Finite Size Effects in the Kitaev Honeycomb Lattice Model

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Overview

- Finite size effects in Kitaev honeycomb lattice model.
- Other work.
  - Thin torus limit.
  - Fendley quantum loop gas models.
- Numerical tools.
- Conclusions.
Kitaev Honeycomb Lattice Model

- Exhibits two topological phases.
- In the A phases the model is gapped and there is an abelian topological phase \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).
- In the B phase there is a gapless phase.
- In the B phase in the presence of an external magnetic field there is a non-abelian topological phase exhibiting Ising anyonic excitations \( SU(2)_2 \).

\[
H = -J_x \sum_{x\text{-links}} \sigma_i^x \sigma_j^x - J_y \sum_{y\text{-links}} \sigma_i^y \sigma_j^y - J_z \sum_{z\text{-links}} \sigma_i^z \sigma_j^z - \sum_k B \sigma_k
\]
Toric Code Mapping

A phases can be mapped to the Toric code model. Will explore mapping in $A_z$ phase here ($A_x$ and $A_y$ phases are unitarily equivalent).

- In the $A_z$ phase where $J_z = 1$, $J_x = J_y = 0$, the model becomes a system of non interacting $z$-dimers whose ground state degeneracy is $2^{N/2}$ (where $N$ is the number of spins).

\[
H = H_0 + V
\]

\[
H_0 = -J_z \sum_{z- \text{links}} \sigma_i^z \sigma_j^z
\]

\[
V = -J_x \sum_{x- \text{links}} \sigma_i^x \sigma_j^x - J_y \sum_{y- \text{links}} \sigma_i^y \sigma_j^y
\]

- Ground state of $H_0$ made up of ferromagnetic dimers can be treated as effective spins.

\[
| \uparrow \uparrow \uparrow \uparrow \rangle \rightarrow | \uparrow \rangle
\]

\[
| \downarrow \downarrow \rangle \rightarrow | \downarrow \rangle
\]
Non constant elements of the fourth order effective Hamiltonian are given by

\[ \langle a | H^{(4)} | b \rangle = \sum_{jkl} \frac{\langle a | V | j \rangle \langle j | V | k \rangle \langle k | V | l \rangle \langle l | V | b \rangle}{(E^T_{0j} - E^T_{0j})(E^T_{0k} - E^T_{0k})(E^T_{0l} - E^T_{0l})}, \]

where

\[ a \neq b \]
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where

$$a \neq b$$

$$H^{(4)} = \sum_{p} \frac{(J_x \sigma_x^1 \sigma_x^2)}{(4J_z)} + \ldots$$

$$+ \sum_{p} \ldots + \ldots$$

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where

$$a \neq b$$

$$H^{(4)} = \sum_p \frac{(J_x \sigma_1^x \sigma_2^x)(J_x \sigma_4^x \sigma_5^x)}{(4J_z)(8J_z)} + \ldots$$

$$+ \sum_p \ldots + \ldots$$

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where

$$a \neq b$$

$$H^{(4)} = \sum_p \frac{(J_x \sigma_1^x \sigma_2^x)(J_x \sigma_4^x \sigma_5^x)(J_y \sigma_3^y \sigma_4^y)}{(4J_z)(8J_z)(4J_z)} + \ldots$$

$$+ \sum_p \frac{(J_y \sigma_3^y \sigma_4^y)}{(8J_z)} + \ldots$$

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where

$$a \neq b$$

$$H^{(4)} = \sum_p \frac{(J_x \sigma_1^x \sigma_2^x)(J_x \sigma_4^x \sigma_5^x)(J_y \sigma_3^y \sigma_4^y)(J_y \sigma_1^y \sigma_6^y)}{(4J_z)(8J_z)(4J_z)} + \ldots$$

$$+ \sum_p \ldots + \ldots$$

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where \( a \neq b \)

\[ H^{(4)} = \sum_p \frac{(J_x \sigma_1^x \sigma_2^x)(J_x \sigma_4^x \sigma_5^x)(J_y \sigma_3^y \sigma_4^y)(J_y \sigma_1^y \sigma_6^y)}{(4J_z)(8J_z)(4J_z)} + \ldots \]

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where

$$a \neq b$$

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where

\[ a \neq b \]

\[ H^{(4)} = \sum_p \left( \frac{J_x \sigma^x_1 \sigma^x_2}{4J_z} \right) \left( \frac{J_x \sigma^x_4 \sigma^x_5}{8J_z} \right) \left( \frac{J_y \sigma^y_3 \sigma^y_4}{4J_z} \right) \left( \frac{J_y \sigma^y_1 \sigma^y_6}{4J_z} \right) + \ldots + \sum_p \left( \frac{J_x \sigma^x_1 \sigma^x_2}{4J_z} \right) \left( \frac{J_x \sigma^x_4 \sigma^x_5}{8J_z} \right) \left( \frac{J_y \sigma^y_3 \sigma^y_4}{4J_z} \right) \left( \frac{J_y \sigma^y_1 \sigma^y_6}{4J_z} \right) + \ldots + \sum_p \left( \frac{J_x \sigma^x_1 \sigma^x_2}{4J_z} \right) \left( \frac{J_y \sigma^y_3 \sigma^y_4}{4J_z} \right) \left( \frac{J_y \sigma^y_1 \sigma^y_6}{4J_z} \right) + \ldots \]
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\[ + \sum_p \frac{(J_x \sigma_1^x \sigma_2^x)(J_y \sigma_3^y \sigma_4^y)(J_y \sigma_1^y \sigma_6^y)(J_x \sigma_4^x \sigma_5^x)}{(4J_z)(4J_z)(4J_z)} + \ldots \]

\[ + \sum_p \text{[Diagram]} + \ldots \]
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\]

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+ \sum_p \frac{(J_x \sigma_1^x \sigma_2^x)(J_y \sigma_3^y \sigma_4^y)(J_y \sigma_1^y \sigma_6^y)(J_x \sigma_4^x \sigma_5^x)}{(4J_z)(4J_z)(4J_z)} + \ldots
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+ \sum_p \frac{(J_y \sigma_3^y \sigma_4^y)(J_x \sigma_1^x \sigma_2^x)(J_y \sigma_1^y \sigma_6^y)}{(4J_z)(4J_z)(4J_z)} + \ldots
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Toric Code Mapping

Can project each of these terms to the effective spin basis with

\[ \mathcal{P}[\sigma^x \otimes \sigma^y] \rightarrow + \sigma^y, \quad \mathcal{P}[\sigma^x \otimes \sigma^x] \rightarrow + \sigma^x, \]
\[ \mathcal{P}[\sigma^y \otimes \sigma^y] \rightarrow - \sigma^x, \quad \mathcal{P}[\sigma^z \otimes I] \rightarrow - \sigma^z, \]
\[ \mathcal{P}[\sigma^z \otimes \sigma^z] \rightarrow + I. \]

The results of this projection is

\[ H^{(4)}_{\text{eff}} = \sum_p \left( -\frac{8 J_x^2 J_y^2 \sigma_1^z \sigma_2^y \sigma_3^z \sigma_4^y}{128 J_z^3} + \frac{8 J_x^2 J_y^2 \sigma_1^z \sigma_2^y \sigma_3^z \sigma_4^y}{64 J_z^3} - \frac{8 J_x^2 J_y^2 \sigma_1^z \sigma_2^y \sigma_3^z \sigma_4^y}{64 J_z^3} \right) \]

\[ H^{(4)}_{\text{eff}} = -J_{\text{eff}} \sum_p Q_p \]

where

\[ Q_p = \sigma_d^z \sigma_r^y \sigma_u^z \sigma_l^y \]

d=down, r=right, u=up, l=left.

and

\[ J_{\text{eff}} = \frac{J_x^2 J_y^2}{16 |J_z^3|} \]

These operators act on the effective spins.

forth order terms in effective spin picture
• When the lattice of effective spins can be bicolored a suitably chosen unitary transformation can be applied.

\[ U = \prod_{\text{horizontal links}} X_j \prod_{\text{vertical links}} Y_k \]

• When applied we get the Toric code Hamiltonian.

\[ H'_{\text{eff}} = U H_{\text{eff}} U^+ = -J_{\text{eff}} \left( \sum_{\text{vertices}} A_s + \sum_{\text{plaquettes}} B_p \right) \]

\[ B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z \quad \text{and} \quad A_s = \prod_{j \in \text{star}(s)} \sigma_j^x \]

• On a torus if the lattice cannot be bicolored we get the Wen model where the ground state is known to be two fold degenerate.

• In the A phase on a lattice which can be bicolored a mapping can be made to the Toric code thus in the thermodynamic limit we know that the ground state is four fold degenerate on a torus.

Finite Lattice Configurations

Each configuration is labelled by two lattice vectors. \( n = (i + \sqrt{3}j)/2 \) we see that the configurations:

(a) \((2i, 2j)\) and (b)\((2i, 2n)\) contain 8 spins.
(c) \((2i, 4j)\), (d) \((4i, 2j)\) and (e)\((4i, 2n)\) contain 16-spins.
(f ) \((3i, 4j)\) is a 24-spin system.
(g) \((3i, 3n)\) is an 18-spin system.
(h) \((4i, 4j)\) and (i) \((4i, 4n)\) contain 32-spins.
When the effective lattice can be bicolored the fourth order perturbative term can be mapped to the Toric code Hamiltonian.

When it cannot it can be mapped to the Wen model.

This table shows for each configuration in each A phase which it can be mapped to.

\( H_W \) signifies Wen’s model and \( H_K \) signifies the Kitaev’s toric code model.

<table>
<thead>
<tr>
<th>( N )</th>
<th>Configuration</th>
<th>( A_x )</th>
<th>( A_y )</th>
<th>( A_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>( (2i, 2j) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>8</td>
<td>( (2i, 2n) )</td>
<td>( H_K )</td>
<td>( H_K )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>12</td>
<td>( (3i, 2j) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>12</td>
<td>( (3i, 2n) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>16</td>
<td>( (2i, 4j) )</td>
<td>( H_K )</td>
<td>( H_K )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>16</td>
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<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>16</td>
<td>( (4i, 2n) )</td>
<td>( H_K )</td>
<td>( H_K )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>18</td>
<td>( (3i, 3n) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_W )</td>
</tr>
<tr>
<td>20</td>
<td>( (5i, 2j) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>20</td>
<td>( (5i, 2n) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>24</td>
<td>( (2i, 6j) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>24</td>
<td>( (3i, 4j) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
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<td>( (6i, 2n) )</td>
<td>( H_K )</td>
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<td>( H_K )</td>
</tr>
<tr>
<td>28</td>
<td>( (5i, 2j) )</td>
<td>( H_W )</td>
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<td>( H_K )</td>
</tr>
<tr>
<td>28</td>
<td>( (5i, 2n) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_K )</td>
</tr>
<tr>
<td>30</td>
<td>( (3i, 5n) )</td>
<td>( H_W )</td>
<td>( H_W )</td>
<td>( H_W )</td>
</tr>
</tbody>
</table>
For the 8 spin \((2i, 2n)\) configuration there exist the second order non-constant terms given by

\[
\frac{\langle a | H^{(2)} | b \rangle}{\sum_j \langle a | V | j \rangle \langle j | V | b \rangle} = \sum_j \frac{\langle a | V | j \rangle \langle j | V | b \rangle}{E_0 - E_j},
\]

where

\[
H^{(2)} = \frac{1}{2 |J_z|} \left[ J_{x}^2 (\sigma_1^{x} \sigma_2^{x} + \sigma_3^{x} \sigma_4^{x}) + J_{y}^2 (\sigma_2^{x} \sigma_3^{x} + \sigma_1^{x} \sigma_4^{x}) \right].
\]

Also for any \((ai, 2j)\) configuration in the \(A_z\) phase the second order effective system is governed by a simple Ising spin chain Hamiltonian:

\[
H^{(2)} = \frac{1}{2 |J_z|} \sum_{n=1}^{N/2} \sigma_n^{y} \sigma_{n+1}^{y}.
\]
Third Order Terms

Third order terms are found in much the same way as the second order terms. For \((a_i, 2j)\) configurations with \(a > 2\) in the \(A_x\) and \(A_y\) phases we have third order non-constant terms as illustrated in the figure below.

\[
\langle a | H^{(3)} | b \rangle = \sum_{j,k} \frac{\langle a | V | j \rangle \langle j | V | k \rangle \langle k | V | b \rangle}{(E_0 - E_j)(E_0 - E_k)}
\]

In all the \(A\) phases of the 18-spin \((3i, 3n)\) configuration there are third order non-constant terms. This 18-spin system is \(3 \times 3\) plaquettes and it cannot be mapped to the Toric code in any of its \(A\) phases.

graphical representations of two of the six third-order finite size corrections terms for the 18-spin \((3i, 3n)\) configuration.
Fourth Order Terms

We will now look at additional non-constant fourth order terms which appear in finite sized systems. As an example we consider the 16-spin (2i, 4j) configuration in the $A_n$ phase.

Some different four terms sequences that non-trivially connect up the dimer basis vectors on the 16-spin (2i, 4j) configuration lattice. Type (a) is a plaquette term $Q_n$ and is valid for all non-horizontal configurations. Types (b) and (c) are horizontal string terms $R_n$ and $Z_n$ respectively. Type (d) and (e) are vertical strings $Y_n$ and $A_n$ respectively. Types (f) and (g) are vertical $X_n$ strings.

\[
H_{FS}^{(4)} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_{n} (Q_n + R_n - 5A_n) - \frac{J_x^2 J_y^2}{16|J_z|^3} \sum_{n} (Z_n + 5Y_n)
\]

\[
-\frac{5}{16|J_z|^3} (J_x^4 \sum_{n=1}^{2} X_n + J_y^4 \sum_{n=3}^{4} X_n)
\]
Numerical Case Study

- Taking a 24-spin $(6i,4j)$ toroidal configuration in the $A_2$ phase.
- Effective Hamiltonian given by:

\[
H_{\text{eff}} = cI + J_{\text{eff}}(H_K + H_{FS}^{(4)}) + O(J^6) + \ldots
\]

$H_K$ is toric code Hamiltonian on effective spins

\[
H_{FS}^{(4)} = 5(\quad )
\]

\[
J_{\text{eff}} = \frac{J^4}{16|J_z|^3} \quad \text{with} \quad J = J_x = J_y
\]

- For small values of $J$ diagonalise the full Hamiltonian with everything raised by dividing by $J_{\text{eff}}$.
- Diagonalised effective fourth order finite size terms and subtracted.
- Expect to be left with toric code spectrum and sixth order effects.
- $\sigma(M)$ is defined as appropriately ordered spectrum of operator $M$.

\[
\frac{\sigma(H) - E_0}{J_{\text{eff}}} = \sigma(H_{FS}^{(4)}) = \sigma(H_K) + O(J^2).
\]
Numerical Case Study

- Taking a 24-spin (6i,4j) toroidal configuration in the A2 phase.
- Effective Hamiltonian given by:

\[
H_{\text{eff}} = cI + J_{\text{eff}} \left( H_K + H_{FS}^{(4)} \right) + O(J^6) + \ldots
\]

where \( H_K \) is toric code Hamiltonian on effective spins.

\[
H_{FS}^{(4)} = 5\left( \sum_{n} Y_n \right)
\]

\[
J_{\text{eff}} = \frac{J^4}{16|J_z|^3}
\]

with \( J = J_x = J_y \)

- For small values of \( J \) diagonalise the full Hamiltonian with everything raised by dividing by \( J_{\text{eff}} \).
- Diagonalised effective fourth order finite size terms and subtracted.
- Expect to be left with toric code spectrum and sixth order effects.
- \( \sigma(M) \) is defined as appropriately ordered spectrum of operator \( M \).

\[
\frac{\sigma(II) - E_0}{J_{\text{eff}}} - \sigma(H_{FS}^{(4)}) = \sigma(H_K) + O(J^2).
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Numerical Case Study

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H_{FS}^{(4)} = 5 \left( \sum_n Y_n - \sum_n A_n \right)
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\frac{\sigma(H)}{J_{\text{eff}}} - E_0 = \sigma(H_{FS}^{(4)}) = \sigma(H_K) + O(J^2).
\]
Avoiding Finite Size Effects

- The smallest finite configuration where the fourth-order perturbative expansion is equivalent to the toric code Hamiltonian with no lower order terms is the 36-spin (3i,6j) configuration.
- An interesting observation is that this is also the smallest configuration satisfying the requirement that plaquette terms share only one effective spin.
Thin Torus Limit

- Model compressed in z-link direction until only two plaquettes wide.
- Resulting model is Quasi 1D.
- Initial exact diagonalisation calculations were interesting.
- Can be treated more effectively numerically (DMRG) and analytically (CFT).
There is a quantum phase transition at $J_{xx} = 0.5$, $J_{yy} = J_{zz} = 0.25$. 
Xiao-Yong Feng et al., 2007 PRL 98 087204

At this critical point using DMRG (from ALPS) it was found that the energy gap goes to zero linearly with one over the system size.

This relation indicates that the critical point can be described by a conformal field theory.

It was discovered that the central charge is 0.5 indicating that the critical point is described by an Ising conformal field theory.

$S(L) \propto \frac{c}{3} \log(L)$

$c = 0.5 \pm 0.01$
Thin Torus Limit

- Early investigations indicate that when an external magnetic field is switched on a gap opens in the centre of the phase diagram.
Fendley Quantum Loop Gas Models

- Freedman loop gas models have a $d = \sqrt{2}$ barrier.
- Prevents realization of topological phases with $k > 2$ (e.g. Fibonacci anyons).
- Barrier can be overcome by using choosing a different inner product.
- Perturbed toric code.

$$H = H_{toric} + uH_u$$

Can add Jones-Wenzl projectors provides $SO(3)_k$ theory at arbitrary level of theory $k$.

$$d = 2 \cos(\pi/(k+2))$$
$$\Delta_{-1} = 0$$
$$\Delta_0 = 1$$
$$\Delta_{n+1} = d \Delta_n - \Delta_{n-1}$$
Fendley Quantum Loop Gas Models

\[ H = H_{toric} + u H_u \]

\[ H_{toric} = \sum_V W_V + \sum_F \overline{W}_F \]

\[ W_V = \frac{1}{2} (1 - \sigma^{z}_{V1} \sigma^{z}_{V2} \sigma^{z}_{V3} \sigma^{z}_{V4}) \]

\[ \overline{W}_F = \frac{1}{2} (1 - \sigma^{x}_{F1} \sigma^{x}_{F2} \sigma^{x}_{F3} \sigma^{x}_{F4}) \]

\[ H_u = \sum_V W_V \sum_{a=1}^{4} \sigma^{z}_{V_a} + \sum_F \overline{W}_F \sum_{a=1}^{4} \sigma^{x}_{F_a} \]
Numerics

- Table summarising numerical tools.

<table>
<thead>
<tr>
<th>Method</th>
<th>1D systems</th>
<th>2D systems</th>
<th>Fermionic systems (sign)</th>
<th>Parallel</th>
<th>Momentum sectors</th>
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<th>Established technique</th>
<th>Periodic boundary conditions</th>
<th>No. Sphs possible</th>
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*UD* indicates that the method is exact, *S* denotes the number of steps, and *exp(N)* and *poly(N)* indicate scaling with *N*. The table provides a summary of the capabilities of various numerical tools for topological quantum computation.
Numerics

Exact diagonalisation code has been developed.

- The code is written in C.
- It makes use of the PETSc and SLEPc libraries.
- It is capable of running efficiently on massively parallel distributed memory machines.
- Can be used for large range of spin 1/2 systems without modification.
Conclusions

‣ Showed non-constant finite size corrections appearing up to the order of the toric code mapping in Kitaev honeycomb model.

‣ Numerical case study demonstrates accuracy of numerics and demonstrates how corrections can be applied.

‣ Can be used to aid in understanding of fermionization approaches. (arXiv:0903.5211)

‣ Interesting critical point in thin torus limit described by Ising conformal field theory.

‣ Evidence of gap opening with magnetic field.
Recent Publications

G. Kells, N. Moran and J. Vala,
Finite size effects in the Kitaev honeycomb lattice model on a torus,

E. Rico, R. Hübener, S. Montangero, N. Moran, B. Pirvu, J. Vala,
H.J. Briegel,
Valence Bond States: Link models,
Submitted to Ann. of Phys.
arXiv:0811.1049

Exact diagonalisation on Blue Gene/P
was used to verify approximate
DMRG calculations.

G. Kells, A. T. Bolukbasi, V. Lahtinen, J. K. Slingerland, J. K. Pachos, J. Vala,
Topological degeneracy and vortex manipulation in Kitaev's honeycomb model.

Gap Scaling using DMRG

Gap Scaling using ED
Collaborators

Advisor:
- Jiri Vala (NUIM)

Postdoctoral fellow:
- Graham Kells (NUIM)

Acknowledgments:
- Adrian Feiguin (StationQ)
- Paul Fendley (Univ. of Virginia)
- Simon Trebst (StationQ)
Thank You!

Questions?