

Jacks polynomials and unitary conformal field theories

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in collaboration with:

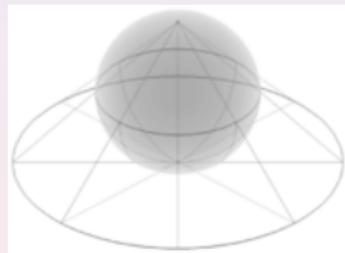
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Preliminaries:

Particles in the lowest Landau Levels on the sphere:



- stereogr. project.: z_i
- N_ϕ : radial magnetic flux
- single particle basis:

$$z^m / (1 + |z|^2 / 4R^2)^{-1 - N_\phi / 2}$$

Many-particles states:

$$\tilde{\Psi}(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) = P_n(z_1, \dots, z_n) \prod_{i=1}^n 1 / (1 + |z_i|^2 / 4R^2)^{-1 - N_\phi / 2}$$

$P_n(z_1, \dots, z_n)$: homogenous polynom. in $\{z_i\}$, $m \leq N_\phi$

quantum Hall ground states

$P_n(\{z_i\})$ translationally and rotationally invariant:

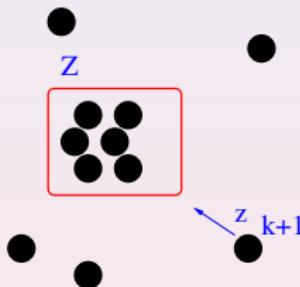
$$\sum_{i=1}^n \partial_i P_n(\{z_i\}) = 0 \quad \text{HW condition}$$

$$\sum_{i=1}^n z_i^2 \partial_i P_n(\{z_i\}) = -N_\phi \left(\sum_i z_i \right) P_n(\{z_i\}) \quad \text{LW condition}$$

$$\sum_{i=1}^n z_i \partial_i P_n(\{z_i\}) = \frac{n N_\phi}{2} P_n(\{z_i\}) \quad \text{total degree}$$

Projection Hamiltonians zero-energy states and (k, r)-Clustering properties

(S.Simon, E.H.Rezayi, N.R.Cooper cond-mat/0608..,cond-mat/0701..)



- L_{k+1} : relative angular momentum ($k + 1$) cluster
- $L_{k+1} \geq r$

$$P_n^{(k,r)}(z_1 = z_2 = \dots = z_{k+1}, z_{k+2}, \dots, z_n) = 0$$

$$P_n^{(k,r)}(Z, z_{k+1}, \dots, z_n) = \prod_{i=k+1}^n (Z - z_i)^r P_{n-k}^{(k,r)}(z_{k+1}, z_{k+2}, \dots, z_n)$$

$$N_\phi = \frac{r(n-k)}{k} \quad \text{Highest density zero energy eigenstate of } \mathcal{P}_k^r$$

Highest density zero-energy (improper) ground states of P_k^r projection Hamiltonians

	$p = 1, 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$	$p = 8$	$p = 9$	$p = 10$
$g = 1$	$J^2 : \frac{1}{2}$	$J^4 : \frac{1}{4}$	$J^4 : \frac{1}{4}$	$J^6 : \frac{1}{6}$	$J^6 : \frac{1}{6}$	$J^8 : \frac{1}{8}$	$J^8 : \frac{1}{8}$	$J^{10} : \frac{1}{10}$	$J^{10} : \frac{1}{10}$
$g = 2$	$P: 1$	$G : \frac{2}{3}$	$*H : \frac{1}{2}$	$J^2 : \frac{1}{2}$	$J^2 : \frac{1}{2}$	$PJ^2 : \frac{1}{3}$	$PJ^2 : \frac{1}{3}$	$GJ^2 : \frac{2}{7}$	$*HJ^2 : \frac{1}{4}$
$g = 3$	$R_3 : \frac{3}{2}$	$*P: 1$	$P: 1$						
$g = 4$	$R_4 : 2$	$R_3 : \frac{3}{2}$	$R_3 : \frac{3}{2}$						
$g = 5$	$R_5 : \frac{5}{2}$	$R_4 : 2$	$R_4 : 2$						
$g = 6$	$R_6 : 3$	$R_5 : \frac{5}{2}$	$R_5 : \frac{5}{2}$						

J : Jastrow P : Pfaffian, R_k : Z_k RR, G : Gaffnian, H : Haffnian

(S.Simon, E.H.Rezayi, N.R.Cooper cond-mat/0608378)

FQH-CFT connection

Construct $P_n^{(k,r)}$ by using parafermionic theories:

$$P_n^{(k,r)}(\{z_i\}) \hat{=} \langle \Psi(z_1) \dots \Psi(z_n) \rangle \prod_{i < j} (z_i - z_j)^{2\Delta_1 - \Delta_2}$$

parafermionic theories: 2D FT with CONFORMAL+ Z_k symmetry:

- Global conformal invariance \leftrightarrow HW+LW conditions

primary operator: $\Phi(z) \rightarrow \Phi(w(z)) = w'(z)^{-\Delta_\Phi} \Phi(z)$

- Z_k symmetry: k -clustering

conserved current OPE: $\partial_{\bar{z}} \Psi_q(z) = 0 \quad q \rightarrow Z_k$ charges
 $\Psi_q \Psi_{q'} \sim \Psi_{q+q'} \quad \Psi_q \Psi_{k-q} \sim I$

The parafermionic algebra:

- General form:

$$\Psi_q(z)\Psi_{q'}(w) = \frac{\gamma_{q,q'}}{(z-w)^{\Delta_q + \Delta_{q'} - \Delta_{q+q'}}} [\Psi_{q+q'}(w)]$$

$$\Psi_q(z)\Psi_{k-q}(w) = \frac{1}{(z-w)^{2\Delta_q}} \left(1 + (z-w)^2 \frac{3\Delta_q}{c} T(w) + \dots \right)$$

- properties: includes Virasoro ($T(z)$) with central charge c
- requirements: $\langle \Psi(z_1)..\Psi(z_1) \rangle$ is determined and well defined
 - specify all singular terms in the above fusions
- satisfy associativity: $\Psi(z_1) \underbrace{(\Psi(z_2)\Psi(z_3))}_{z_2 \rightarrow z_3} = \underbrace{(\Psi(z_1)(\Psi(z_2))}_{z_1 \rightarrow z_2} \Psi(z_3)$

Associative solutions:

First associativity requirements:

$$\Delta_q = \frac{rq(k-q)}{2k} \quad P_n^{(k,r)} = \langle \Psi(z_1) \dots \Psi(z_n) \rangle \prod_{i < j} (z_i - z_j)^{r/k}$$

$r = 2$: Z_k Read-Rezayi states

$$\begin{aligned}\Psi_1(z)\Psi_1(w) &= \frac{1}{(z-w)^{2/k}}\Psi_2(w) + \dots \\ \Psi_1(z)\Psi_{k-1}(w) &= \frac{1}{(z-w)^{2(k-1)/k}} \left(I + \frac{k+2}{k}(z-w)^2 T(w) + \dots \right)\end{aligned}$$

- fixed central charge $c = 2(k-1)/k$
- **Unitary** representations: $k = 2$ Ising model, $k = 3$ three states Potts \dots ..

$r > 2$, fixed (negative) c : Non Unitary paraferm.

P.Jacob and P.Mathieu hep-th/0506..

B.Feigin, M.Jimbo, T.Miwa and E.Mukhin math/0112..

A. Bernevig, D.Haldane, V. Gurarie, S.Simon, E. Ardonne 07-09

- Associative solutions for:

$$\Delta_q = \frac{rq(k-q)}{2k} \quad c = (k-1) \left(1 - \frac{k(r-1)^2}{k+r} \right)$$

- related to $WA_{k-1}(k+1, k+r)$ CFTs
 - extended CFTs → conserved currents $s = 2, 3 \dots k$
 - associated to Lie simple algebra A_{k-1}
- $P_n^{(k,r)} \rightarrow$ single Jack polynomials

Jack polynomials at negative parameter $\alpha = -(k+1)/(r-1)$

Homogenous symmetric polynomials of degree M

$$J_\lambda^\alpha = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu}(\alpha) m_\mu$$

- $\lambda = [\lambda_1, \lambda_2 \dots \lambda_n]$ partition of M
- $m_\lambda = \text{Symm}(z_1^{\lambda_1} \cdots z_n^{\lambda_n})$
- eigenfunctions of CS Hamiltonians

Exemple: $n = 3, \lambda = [2, 1]$

$$\begin{aligned} J_{[2,1]}^\alpha &= m_{[2,1]} + \frac{6}{2+\alpha} m_{[1,1,1]} \\ &= z_1^2 z_2 + z_1^2 z_3 + z_2^2 z_1 + \dots + \frac{6}{2+\alpha} z_1 z_2 z_3 \end{aligned}$$

k clustering and Jacks

(B.Feigin, M.Jimbo, T.Miwa and E.Mukhin math/0112..)

$k+1$ and $r-1$ co-prime. λ is (k, r, n) -admissible if

$$\lambda_i - \lambda_{i+k} \geq r \quad (1 \leq i \leq n).$$

- the coefficients $u_{\lambda\mu}(\alpha)$ does not have a pole for negative values of $\alpha = -(k+1)/(r-1)$.
- $J_{\lambda}^{-(k+1)/(r-1)}(z_1, \dots, z_n)$ vanishes when $z_1 = z_2 = \dots = z_{k+1}$.
- The space $J_{\lambda}^{-(k+1)}$ is a basis for k -clustering poly

General result:

$$P_n^{(k,r)}(\{z_i\}) = J_{\lambda}^{-(k+1)/(r-1)}(\{z_i\})$$
$$\lambda = [\underbrace{N_{\phi}, N_{\phi}, \dots}_{\text{k times}}, \underbrace{N_{\phi} - r, N_{\phi} - r, \dots}_{\text{k times}}, \dots, \underbrace{r, r, \dots}_{\text{k times}}];$$

(B. Bernevig, D.Haldane 07-09)

(B. Bernevig, N.Regnault cond-mat/0902..)

Unitary vs Non-Unitary parafermions

Non-Unitary CFTs cannot describe gapped systems

(N. Read: cond-mat/0711.../0805.../0807...)

..but Unitary parafermions for $r > 3$ exists!

associative solutions with c as free parameter

- $k = 2, r = 6, 10, 14: \rightarrow WB_{(r-2)/4}$ theories
- k arbitrary $r = 4$: unitary sequence associated to coset $SO_m \times SO_2 / SO_{m+2}$
 - (V.Fateev,A.Zamolodchikov (86-87), P. Furlan, R.Paunov,I.V Todorov(92))
 - (V. Dotsenko, J.Jacobsen, R.S. (02-03))
 - (V.Dotsenko, B. Estienne (06-08))
- $k = 3, r = 8$: a new associative solution
 - (V. Dotsenko, R.S. (05))

$$k = 2 \ r = 6$$

$\Delta = 3/2$, Superconformal $N = 1$ algebra:

$$\Psi(z)\Psi(0) = \frac{1}{z^3} \left(1 + z^2 \frac{3}{c} T(0) \right)$$

Unitary sequence:

$$c = \frac{3}{2} \left(1 - \frac{12}{m(m+1)} \right) \quad m = 3, 4, \dots$$

$c = 7/10$ ($m = 3$) \rightarrow Tri-critical Ising model
(concides minimal model $M(4, 5)$)

Superconformal current correlators

(S. Simon hep-th/0811..)

We found the following expansion for the $n = 4$ particles function:

$$P_4^{(k=2,r=6)}(\{z_i\}) = J_{[6,6]}^{-3/5}(\{z_i\}) + \frac{3(21+4c)}{14c} J_{[6,4,2]}^{-2}(\{z_i\})$$

- $c = -21/4$: result coincide with the **non-unitary** $M(3,8)$ models
- correction term (**non-unitary**) → (**unitary**) correlation function:
it is a Laughlin state $\nu = 1/2$ (is unique as
 $\rightarrow N_\phi = 6 = 2(n-1)/1$)

$$P_4^{(k=2,r=6)}(\{z_i\}) = \underbrace{A_4^1(\{z_i\})}_{k=2 \text{ clustering}} + \frac{1}{c} \underbrace{A_4^2(\{z_i\})}_{k-1=1 \text{ clustering}} .$$

$r = 10 : WB_2$ theory

$$\begin{aligned}\Psi(z)\Psi(w) &= \frac{1}{(z-w)^5} + \frac{1}{(z-w)^3} \frac{5}{c} T(w) + \frac{5}{2c(z-w)^2} \partial T(w) + \dots \\ &+ \frac{1}{z-w} \left(\frac{135}{2c(22+5c)} \Lambda(w) + \frac{15(c-1)}{4(22+5c)} \partial^2 T(w) + \gamma W(w) \right).\end{aligned}$$

$\Psi\Psi \rightarrow I + W$: Others primary operators in the fusion.

Ψ still Abelian!

Unitary sequence:

$$c = \frac{5}{2} \left(1 - \frac{12}{m(m+1)} \right) \quad m = 3, 4, \dots$$

WB_2 correlator

$$P_4^{(k=2,r=10)}(\{z_i\}) = J_{[10,10]}^{-1/3}(\{z_i\}) + \frac{25(22+5c)}{44c} A_4^2(\{z_i\})$$

- $c = -22/5 \rightarrow$ non-Unitary $M(3, 12)$
- $A_4^2(\{z_i\})$ $k-1=1$ clustering polynomial

$$\begin{aligned} P^2(\{z_i\}) &= J_{[10,8,2]}^{-2}(\{z_i\}) - \frac{3}{10} J_{[10,7,3]}^{-2} + \frac{7}{39} J_{[10,6,4]}^{-2} + \\ &\quad + \frac{25}{546} J_{[9,7,3,1]}^{-2} - \frac{125}{5148} J_{[9,6,4,1]}^{-2} + \frac{5}{567} J_{[8,6,4,2]}^{-2} \end{aligned}$$

- Develops on the space of Laughlin $\nu = 1/2$ with 4 added flux ($N_\phi = 10 = 6 + 4$)
- An observation: associativity \leftrightarrow dimension of $P_n^{(k,r)}$ space.

$$k = 3, r > 4$$

Unitary sequence:

$$c = 2 \left(1 - \frac{12}{m(m+4)} \right) \quad m = 3, 4, \dots$$

- $c = 6/7$: Tricritical Potts model (coincides with minimal $M(6, 7)$)

6-particles correlation functions:

$$P_6^{(k=3, r=4)}(\{z_i\}) = J_{[4,4,4]}^{-4/3}(\{z_i\}) + \frac{4(40 + 7c)}{45c} J_{[4,4,2,2]}^{-3}(\{z_i\})$$

- $c = -40/7 \rightarrow$ non-Unitary $WA_2(4, 6)$
- Correction term: Pfaffian state ($k = 2$ clustering) $\nu = 1$

General structure of unitary $P_n^{(k,r)}$ polynomials

By using current algebra:

$$P_{lk}^{(k,r)}(\{z_i\}) = A_1^{(k,r,lk)}(\{z_i\}) + \frac{1}{c} A_2^{(k,r,lk)}(\{z_i\}) + \dots + \frac{1}{c^{l-1}} A_l^{(k,r,lk)}(\{z_i\})$$

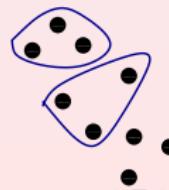
$$A_1^{k,r,lk} = J_{\lambda_r}^{-(k+1)/(r-1)} + \dots$$

$A_l^{k,r,lk}$ satisfy (k, r) - generalized clustering conditions:

(B.Bernevig, D.Haldane cond-mat/0711..)

- $A_j^{k,r,lk}$ satisfy HW and LW conditions
- $A_j^{k,r,lk}$ vanishes when $l - j + 1$ cluster of k particles are formed

$k=3, n=9$



Conclusion

- Discussed the **unitary** generalization of RR states
- Computed correlation function of different parafermionic CFTs
 - Their fine structure vs $(k - r)$ clustering properties
 - Their form in terms of Jacks polynomials
- Everything for these CFTs is known →: Kac Table, fusion rules (monodromy properties)..
- Ex.: $\nu = 1$: Moore Read → Ising model.... $\nu = 1/3$: Tricritical Ising model?