The Toric Code Model in a Magnetic Field

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Outline

1. The Toric Code Model
2. The Parallel Case
3. The Transverse Case
4. The General Case ?
The Toric Code Model

\[ H = -J \sum_s A_s - J \sum_p B_p \]

with \( A_s = \prod_{i \in s} \sigma^x_i \), and \( B_p = \prod_{i \in p} \sigma^z_i \)

Conserved quantities

• $[H, A_s] = [H, B_p] = [A_s, B_p] = 0$,

• $A_s^2 = B_p^2 = I$

• $\prod_{A_s} A_s = \prod_{B_p} B_p = I$

• $N_s + N_p = N$

• 2 $\mathbb{Z}_2$ operators conserved, e.g., $Z_1 = \prod_{i \in C_1} \sigma_i^z$ and $Z_2 = \prod_{i \in C_2} \sigma_i^z$

$\Rightarrow$ Exactly solvable AND trivially solved!
Spectrum and Eigenstates

- Equidistant energy spectrum: $-NJ, -NJ + 4J, -NJ + 8J, \ldots$

- $4$-fold degenerate ground-state:

$$|\psi_0, \pm 1, \pm 1\rangle = \mathcal{N} \left( \frac{\mathbb{I} \pm Z_1}{2} \right) \left( \frac{\mathbb{I} \pm Z_2}{2} \right) \prod_s \left( \frac{\mathbb{I} \pm A_s}{2} \right) \prod_p \left( \frac{\mathbb{I} \pm B_p}{2} \right) |\text{Ref}\rangle$$

- Elementary excitations:

  - “One pair of charges”:
    $$|A_i, A_j, \pm 1, \pm 1\rangle = \sigma^z_{k \in (i,j)} |\psi_0, \pm 1, \pm 1\rangle$$

  - “One pair of fluxes”:
    $$|B_i, B_j, \pm 1, \pm 1\rangle = \sigma^x_{k \in (i,j)} |\psi_0, \pm 1, \pm 1\rangle$$
Quantum Statistics of the Excitations

$A$’s and $B$’s are individually hard-core bosons but...

$$|\psi_1\rangle = \prod_{i \in S_1} \sigma_i^z \prod_{i \in S_2} \sigma_i^x |\psi_0\rangle$$

$$|\psi_2\rangle = \prod_{i \in S_3} \sigma_i^z |\psi_1\rangle = -|\psi_1\rangle$$

(since $\sigma_i^z \sigma_i^x = -\sigma_i^x \sigma_i^z$)

Charge going around a $\pi$ flux = double exchange and $e^{i\pi} = -1$

Excitations are Abelian anyons (semions)
Topological Degeneracy and its Robustness

- Ground-state degeneracy $= 4^g$ on a genus $g$ surface

- Topological order “destroyed” by thermal fluctuations (vanishing topological entropy**, vanishing expectation values of cycle operators***, ...)

- Topological degeneracy robust against local perturbation but:
  “Of course, the perturbation should be small enough, or else a phase transition may occur.”*

** Breakdown of the topological phase at $T = 0$ ?

*** Z. Nussinov and G. Ortiz, Phys. Rev. B 77, 064302 (2008),
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TCM in a parallel field : the single-component case

\[ H = -J \sum_s A_s - J \sum_p B_p - h_z \sum_i \sigma_i^z \]

- \( B_p \)'s are still conserved quantities but \( A_s \)'s can move

- Low-energy effective theory \((B_p = +1, \forall p) \Leftrightarrow \) 2D transverse field Ising model

\[ \sigma_{k \in (i,j)}^z = \mu_i^x \mu_j^x, \quad A_i = \mu_i^z \Rightarrow H_{\text{eff}} = -J \sum_i \mu_i^z - h_z \sum_{\langle i,j \rangle} \mu_i^x \mu_j^x - J N_p \]

- Weak (strong) field in TCM \( \Leftrightarrow \) Strong (weak) field in Ising model

Second-order phase transition at \( h_z / (2J) = 0.1642(2) \)

\[ \Rightarrow \] Topological degeneracy lifted for \( h_z / (2J) > 0.1642(2) \)


TCM in a parallel field: the two-component case

\[ H = -J \sum_s A_s - J \sum_p B_p - h_x \sum_i \sigma_i^x - h_z \sum_i \sigma_i^z \]

- \(A_s\)'s and \(B_p\)'s are no more conserved. Anyons can move!

- Mapping onto a classical 3D \(\mathbb{Z}_2\) gauge Higgs model + Monte-Carlo simulations*

- Perturbative analysis (weak-field and strong-field expansions)** using Continuous Unitary Transformations

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Perturbative Continuous Unitary Transformations

\[ H = -\frac{N}{2} + Q + T_0 + T_{+2} + T_{-2} \]

\[ \Rightarrow \quad H_{\text{eff}} = -\frac{N}{2} + Q + T_0 + \frac{1}{2} (T_{+2}T_{-2} - T_{-2}T_{+2}) + \ldots \]

\[ H_{\text{eff}} = U^\dagger H U \quad \text{such that} \quad [H_{\text{eff}}, Q] = 0 \]

Landau-like Quasiparticle description

Weak-field expansion \((J = 1/2)\)

- **0-QP sector** (ground-state energy per spin):

\[
e_0 = -\frac{1}{2} - \frac{1}{2}(h^2_z + h^2_x) - \frac{15}{8}(h^4_z + h^4_x) + \frac{h^2_x h^2_z}{4} - \frac{147}{8}(h^6_z + h^6_x) + \frac{113}{32}(h^2_x h^4_z + h^4_x h^2_z)
\]
\[
+ \frac{6685}{128}(h^2_x h^6_z + h^6_x h^2_z) + \frac{20869}{384} h^4_x h^4_z
\]

- **1-QP sector** (gap of a “dressed charge” or a ‘dressed flux’):

\[
\Delta_{\text{charge}} = 1 - 4h_z - 4h^2_z - 12h^3_z + 2h^2_x h_z - 36h^4_z + 3h^2_x h^2_z + 5h^4_x - 176h^5_z + \frac{83}{4}h^2_x h^3_z
\]
\[
+ \frac{27}{2}h^4_x h_z - \frac{2625}{4}h^6_z + 63h^2_x h^4_z + 71h^4_x h^2_z + 92h^6_x - \frac{14771}{4}h^7_z + \frac{28633}{64}h^2_x h^5_z
\]
\[
+ \frac{925}{4}h^4_x h^3_z + \frac{495}{2}h^6_x h_z - \frac{940739}{64}h^8_z + \frac{118029}{64}h^2_x h^6_z + \frac{19263}{16}h^4_x h^4_z
\]
\[
+ \frac{80999}{96}h^6_x h^2_z + \frac{495}{2}h^6_x h_z + \frac{35649}{16}h^8_x
\]
Single Quasiparticle Dispersion

\((h_z = 0.1 \text{ and } h_x = 0.05)\)

\[
\Delta = \min(\Delta_{\text{charge}}, \Delta_{\text{flux}})
\]

Single-flux dispersion

Single-charge dispersion
Phase diagram of the parallel case

- 2\textsuperscript{nd} order transition line (Ising Universality Class?)
- 1\textsuperscript{st} order transition line
- Multicritical point at $h_x = h_z = 0.1703(2)$ ($\nu_{\text{mult.}} > \nu_{\text{Ising}}$)
- Ising-like critical point at $h_x = h_z = 0.24(1)$

No local order parameter!

Transition $\Leftrightarrow \Delta = 0$
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TCM in a transverse field

\[ H = -J \sum_s A_s - J \sum_p B_p - h_y \sum_i \sigma_i^y \]

- \( A_s \)'s and \( B_p \)'s are no more conserved

- Parity of the number of spins \( \uparrow_y \) per row and per column is conserved

- Spectrum invariant under \( J \leftrightarrow h_y \) : Self-Dual Model

- Perturbative analysis using PCUT + Exact Diagonalizations

Self-Duality and Related Problems

\[ H = -J \sum_s A_s - J \sum_p B_p - h_y \sum_i \sigma^y_i \quad (\tilde{\sigma}^z_j = A_s, \tilde{\sigma}^z_j = B_p, \text{and} \quad \tilde{\sigma}^x_j = \prod_{j>i} \sigma^y_i) \]

\[ \Downarrow \]

\[ H_{XM}^* = -J \sum_j \tilde{\sigma}^z_j - h_y \sum_p \prod_{j \in p} \tilde{\sigma}^x_j \quad (\tau^z_j = \prod_{j \in p} \tilde{\sigma}^x_j, \text{and} \tau^x_j = \prod_{j>i} \tilde{\sigma}^z_i) \]

\[ \Downarrow \]

\[ H_{XM} = -J \sum_j \prod_{j \in p} \tau^x_j - h_y \sum_p \tau^x_j \]

\[ \therefore \]

XM Model has also the same spectrum as the Quantum Compass Model**

\[ H_{QCM} = -J_x \sum_r \sigma^x_r \sigma^x_{r+n_1} - J_y \sum_r \sigma^x_r \sigma^x_{r+n_2} \]

Mapping only valid for open boundary conditions and in the thermodynamical limit


Weak-coupling expansion using PCUT

- **0-QP sector** (ground-state energy per spin):

\[
\begin{align*}
J & = \cos \theta \\
h_y & = \sin \theta \\
m_y & = \frac{\partial}{\partial h_y} e_0 \quad (\text{H.-F. theorem})
\end{align*}
\]

Magnetization Jump at \( \theta = \pi/4 \!\)

First-order transition at the self-dual point \( h_y = J \!\)
Weak-coupling expansion using PCUT

- 1-QP and 2-QP sectors:

- Comparison with Exact Diag.

\[ N = 32 \text{ spins} \quad + \quad \text{PBC} \]

- Formation of 2-QP bound states

Level crossings not captured by perturbative analysis but still a finite gap
Weak-coupling expansion using PCUT

- 4-QP sector (same symmetry as the 0-QP sector):

\[ \Delta_4 \text{ vanishes when } N \text{ increases} \]

The General Case?

1. Second-order (captured by PCUT) vs first-order transition

2. Importance of bound states

3. Robustness of topological phases in other 2D systems

4. Robustness in higher-dimensional systems

5. Implementation in experimental devices