Multiple Condition Statistics
in a
Quantum Hall State

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in collaboration with

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Outline

- **Reminder: FQHE for spherical geometry**
- **Root partition description**
- **New many-body interaction (MBI) state series**
- **Jack polynomial description**
- **3-body ($\nu = 2/5$) Gaffnian state: ${}^{3}_{6}$MBI**
- **4-body ($\nu = 3/7$) state: ${}^{4,3}_{12,5}$MBI Non-Jack FQH state**
- **Summary and outlook**

Spherical geometry

Landau bands:

Centered magnetic monopole with radial field strength B

Sphere radius R, total flux is a multiple $2S$ of h/e
$\rightarrow B = \hbar S/eR^2$

Large R (and S) limit recovers plane setting

magnetic Hamiltonian:

$$H_0 = \frac{1}{2mR^2}(\boldsymbol{L}^2 - \hbar^2 S^2), \quad l = S + n, \quad n \in \mathbb{N}_0 \text{ LB index}$$

spinor coordinates:

$$u = \cos(\frac{\theta}{2}) \exp(i\frac{\varphi}{2}), \quad v = \sin(\frac{\theta}{2}) \exp(-i\frac{\varphi}{2})$$

Spherical geometry

eigenfunctions of the LLB:

$$e_{S,m} = u^{S+m} v^{S-m}, \quad m = S, S-1, \ldots, -S$$

$$\boldsymbol{L}^2 e_{S,m} = \hbar^2 S(S+1) e_{S,m}, \quad L_3 e_{S,m} = \hbar m e_{S,m}$$

Laughlin wave function plane / sphere: $\boldsymbol{z} \longrightarrow \frac{\boldsymbol{u}}{\boldsymbol{v}}$, multiply by all $\boldsymbol{v}$'s

$$\Psi_L^{(P)} = \prod_{i<j} (z_i - z_j)^3 \quad / \quad \Psi_L^{(S)} = \prod_{i<j} (u_i v_j - v_i u_j)^3$$

2nd quantization notation of the many-particle basis states:

$$\overbrace{|\mathbf{10010\ldots0011}\rangle}^{\boldsymbol{S,S-1,\ldots,-S+1,-S}} = \boldsymbol{a}_{\boldsymbol{S}}^{\dagger} \boldsymbol{a}_{\boldsymbol{S-3}}^{\dagger} \ldots \boldsymbol{a}_{\boldsymbol{-S+1}}^{\dagger} \boldsymbol{a}_{\boldsymbol{-S}}^{\dagger} |\mathbf{0}\rangle$$

Root partitions

numerical observation: QH trial states can be characterized by their root partition. Laughlin state:

$$\mathsf{RP}_{\Psi_L} = 1001001001001001001001\ldots1001001$$

corresponding state has the highest variance of all basis states with non-zero coefficients:

$$\mathsf{var}(|b\rangle) = \sum_{m=-S}^{S} m^2 n_m^{(b)}$$

all other non-zero basis states can be generated by pairwise squeezing operations of particles at m_1 and m_2, $m_1 > m_2 + 1$:

$$m_1 \to m_1 - 1, \quad m_2 \to m_2 + 1.$$

translational (on sphere: rotational) invariance fixes coefficients, i.e. $L^-_{\text{tot}} |\Psi\rangle = 0$.

Root partitions

exemplary squeezing operations:

$$\color{red}{1}00\color{red}{1}001001001001001001001$$

$$0110001001001\color{red}{1}00\color{red}{1}001001001$$

$$0110001001000110001001001$$

$$\ldots.$$

$$0000000011111111100000000$$

two-particle Laughlin state:

$$\Psi_L^{L^z=0} = \begin{bmatrix} a\{1001\} \\ b\{0110\} \end{bmatrix}, \quad L^-_{\textbf{tot}} \Psi_L^{L^z=0} = \sqrt{3}(a+b)\{1010\} \stackrel{!}{=} 0$$

$$\boldsymbol{\longrightarrow \Psi_L = \tfrac{1}{\sqrt{2}}[\{1001\} - \{0110\}]}$$

MBI state series

generalization of root partitions to other filling fractions using n-body interactions

$\Psi_L = \Psi_{\text{MBI}}^{(2)} \sim d^{0+3}$ as 2 particles approach each other (Laughlin state):

$$100100100100100\ldots \equiv u_1^0 u_2^3 u_3^6 u_4^9 u_5^{12} \ldots$$

generalization to 3-body interactions: $\Psi_{\text{MBI}}^{(3)} \sim d^{0+1+5}$ as 3 particles approach each other (${}^{3}_{6}$MBI, shorthand 2/5 state, Gaffnian):

$$110001100011000\ldots \equiv u_1^0 u_2^1 u_3^5 u_4^6 u_5^{10} u_6^{11} \ldots$$

generalization to 4-body interactions: $\Psi_{\text{MBI}}^{(4)} \sim d^{12}$ as 4 particles approach each other (${}^{4,3}_{12,5}$ MBI state, shorthand 3/7-state):

$$110010011001001100100110010011001000\ldots$$

Jack Polynomials

Jack polynomials $J_\lambda^\alpha(z)$:

symmetric polynomial in $z \equiv z_1, z_2, \ldots, z_N$ (obtained by fermionic QH states divided by a Vandermonde Determinant)

eigenstates of $H_{\mathsf{LB}} = \sum_i \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)$

$\lambda \equiv \lambda(k,r)$, $k, r \in \mathbb{N}$ denotes the partition dominating the polynomial*, written as an occupation-number configuration $k \underbrace{0 \ldots 0}_{r-1} k \underbrace{0 \ldots 0}_{r-1} \ldots k$

$\alpha \equiv \alpha(k,r) = -\frac{k+1}{r-1}$, which thus are Jacks with negative α, mathematically first defined only for $r = 2$**

rich algebraic structure; $\exists$ a decomposition scheme into bosonic monomials

* B. A. Bernevig and F. D. M. Haldane, PRL 100, 246802 (2008).

** B. Feigin *et al.*, Int. Math. Res. Not. 2002, 1223.

Jack Polynomials and FQH states

Laughlin state $100100100100\ldots$ **in Jack description** $(\boldsymbol{k}, \boldsymbol{r}) = (\mathbf{1}, \mathbf{2})$:

$$J^{\alpha}_{\lambda,L} = \prod_{i<j} (z_i - z_j)^2, \quad \alpha = -2, \quad \lambda = 1010101010101010....1$$

Root partition change under Vandermonde division: Sum packets of neighboring "1"to one total number and omit one subsequent "0"

2/5 (Gaffnian) state $110001100011000\ldots$ **is the Jack** $(\boldsymbol{k}, \boldsymbol{r}) = (\mathbf{2}, \mathbf{3})$:

$$J^{\alpha}_{\lambda,G}, \quad \alpha = -\frac{3}{2}, \quad \lambda = 2002002002002002002002\ldots2$$

similar description for Moore Read, Read Rezayi, and related constructions

for Jain states*

1st exception: the $\mathbf{3/7}$ **state is no Jack (**$20102010201\ldots$**)**

***N. Regnault, B. Andrei Bernevig, and F. D. M. Haldane, arXiv:0901.2121.**

$2/5$ MBI (Gaffnian) state

trial state for the filling $\nu = 2/5$

supposed to have non-Abelian excitations, in competition with the Abelian hierarchy trial state at $\nu = 2/5$

associated with the non-unitary $M(5,3)$ conformal field theory
$\longrightarrow$ gapless state ?*

root partition $110001100011000\ldots.$ (bosonic: $200200200200\ldots2$)

$$\text{exact ground state of} \quad H = \overbrace{V_3^3 + V_5^3}^{\text{3-body}}_{J_{\text{rel}} = 3,5}$$

* RT, B. A. Bernevig, and M. Greiter, *State counting, Excitations, Criticality: A root partition analysis of the many-body interaction state series*, in preparation.

Quasihole state counting

start from a QH ground state and add n holes (empty orbitals) to the system

Abelian excitations: pure configuration space state counting

Non-Abelian excitations: topologically protected internal space in addition

Pauli principle for (k, r) Jack states for admissible partitions: Not more than k particles in r consecutive orbitals.

numerically: start from a sphere setting with a unique (0 energy) ground state being the wanted QH trial state, increase $2S$ by n and analyze the multiply degenerate 0 energy subspace.

Abelian state counting

hierarchy wave function for $\nu = 2/5^*$, N_1 number of quasielectrons, q number of quasiholes in the hierarchy liquid:

$$\Psi_{[3,2]} = \int D[a,b] \prod_{k<l}^{N_1} \left(\bar{a}_k \bar{b}_l - \bar{a}_l \bar{b}_k\right)^2 \prod_{k=1}^{N_1} \prod_{i=1}^{N} \left(\bar{b}_k \frac{\partial}{\partial u_i} - \bar{a}_k \frac{\partial}{\partial v_i}\right) \Psi_L(u,v),$$

$$2S = \frac{5}{2}N - 4 - \frac{q}{2}, \quad 2(N_1 - 1) + q = N.$$

6 particles, no quasiholes: $2S = 11, \quad (\equiv 110001100011)$.

One flux added:

$2S = 12, \quad q = 2, \quad N_1 = 3 \longrightarrow \binom{N_1+2}{2} = 10$ excitation states

Two fluxes added:

$2S = 13, \quad q = 4, \quad N_1 = 2 \longrightarrow \binom{N_1+4}{4} = 15$ excitation states

*M. Greiter, Phys. Lett. B 336 (1994).

Partition state counting

adapted counting from bosonic Jack partitions:

$$\{\lambda|\lambda_i - \lambda_{i+2} > 2, \quad \lambda_i \in [1, 2S+1]\}$$

$$\underbrace{2002002}_{7} \equiv \{7, 7, 4, 4, 1, 1\}$$

one flux added: 10 excitation states

two fluxes added: 50 excitation states!

numerical check: add quasiholes and extract the root partitions in the 0 energy subspace

perfect matching, counting exceeds the abelian counting $\longrightarrow$ $2/5$ MBI state has non-Abelian excitations and differs from the Abelian hierarchy state

Jack description is correct and partition counting can be applied to the thermodynamic limit

$3/7$ MBI state

new trial state for $\nu = 3/7$

root partition $11001001100\ldots$ (bosonic $20102010\ldots$)

exact ground state of $\quad H = \sum_{i=6}^{12} V_i^4 + V_3^3$

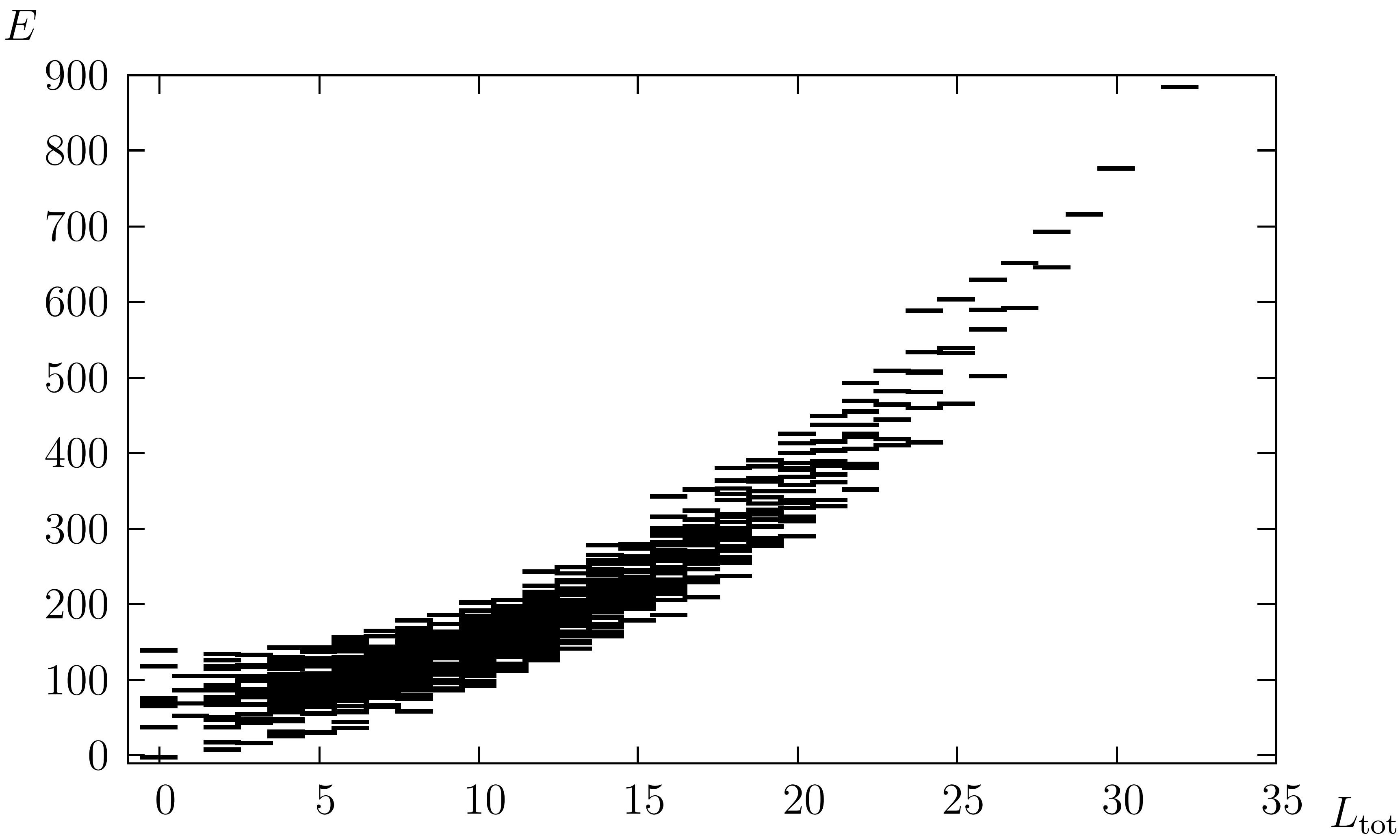
possesses a new type of non-Abelian excitations

state counting from partitions is given by a
Multiple Condition Pauli Principle*

$$\begin{aligned}\{\lambda| \qquad & \lambda_i - \lambda_{i+2} > 1; \\ & \lambda_{i-3} + \lambda_{i-2} + \lambda_{i-1} - 3\lambda_i > 5; \\ & \nexists\ \lambda_i - \lambda_{i+1} = \lambda_{i+1} - \lambda_{i+2} = \lambda_{i+2} - \lambda_{i+3} = 1.\}\end{aligned}$$

* RT, B. A. Bernevig, and M. Greiter, *Multiple Condition Statistics in a Quantum Hall state*, in preparation.

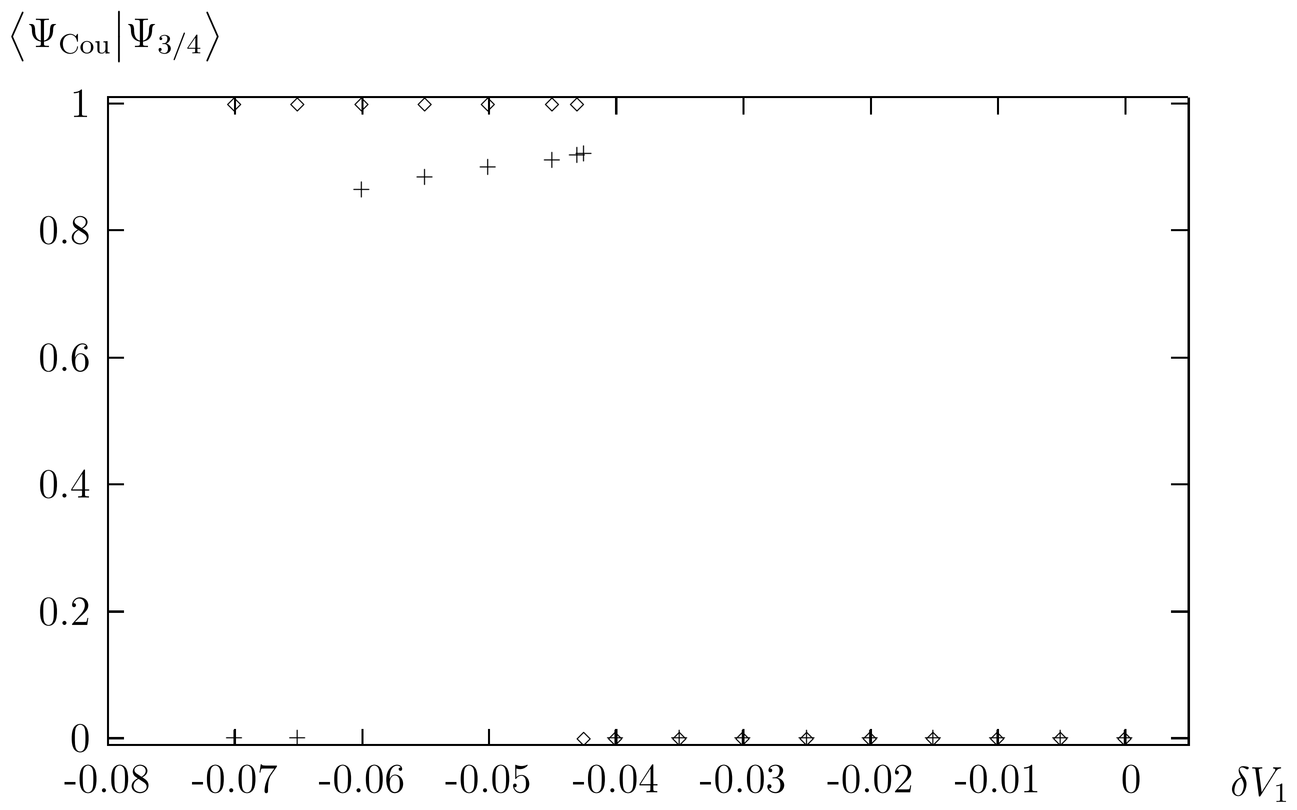
3/7 MBI state



Spectrum of H for $N_p = 8$, $S = 7.5$. The 3/7 MBI state is rotationally invariant and the unique 0 energy ground state.

$3/7$ MBI state

overlap with Coulomb:



5 and 8 particles denoted by $\diamond$ and $+$

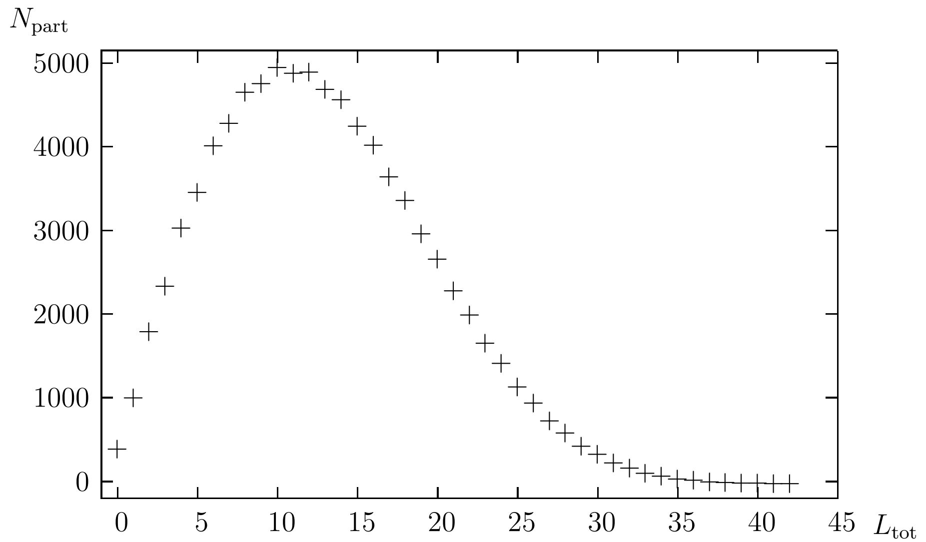
sharp transition at $\delta V_1 = -0.043$

overlap is comparable to other competitive trial states at this filling

3/7 MBI state

L_{tot}	0	1	2	3	4	5	6	7	8	9	10
PP	2	1	4	3	5	4	5	3	3	1	1
Num	2	1	4	3	5	4	5	3	3	1	1

Multiplet-resolved root configuration counting for $N_p = 5$, and 4 holes added to the unique ground state setting.



Multiplet distribution of root partitions for $N_p = 14$, 6 holes added; 2352670 states distributed over multiplets

$3/7$ MBI state

Exact solution for one flux added to a unique N_p particle ground state:

$$N_{\text{rp}}^{3/4}(N_p) = \frac{2}{81}N_p^3 + \frac{5}{27}N_p^2 + \frac{20}{27}N_p + \frac{47}{81}$$

overcounting of cubic contribution $(N_p/3)^3/3!$ expected by configuration space counting by a factor 4

state counting exceeds Abelian configuration space counting already for one flux added

$\longrightarrow$ new type of non-Abelian statistics, non-Abelian hierarchy states

Summary

- **New state series: MBI**

- **Root partition description of Jack and Non-Jack states**

- **Analysis of the $2/5$ MBI state (Gaffnian)**

- **Jack Polynomial description of excitations numerically confirmed**

- **Non-Jack $\nu = 3/7$ MBI state obeying multiple condition Pauli principle**

Perspective: Thermodynamic description of Abelian and Non-Abelian Quantum Hall states; new types of fractional excitations