sigma_i^y\right)$$

$$\Downarrow$$

$$H_{\text{XM}}{}^{\star} = -J\sum_j \widetilde{\sigma}^z_j - h_y\sum_{\widetilde{p}}\prod_{j\in\widetilde{p}}\widetilde{\sigma}^x_j \ \left(\tau^z_j = \prod_{j\in\widetilde{p}}\widetilde{\sigma}^x_j, \text{and}\tau^x_j = \prod_{j>i}\widetilde{\sigma}^z_i\right)$$

$$\Downarrow$$

$$H_{\text{XM}} = -J\sum_j\prod_{j\in\widetilde{p}}\tau^x_j - h_y\sum_{\widetilde{p}}\tau^x_j$$

XM Model has also the same spectrum as the Quantum Compass Model**

$$H_{\text{QCM}} = -J_x\sum_{\mathbf{r}}\sigma^x_{\mathbf{r}}\sigma^x_{\mathbf{r}+\mathbf{n_1}} - J_y\sum_{\mathbf{r}}\sigma^x_{\mathbf{r}}\sigma^x_{\mathbf{r}+\mathbf{n_2}}$$

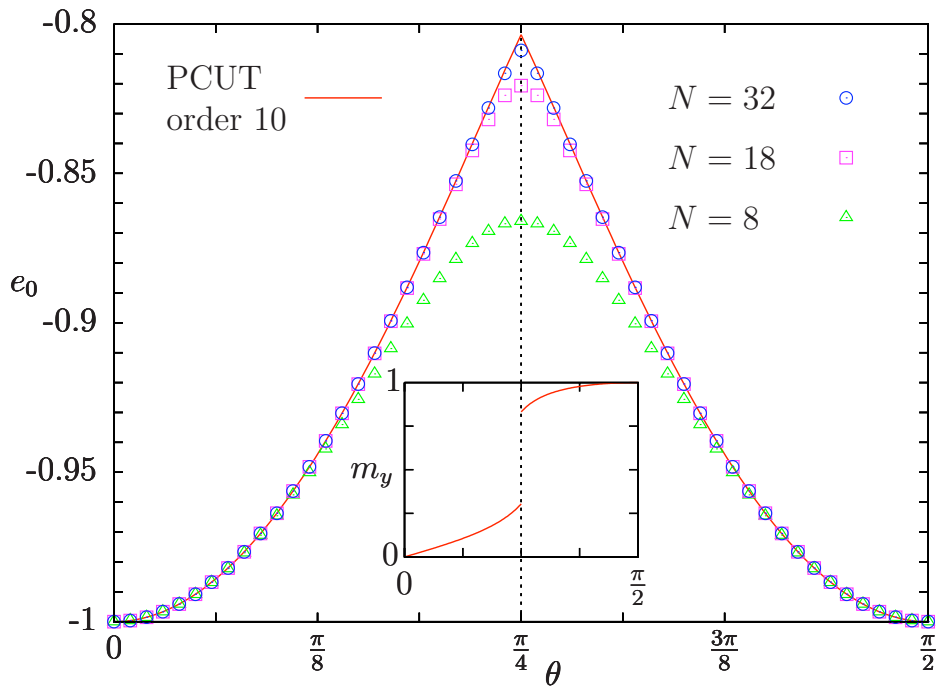
Mapping only valid for open boundary conditions and in the thermodynamical limit

* C. Xu and J. E. Moore, Phys. Rev. Lett. **93**, 047003 (2004), Nucl. Phys. B **716**, 487 (2005)

** Z. Nussinov and E. Fradkin, Phys. Rev. B **71**, 195120 (2005)

Weak-coupling expansion using PCUT

- 0-QP sector (ground-state energy per spin) :



$J = \cos\theta$

$h_y = \sin\theta$

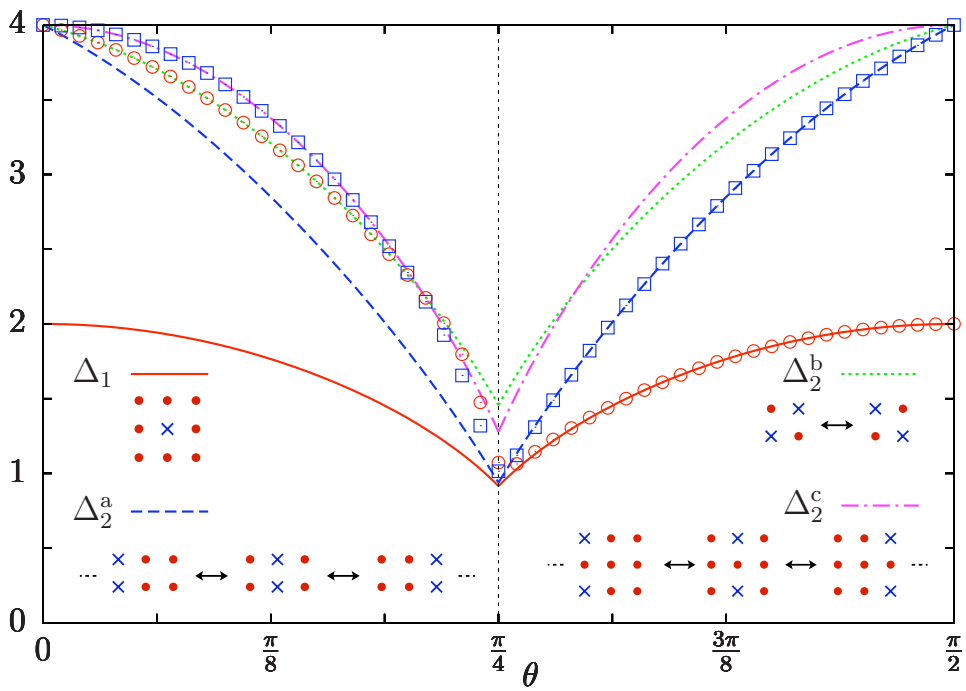
$m_y = \partial_{h_y} e_0$ (H.-F. theorem)

Magnetization Jump at $\theta = \pi/4$!

First-order transition at the self-dual point $h_y = J$!

Weak-coupling expansion using PCUT

- 1-QP and 2-QP sectors :



- Comparison with Exact Diag.

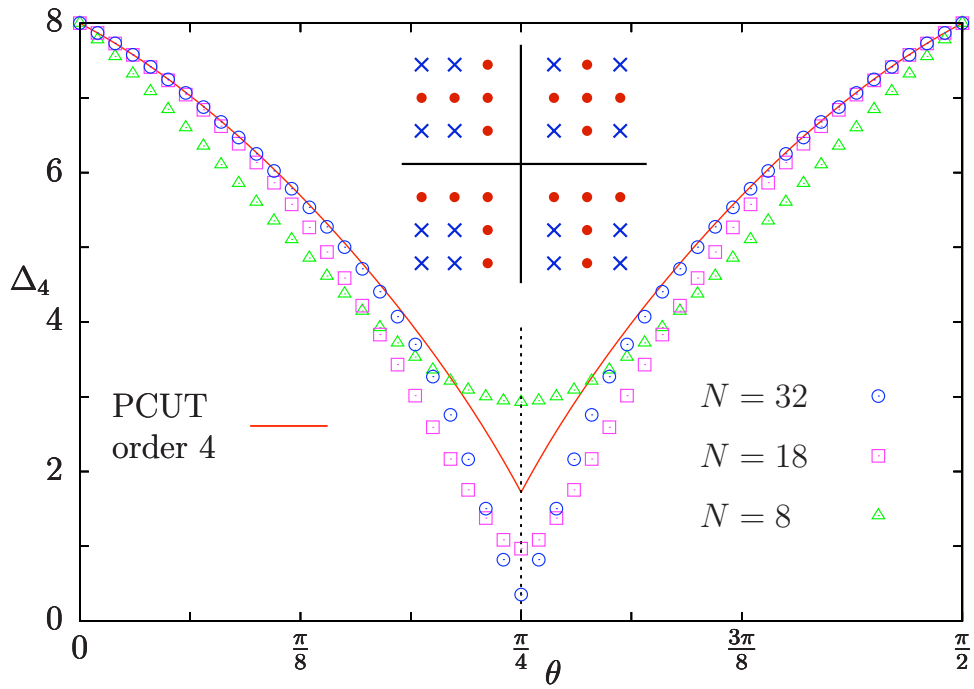
$N = 32$ spins + PBC

- Formation of 2-QP bound states

Level crossings not captured by perturbative analysis but still a finite gap

Weak-coupling expansion using PCUT

- 4-QP sector (same symmetry as the 0-QP sector) :



Formation of a 4-QP bound state

(It's a boson !)

Perturbative expansion valid everywhere except at the transition point

Δ_4 vanishes when N increases*

* J. Dorier, F. Becca, and F. Mila, Phys. Rev. B **72**, 024448 (2005)

The General Case ?

1. Second-order (captured by PCUT) *vs* first-order transition

2. Importance of bound states

3. Robustness of topological phases in other 2D systems

4. Robustness in higher-dimensional systems

5. Implementation in experimental devices