Finite Size Effects in the Kitaev Honeycomb Lattice Model

Niall Moran

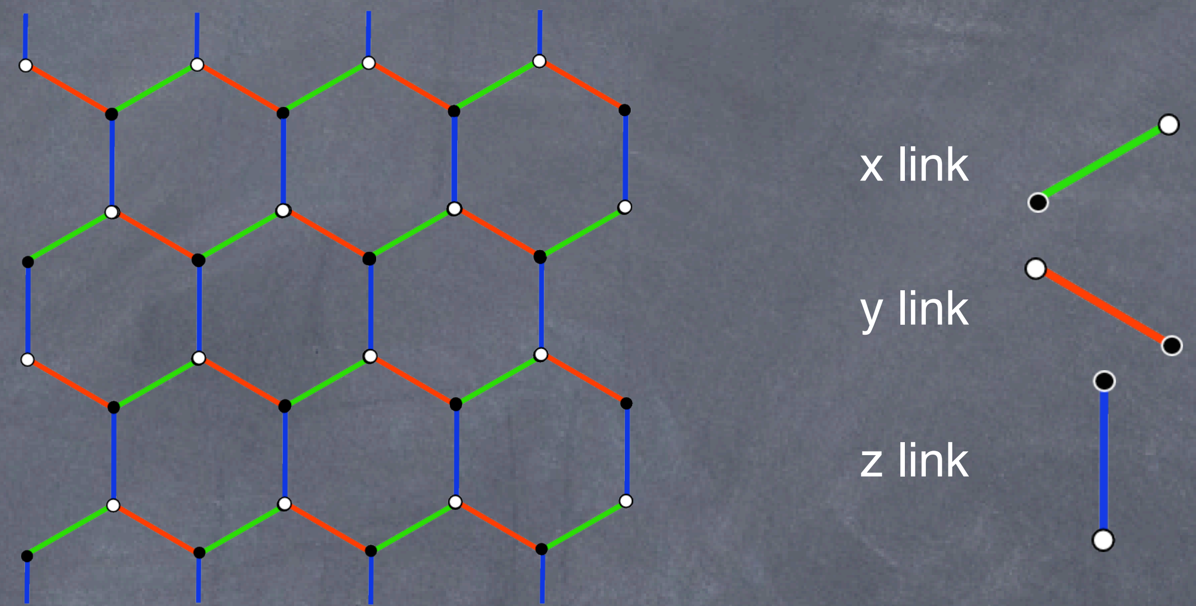
Jiri Vala's Group,
Department of Mathematical Physics,
National University of Ireland, Maynooth.

Overview

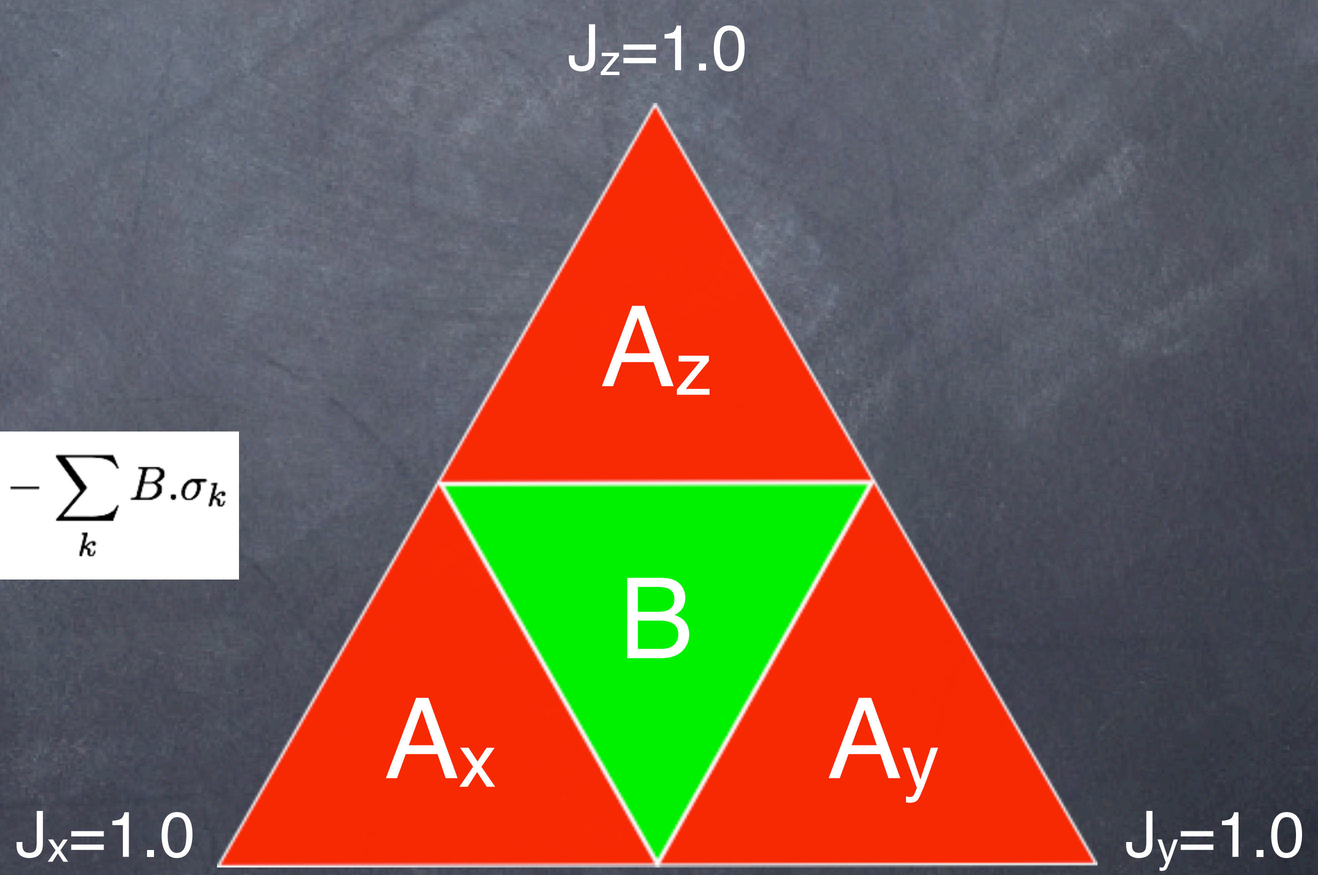
- Finite size effects in Kitaev honeycomb lattice model.
- Other work.
  - Thin torus limit.
  - Fendley quantum loop gas models.
- Numerical tools.
- Conclusions.

Kitaev Honeycomb Lattice Model

- Alexei Kitaev (2006).
- Exhibits two topological phases.
- In the A phases the model is gapped and there is an abelian topological phase $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- In the B phase there is a gapless phase.
- In the B phase in the presence of an external magnetic field there is a non abelian topological phase exhibiting Ising anyonic excitations $SU(2)_2$.



$$H = -J_x \sum_{x-links} \sigma_i^x \sigma_j^x - J_y \sum_{y-links} \sigma_i^y \sigma_j^y - J_z \sum_{z-links} \sigma_i^z \sigma_j^z - \sum_k B.\sigma_k$$



Toric Code Mapping

A phases can be mapped to the Toric code model. Will explore mapping in A_z phase here (A_x and A_y phases are unitarily equivalent).

- In the A_z phase where $J_z = 1$, $J_x = J_y = 0$, the model becomes a system of non interacting z-dimers whose ground state degeneracy is $2^{N/2}$ (where N is the number of spins).

$$H = H_0 + V$$

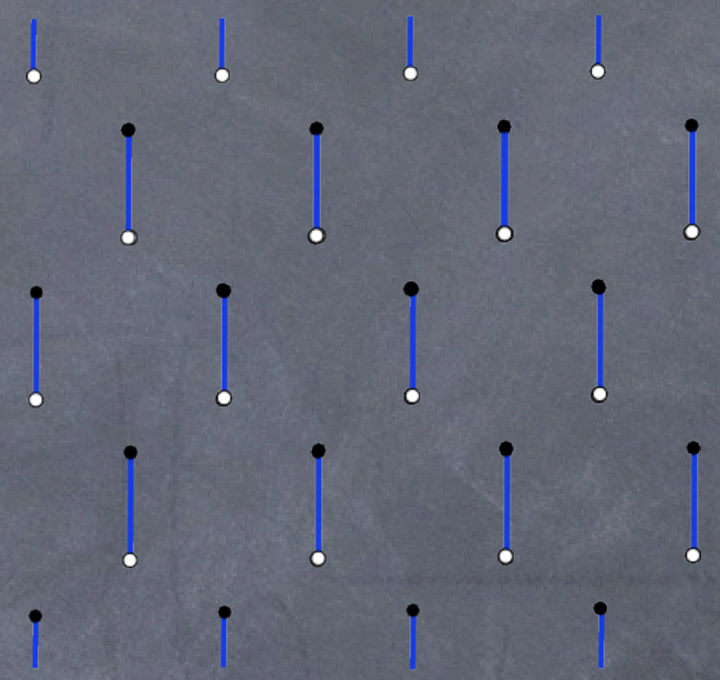
$$H_0 = -J_z \sum_{z-links} \sigma_i^z \sigma_j^z$$

$$V = -J_x \sum_{x-links} \sigma_i^x \sigma_j^x - J_y \sum_{y-links} \sigma_i^y \sigma_j^y$$

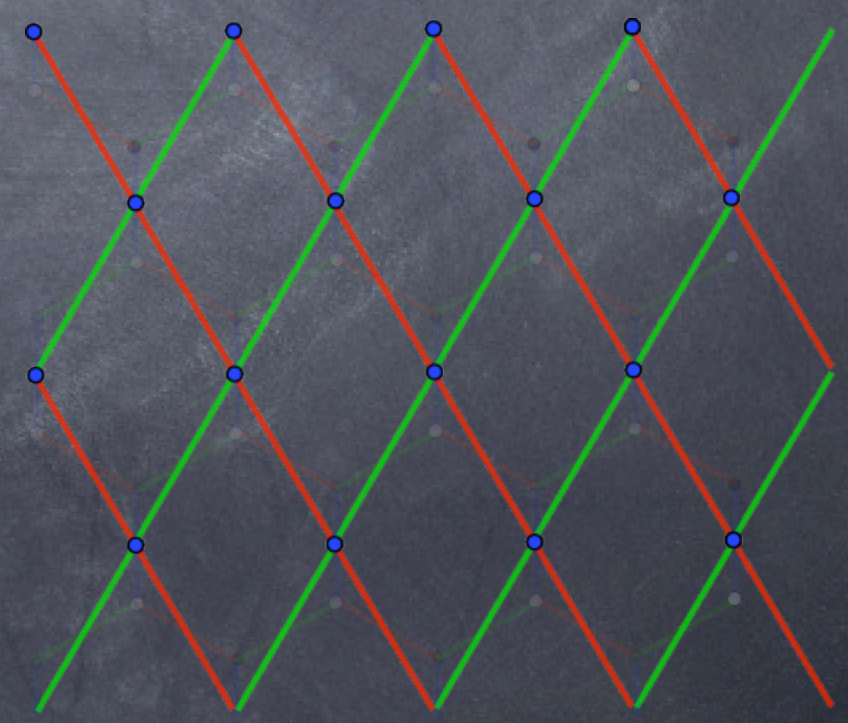
- Ground state of H_0 made up of ferromagnetic dimers can be treated as effective spins.

$$|\uparrow\uparrow> \rightarrow |\uparrow>$$

$$|\downarrow\downarrow> \rightarrow |\downarrow>$$



non-interacting z-dimers.



dimers as effective spins.

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Non constant elements of the fourth order effective Hamiltonian are given by

$$\langle a\,|H^{(4)}|b\rangle = \sum_{jkl} \frac{\langle a\,|V|j\rangle\langle j\,|V|k\rangle\langle k\,|V|l\rangle\langle l\,|V|b\rangle}{(E_0 - E_j)(E_0 - E_k)(E_0 - E_l)},$$

where

$$a \neq b$$

$$H^{(4)} = \sum_p \underline{\hspace{12cm}} + \ldots$$

$$+\sum_p \underline{\hspace{12cm}} + \ldots$$

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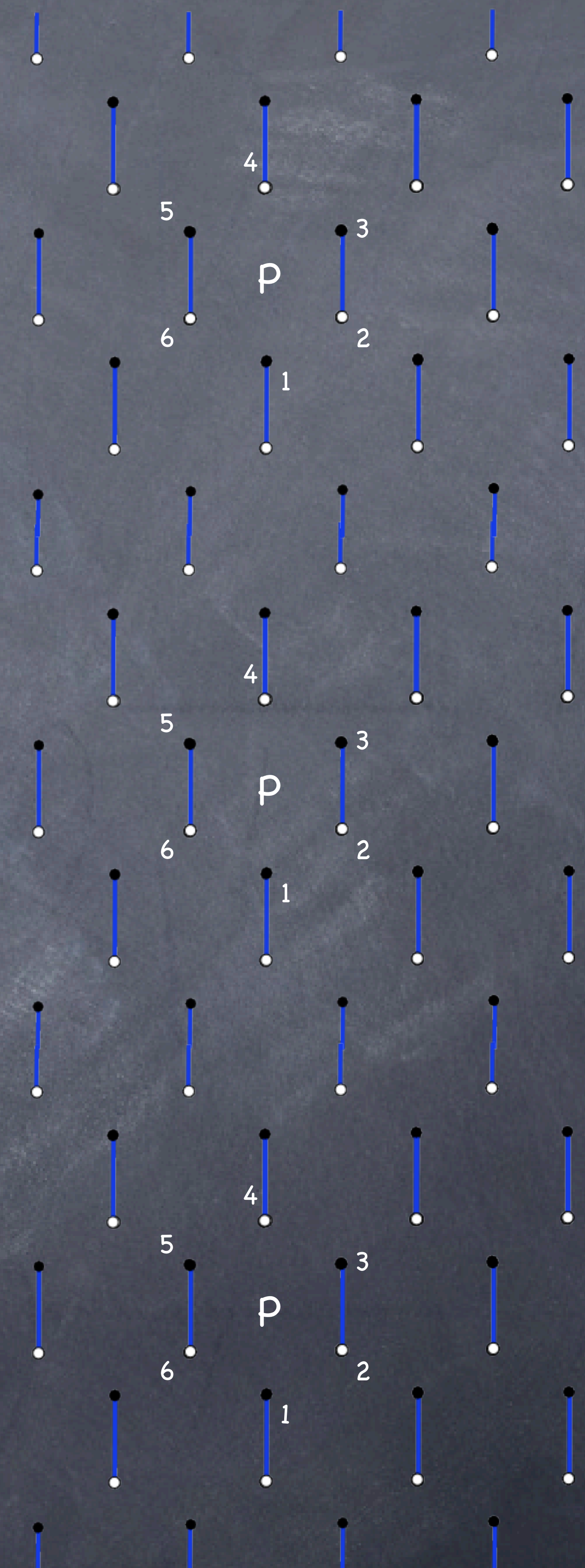

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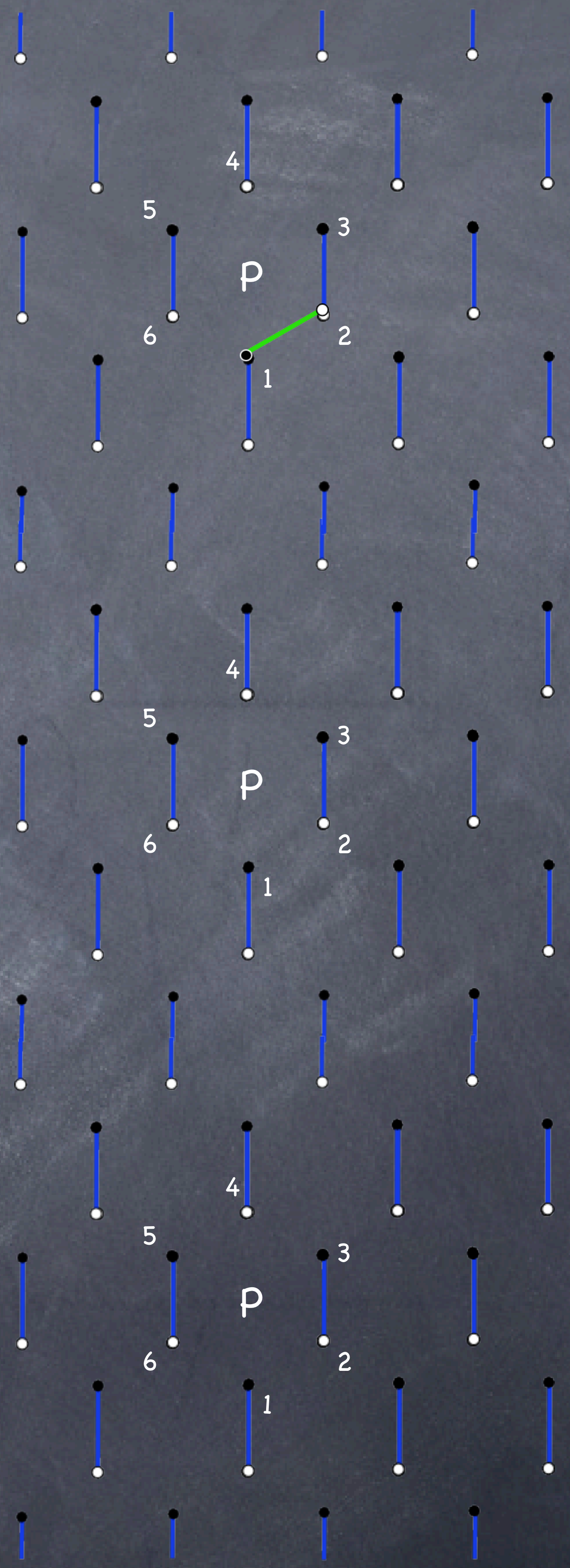

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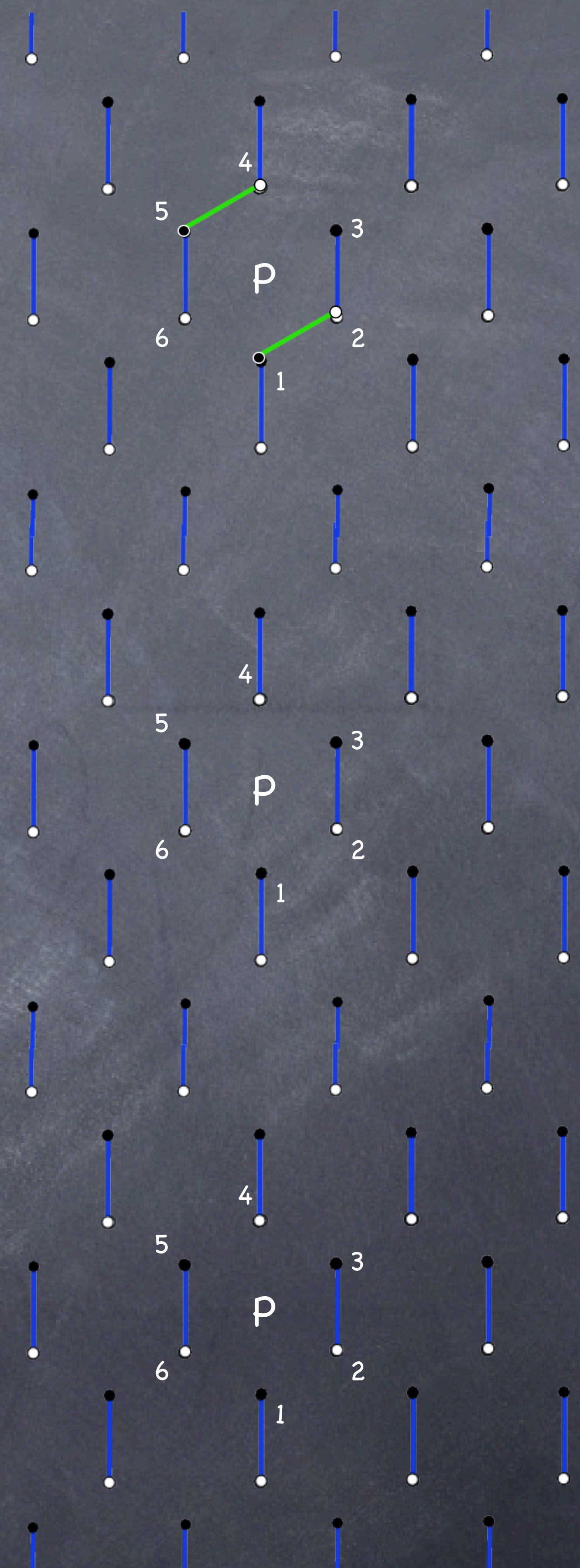

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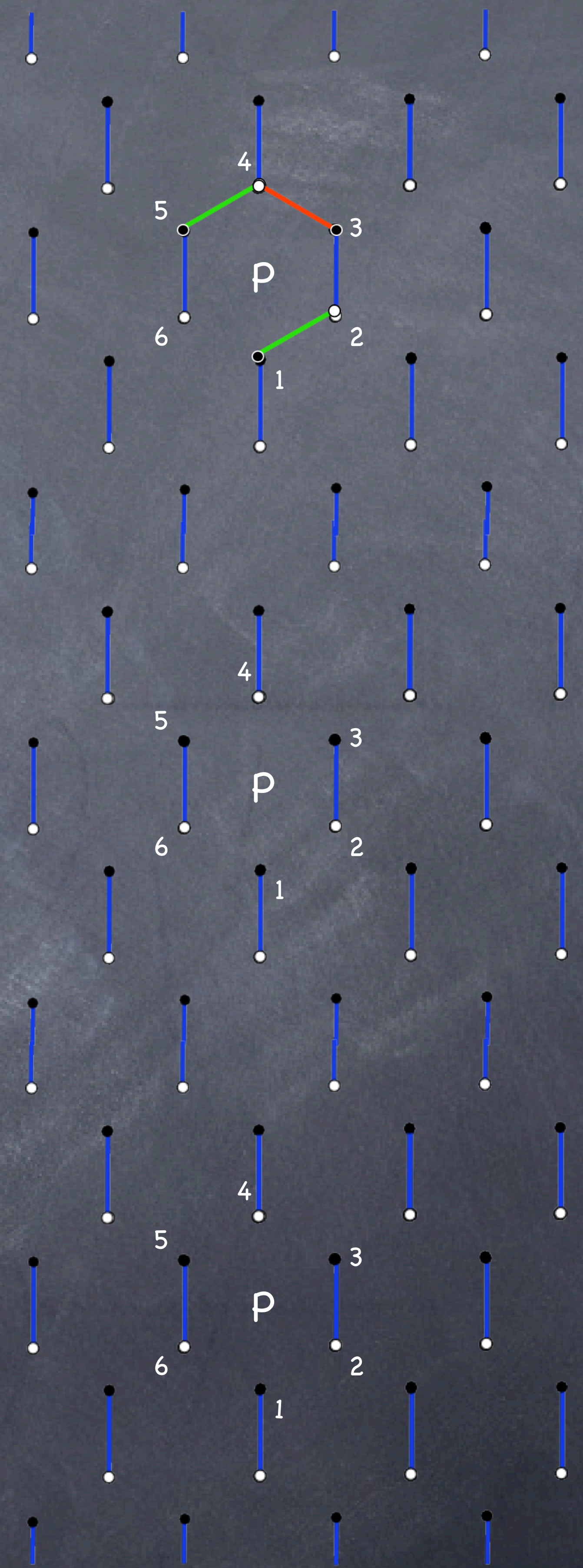

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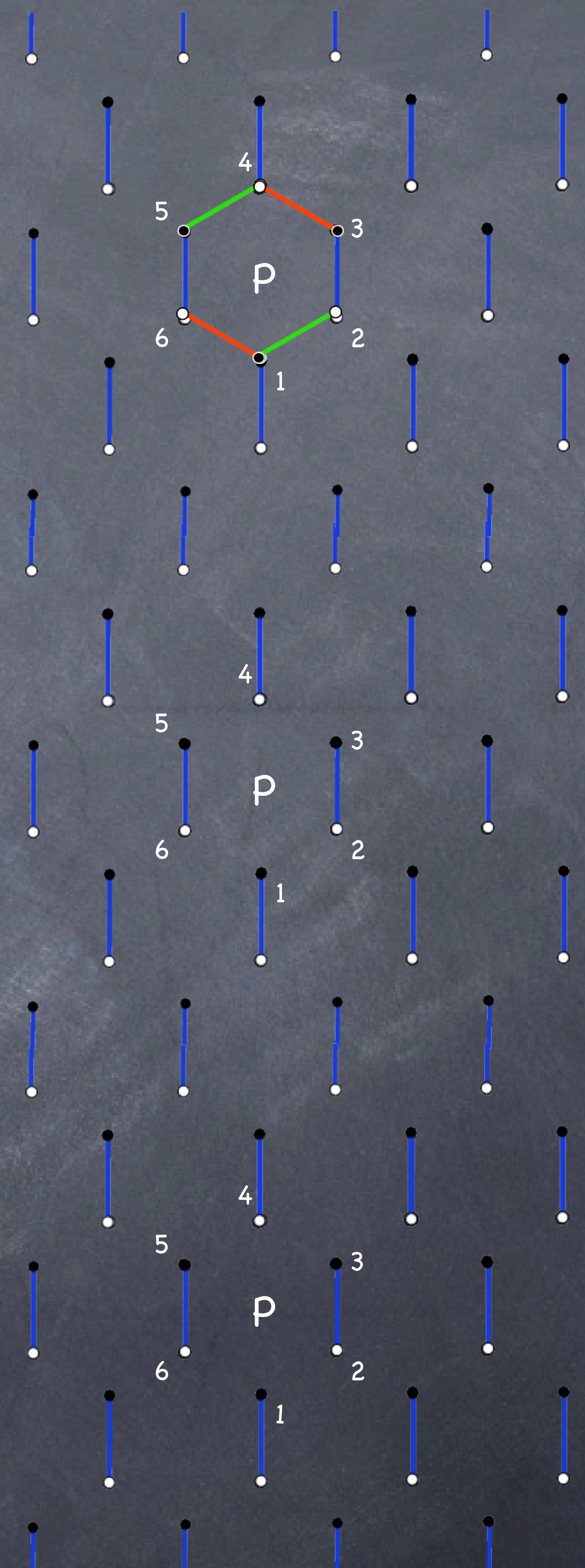

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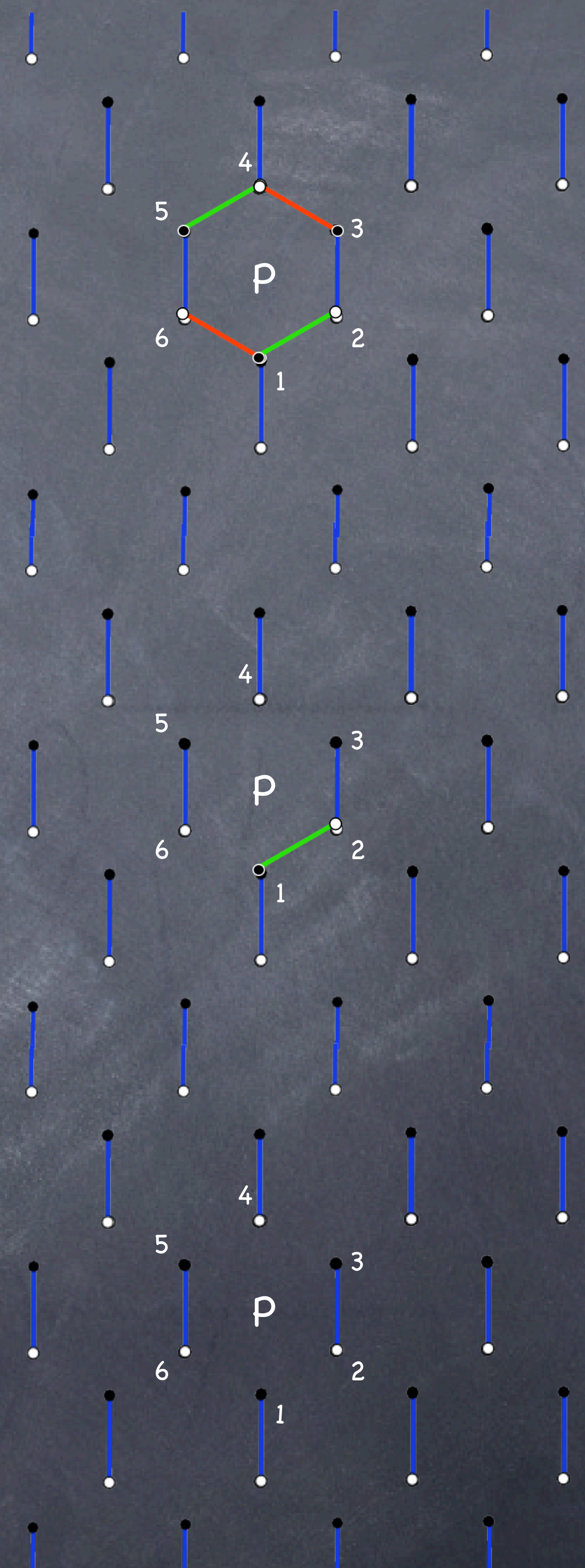

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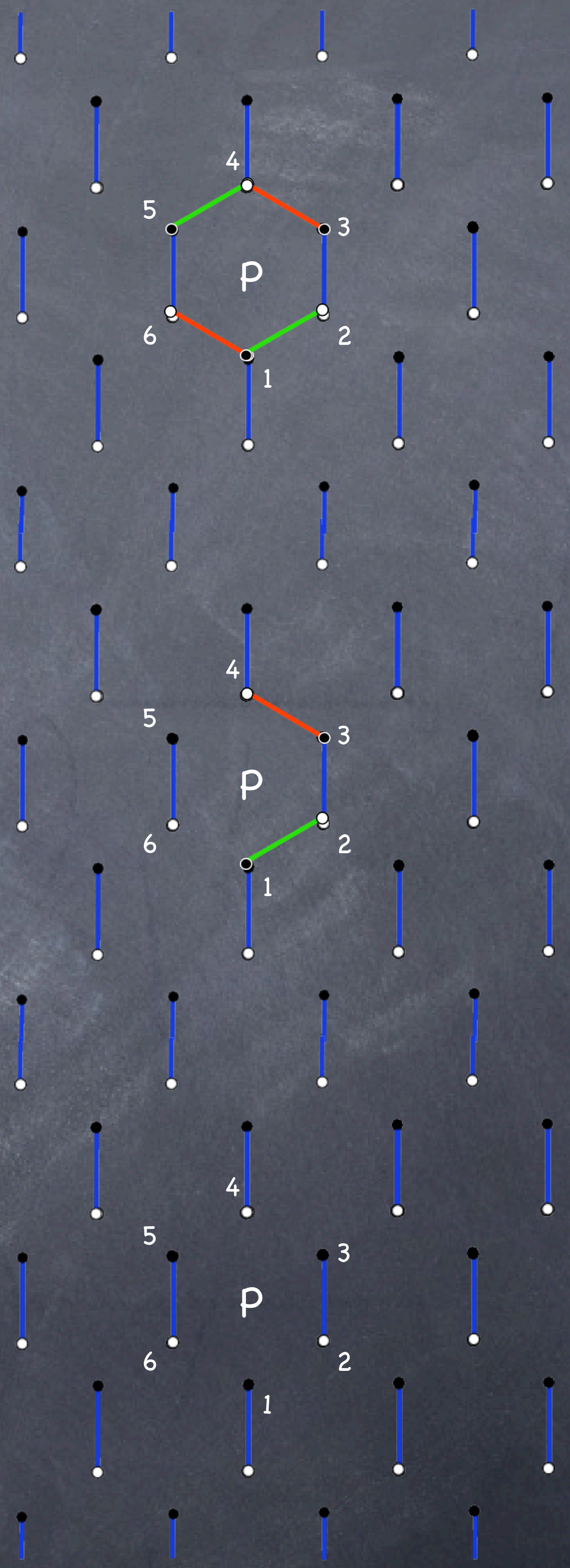

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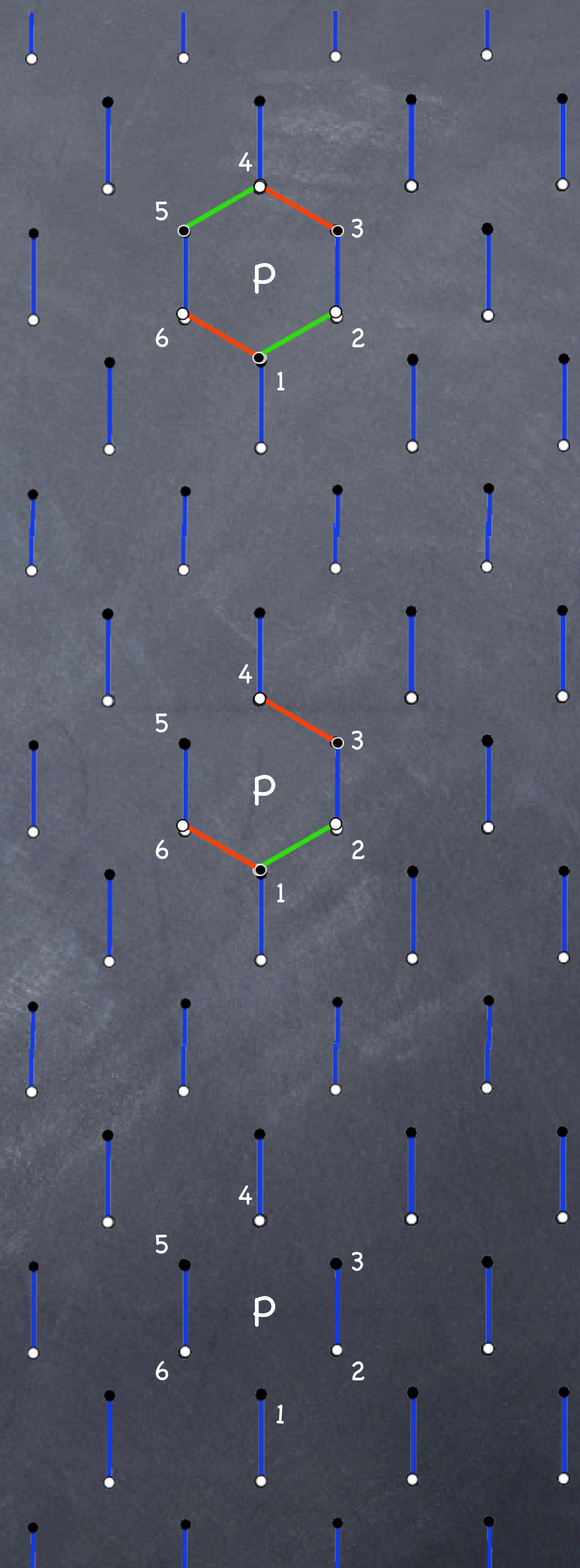

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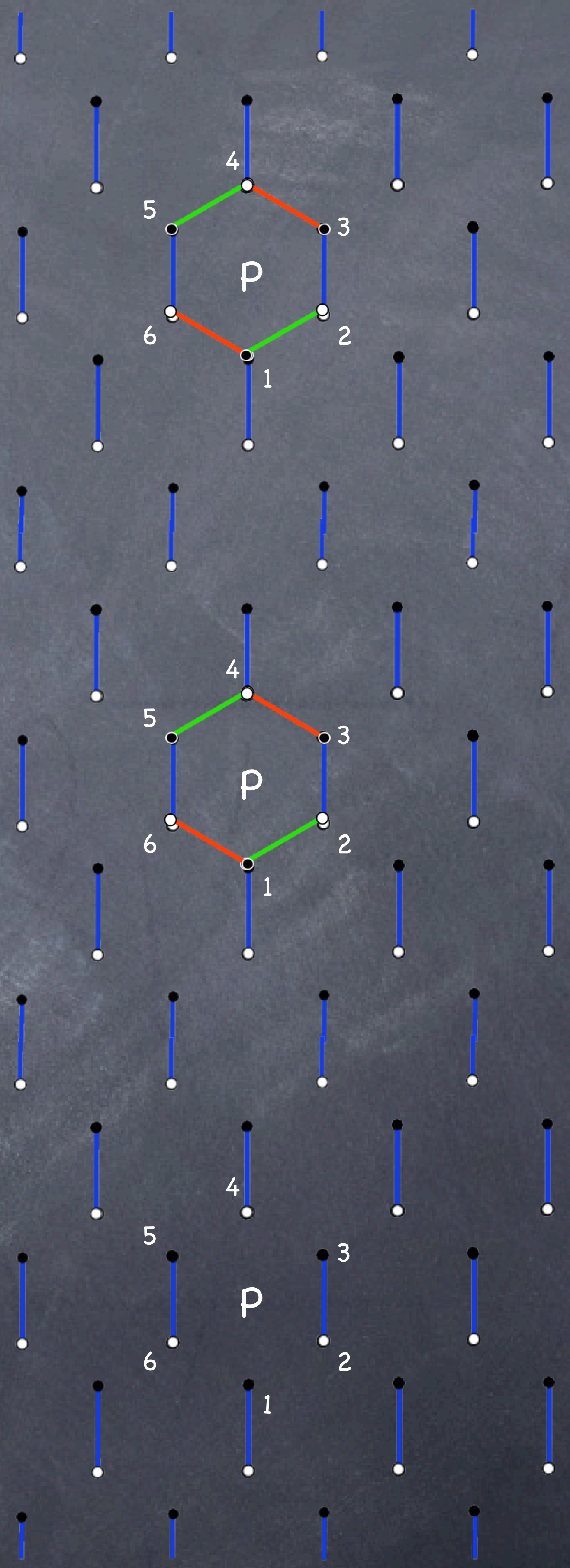

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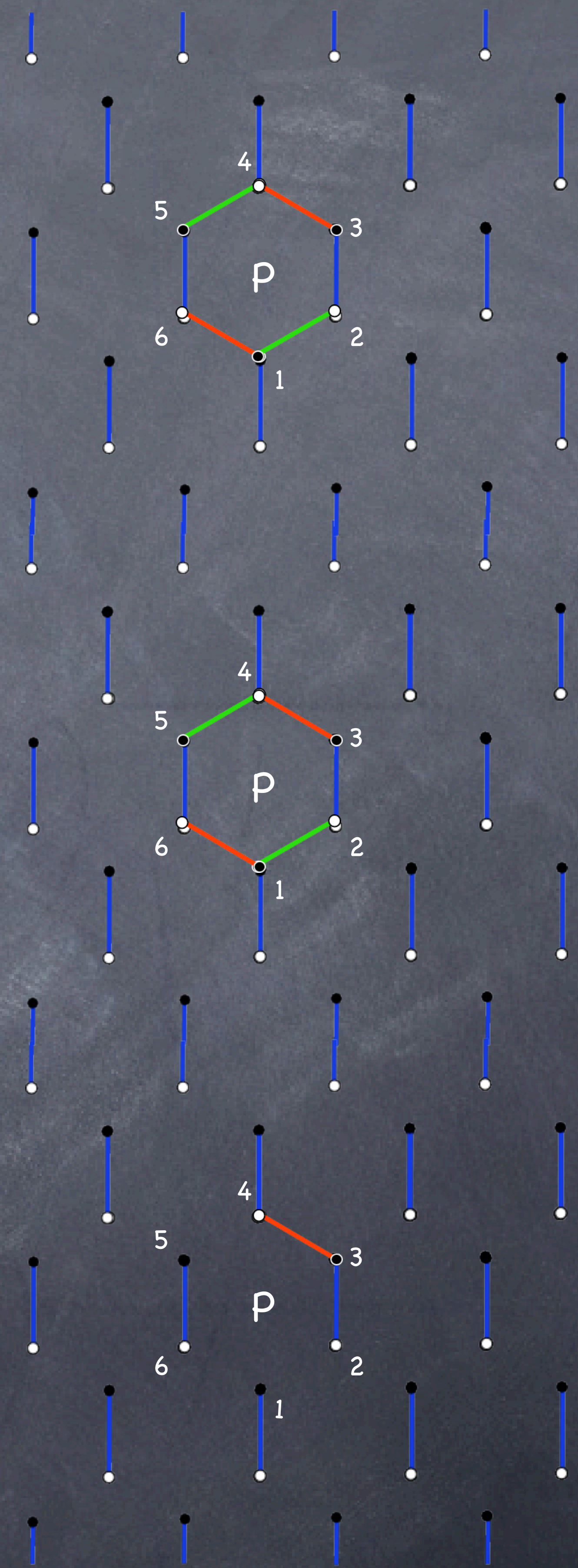

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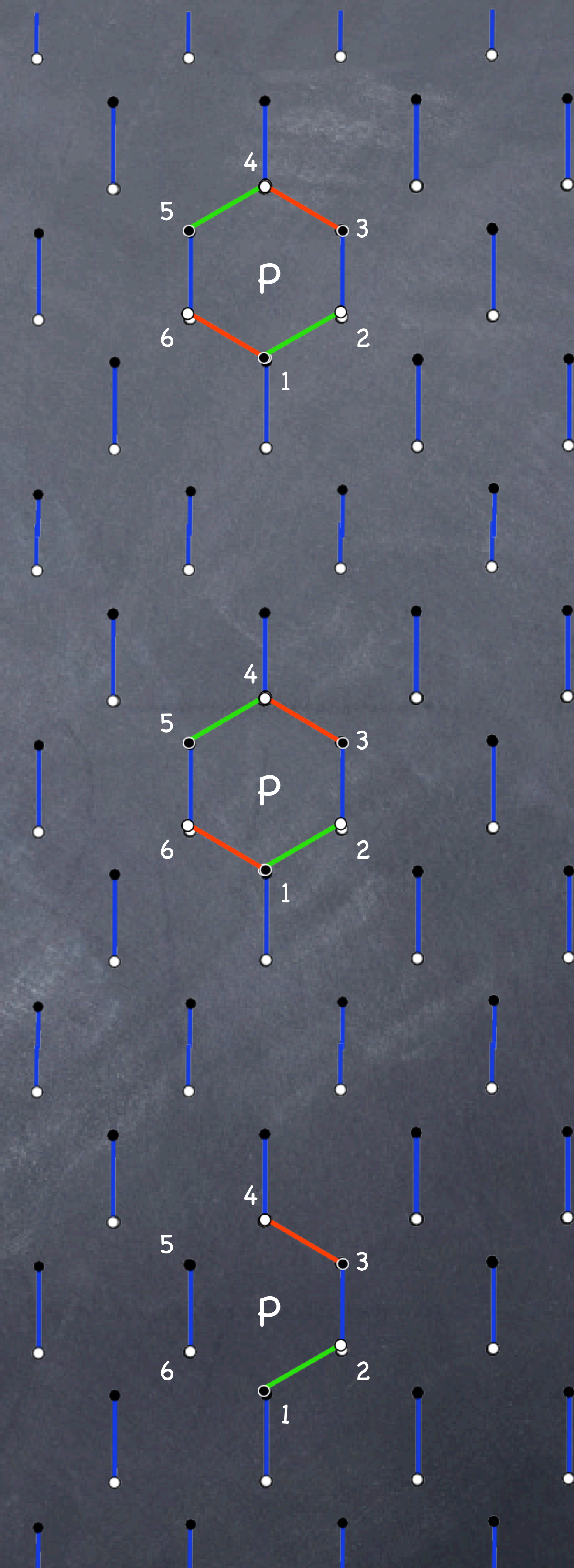

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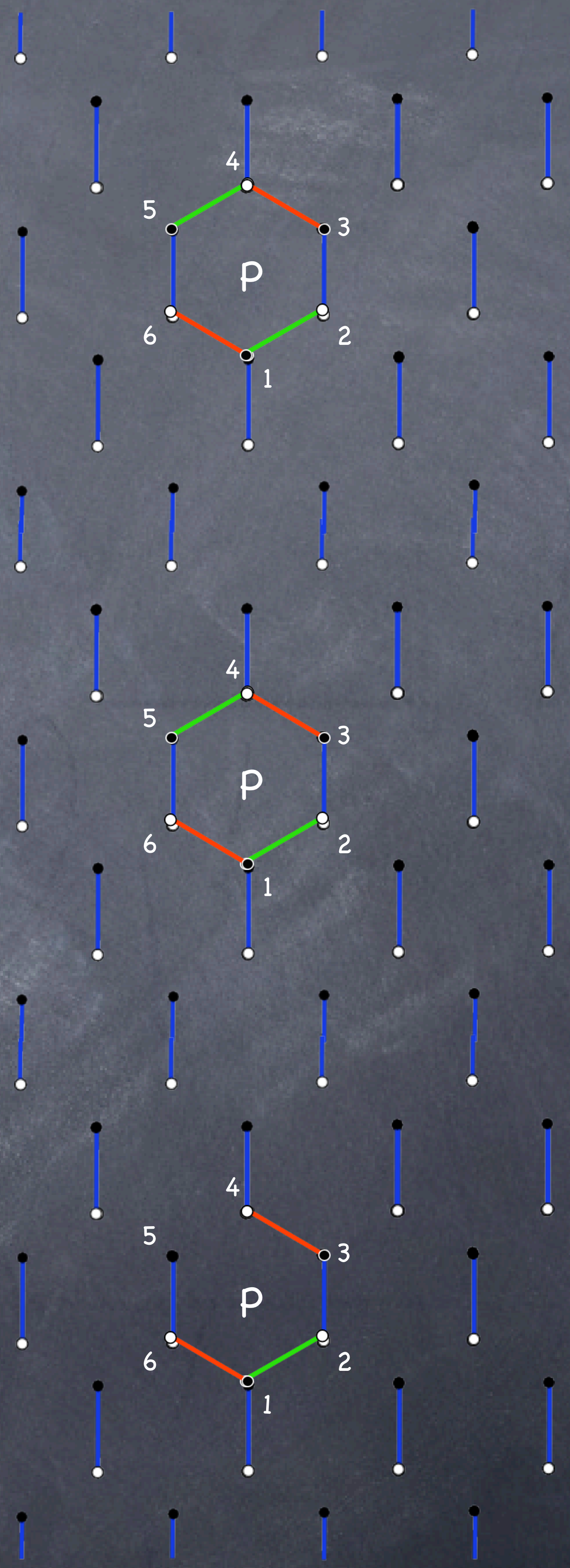

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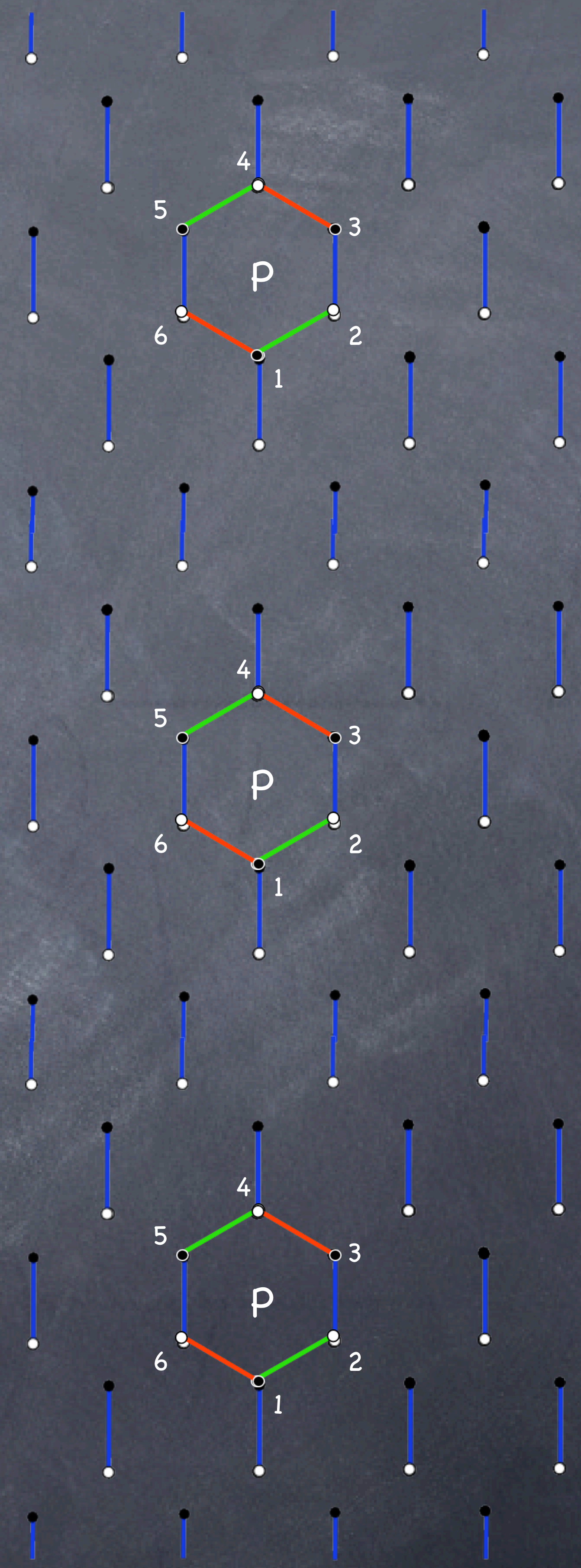
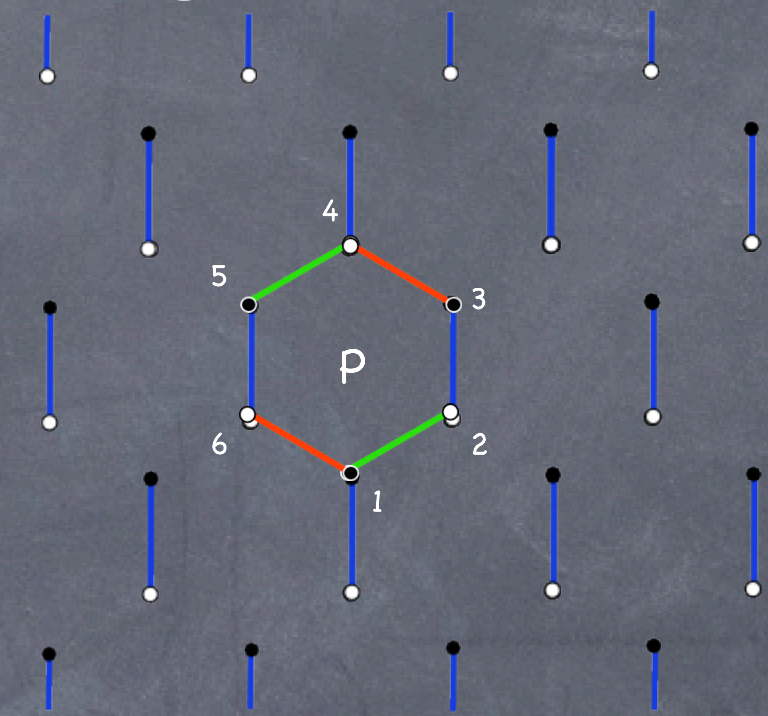


Toric Code Mapping

Can project each of these terms to the effective spin basis with



$$\mathcal{P}[\sigma^x \otimes \sigma^y] \rightarrow +\sigma^y, \qquad \mathcal{P}[\sigma^x \otimes \sigma^x] \rightarrow +\sigma^x,$$
$$\mathcal{P}[\sigma^y \otimes \sigma^y] \rightarrow -\sigma^x, \qquad \mathcal{P}[\sigma^z \otimes I] \rightarrow +\sigma^z,$$
$$\mathcal{P}[\sigma^z \otimes \sigma^z] \rightarrow +I,$$

The results of this projection is

$$H_{eff}^{(4)} = \sum_p \left(-\frac{8J_x^2J_y^2\sigma_1^z\sigma_2^y\sigma_3^z\sigma_4^y}{128J_z^3} + \frac{8J_x^2J_y^2\sigma_1^z\sigma_2^y\sigma_3^z\sigma_4^y}{64J_z^3} - \frac{8J_x^2J_y^2\sigma_1^z\sigma_2^y\sigma_3^z\sigma_4^y}{64J_z^3}\right)$$

$$H_{eff}^{(4)} = -J_{eff}\sum_p Q_p$$

where

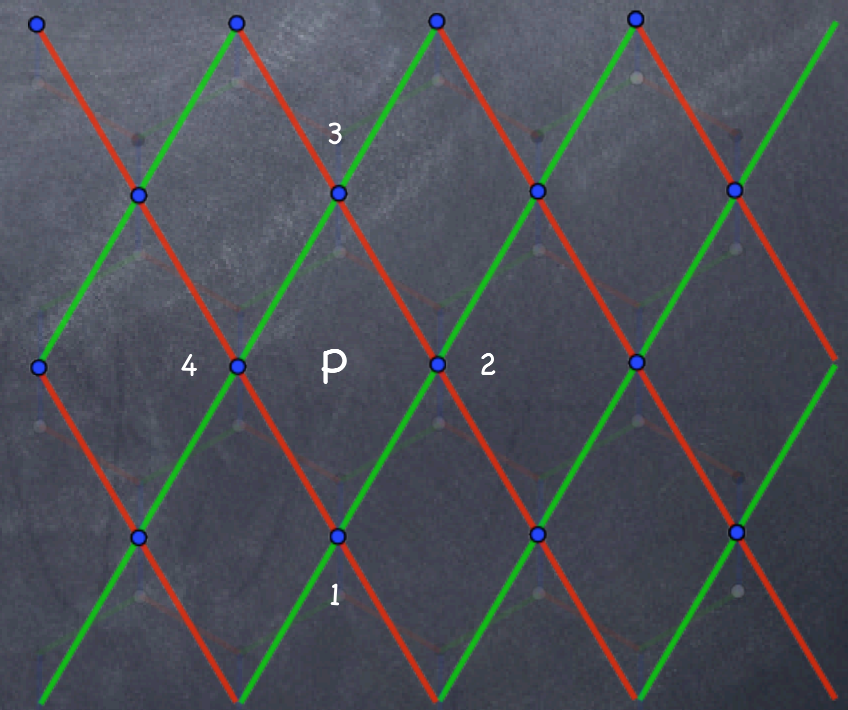
$$Q_p = \sigma_d^z\sigma_r^y\sigma_u^z\sigma_l^y$$

d=down, r=right, u=up,l=left.

and

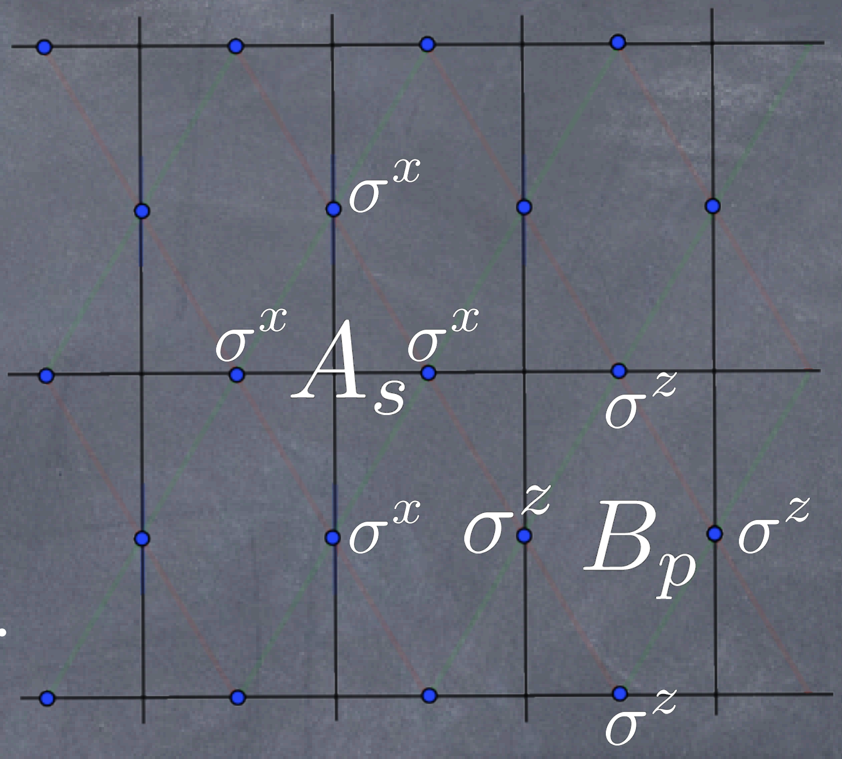
$$J_{eff} = \frac{J_x^2J_y^2}{16|J_z^3|}$$

These operators act on the effective spins.



forth order terms in effective spin picture

- When the lattice of effective spins can be bicolored a suitably chosen unitary transformation can be applied.



$$U = \prod_{horizontal\ links} X_j \prod_{vertical\ links} Y_k$$

- When applied we get the Toric code Hamiltonian.

$$H'_{eff} = UH_{eff}U^+ = -J_{eff}(\sum_{vertices} A_s + \sum_{plaquettes} B_p)$$

$$B_p = \prod_{j \in boundary(p)} \sigma_j^z \quad \text{and} \quad A_s = \prod_{j \in star(s)} \sigma_j^x$$

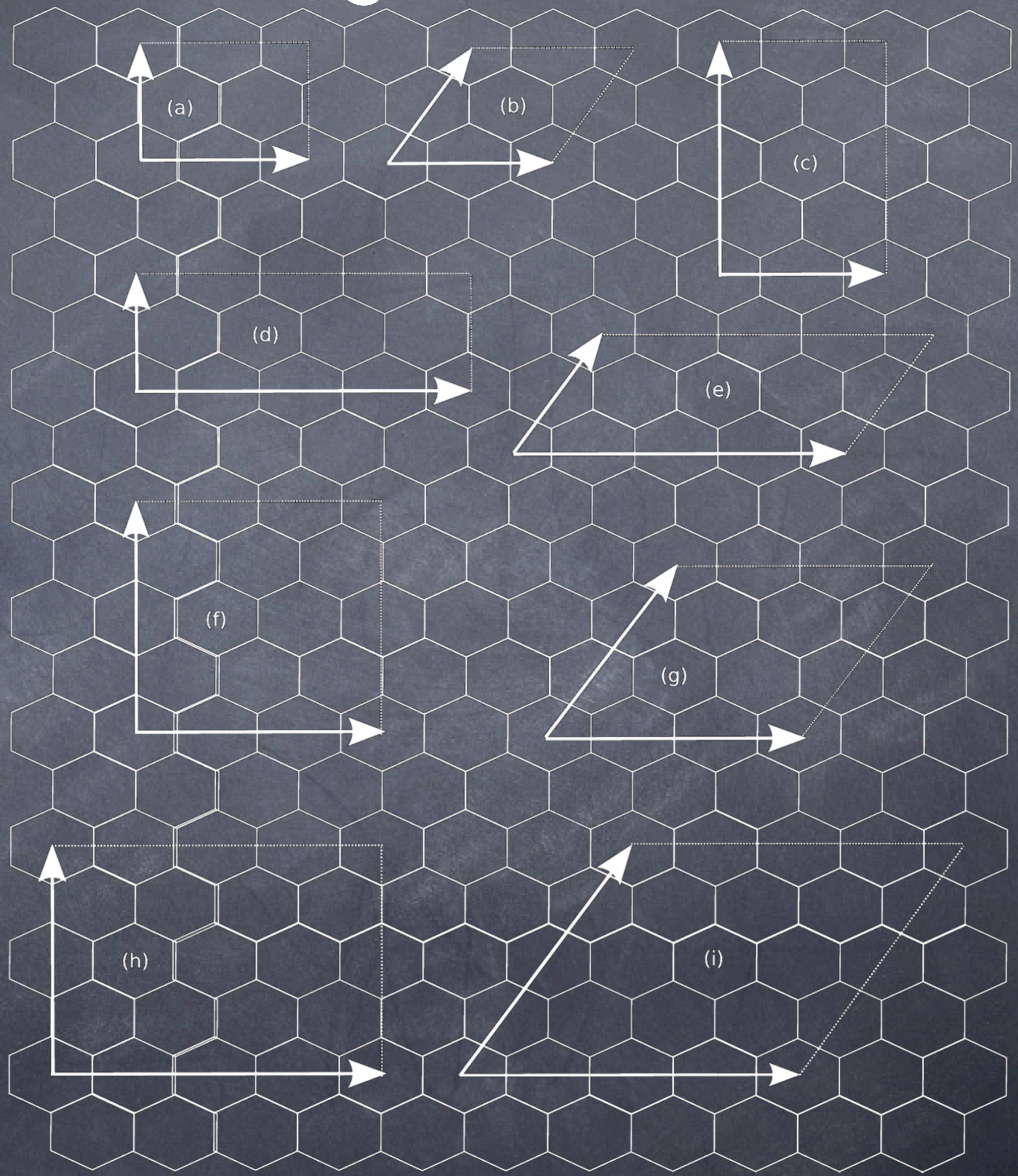
- On a torus if the lattice cannot be bicolored we get the Wen model where the ground state is known to be two fold degenerate.
- In the A phase on a lattice which can be bicolored a mapping can be made to the Toric code thus in the thermodynamic limit we know that the ground state is four fold degenerate on a torus.

Kitaev A, 2006 Ann. Phys. 321 2

Finite Lattice Configurations

- Each configuration is labelled by two lattice vectors. n = (i + √3j)/2 we see that the configurations:

(a) (2i, 2j) and (b)(2i, 2n) contain 8 spins.
(c) (2i, 4j), (d) (4i, 2j) and (e)(4i, 2n) contain 16-spins.
(f) (3i, 4j) is a 24-spin system.
(g) (3i, 3n) is an 18-spin system.
(h) (4i, 4j) and (i) (4i, 4n) contain 32-spins.



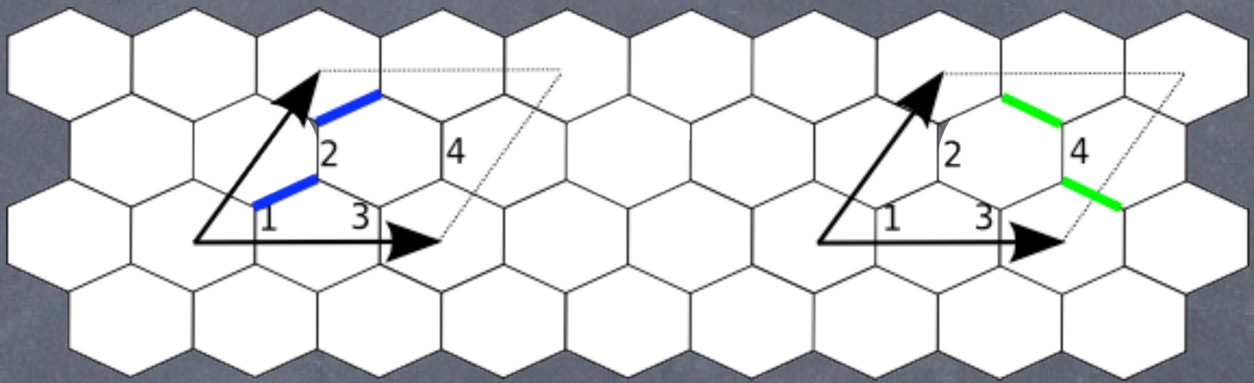
Configuration Mapping

- When the effective lattice can be bicolored the fourth order perturbative term can be mapped to the Toric code Hamiltonian.
- When it cannot it can be mapped to the Wen model.
- This table shows for each configuration in each A phase which it can be mapped to.
- H_W signifies Wen's model and H_K signifies the Kitaev's toric code model.

N	Configuration	A_x	A_y	A_z
8	$(2\mathbf{i}, 2\mathbf{j})$	H_{W}	H_{W}	H_{K}
8	$(2\mathbf{i}, 2\mathbf{n})$	H_{K}	H_{K}	H_{K}
12	$(3\mathbf{i}, 2\mathbf{j})$	H_{W}	H_{W}	H_{K}
12	$(3\mathbf{i}, 2\mathbf{n})$	H_{W}	H_{W}	H_{K}
16	$(2\mathbf{i}, 4\mathbf{j})$	H_{K}	H_{K}	H_{K}
16	$(4\mathbf{i}, 2\mathbf{j})$	H_{W}	H_{W}	H_{K}
16	$(4\mathbf{i}, 2\mathbf{n})$	H_{K}	H_{K}	H_{K}
18	$(3\mathbf{i}, 3\mathbf{n})$	H_{W}	H_{W}	H_{W}
20	$(5\mathbf{i}, 2\mathbf{j})$	H_{W}	H_{W}	H_{K}
20	$(5\mathbf{i}, 2\mathbf{n})$	H_{W}	H_{W}	H_{K}
24	$(2\mathbf{i}, 6\mathbf{j})$	H_{W}	H_{W}	H_{K}
24	$(3\mathbf{i}, 4\mathbf{j})$	H_{W}	H_{W}	H_{K}
24	$(6\mathbf{i}, 2\mathbf{j})$	H_{W}	H_{W}	H_{K}
24	$(6\mathbf{i}, 2\mathbf{n})$	H_{K}	H_{K}	H_{K}
28	$(5\mathbf{i}, 2\mathbf{j})$	H_{W}	H_{W}	H_{K}
28	$(5\mathbf{i}, 2\mathbf{n})$	H_{W}	H_{W}	H_{K}
30	$(3\mathbf{i}, 5\mathbf{n})$	H_{W}	H_{W}	H_{W}

Second Order Terms

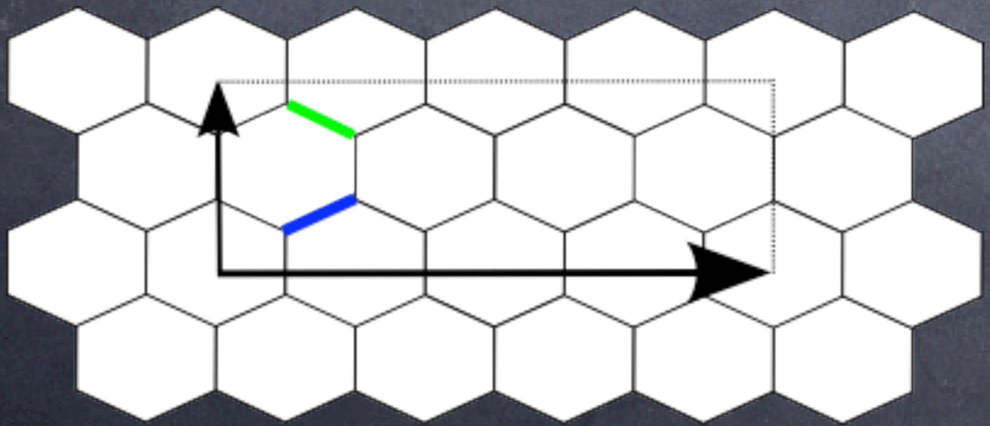
For the 8 spin (2i, 2n) configuration there exist the second order non-constant terms given by



second order non constant terms for (2i, 2n) configuration.

$$\langle a\,|H^{(2)}|b\rangle = \sum_j \frac{\langle a\,|V|j\rangle\langle j\,|V|b\rangle}{E_0 - E_j},$$

$$H^{(2)} = \frac{1}{2|J_z|}[J_x^2(\sigma_1^x\sigma_2^x + \sigma_3^x\sigma_4^x) + J_y^2(\sigma_2^x\sigma_3^x + \sigma_1^x\sigma_4^x)].$$

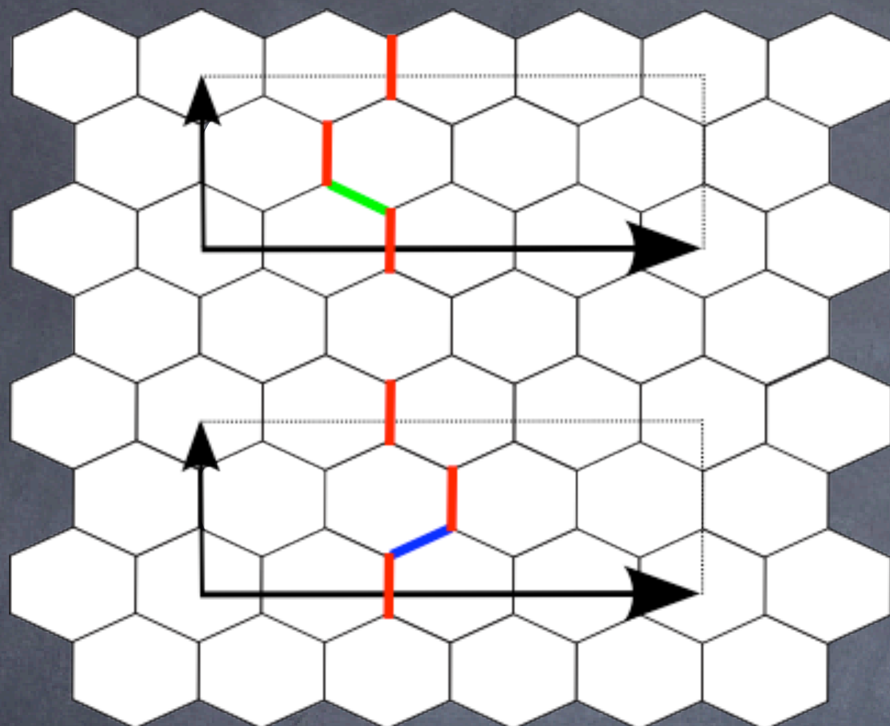


second order non constant terms for (ai, 2j) configuration.

Also for any (ai, 2j) configuration in the A_z phase the second order effective system is governed by a simple Ising spin chain Hamiltonian.

$$H^{(2)} = \frac{1}{2|J_z|}J_xJ_y\sum_{n=1}^{N/2}\sigma_n^y\sigma_{n+1}^y,$$

Third Order Terms



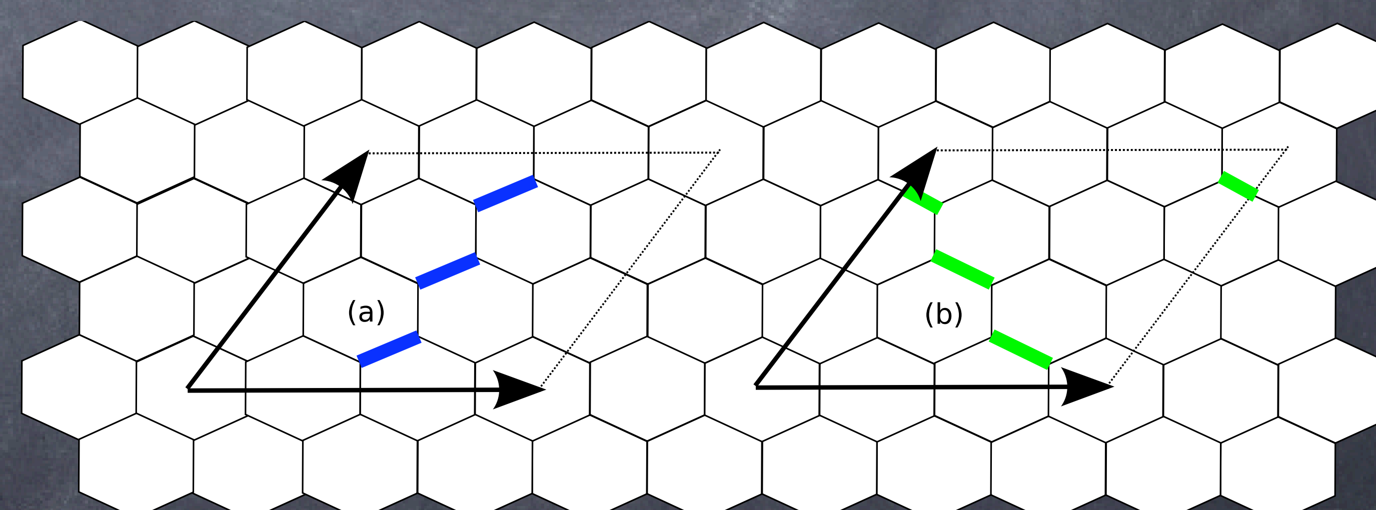
non-constant 3rd order terms for (ai, 2j) configurations in Ax phase (top) and Ay phase (bottom).

Third order terms are found in much the same way as the second order terms. For (ai, 2j) configurations with a > 2 in the A_x and A_y phases we have third order non-constant terms as illustrated in the figure below.

$$\langle a\,|H^{(3)}|b\rangle = \sum_{jk} \frac{\langle a\,|V\,|j\rangle\langle j\,|V\,|k\rangle\langle k\,|V\,|b\rangle}{(E_0 - E_j)(E_0 - E_k)},$$

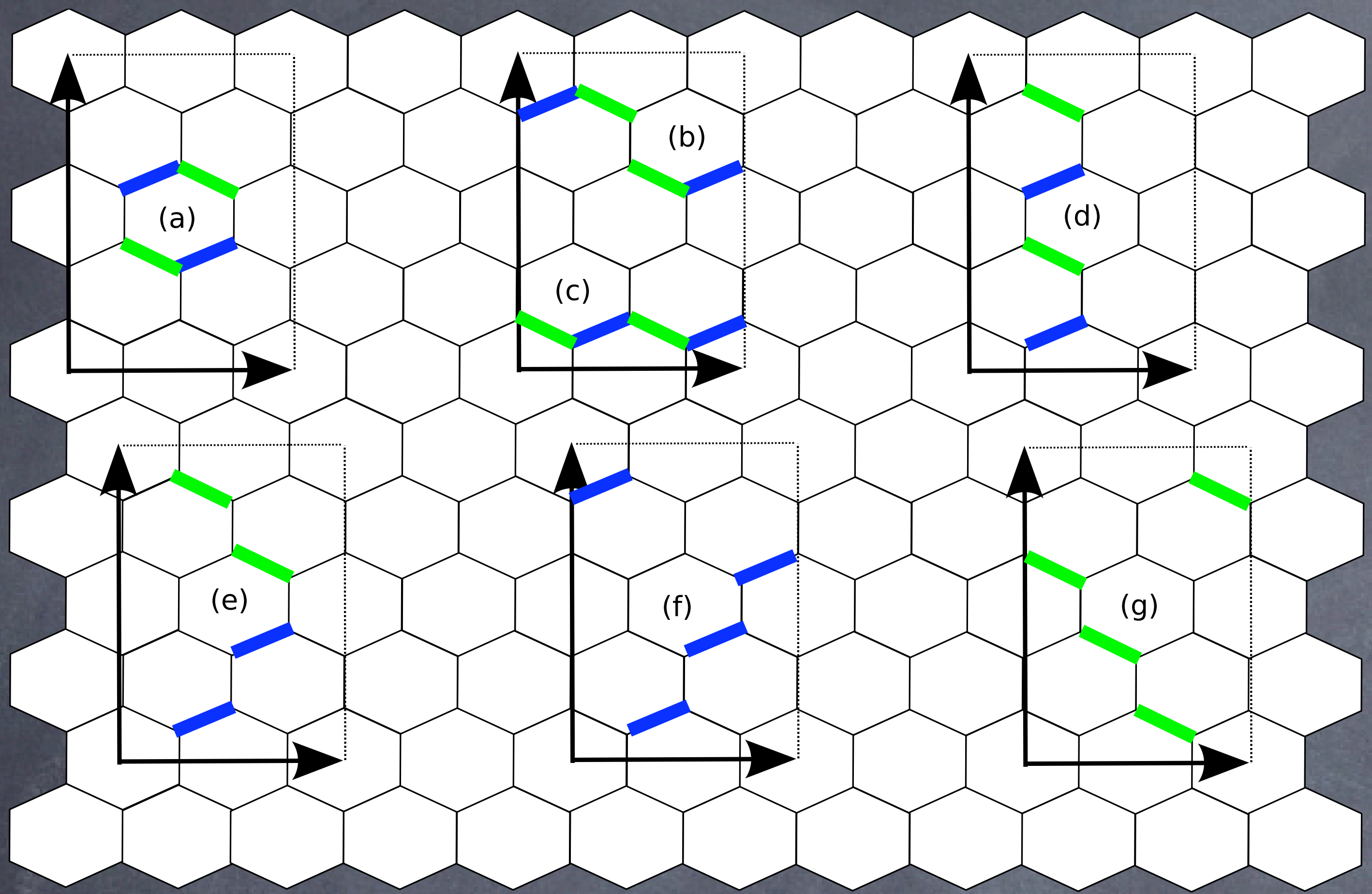
In all the A phases of the 18-spin (3i, 3n) configuration there are third order non-constant terms.

This 18-spin system is 3 × 3 plaquettes and it cannot be mapped to the Toric code in any of its A phases.



graphical representations of two of the six third-order finite size corrections terms for the 18-spin (3i, 3n) configuration.

Fourth Order Terms



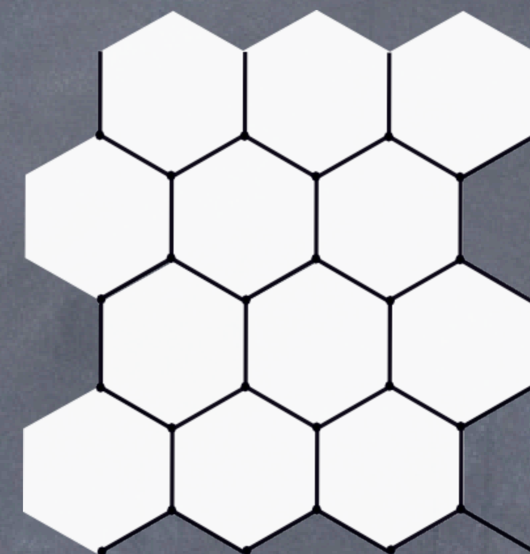
Some different four terms sequences that non-trivially connect up the dimer basis vectors on the 16-spin (2i, 4j) configuration lattice. Type (a) is a plaquette term Q_n and is valid for all non-horizontal configurations. Types (b) and (c) are horizontal string terms R_n and Z_n respectively. Type (d) and (e) are vertical strings Y_n and A_n respectively. Types (f) and (g) are vertical X_n strings.

We will now look at additional non-constant fourth order terms which appear in finite sized systems. As an example we consider the 16-spin (2i, 4j) configuration in the A_z phase.

$$H_{FS}^{(4)} = -\frac{J_x^2 J_y^2}{16|J_z|^3}\sum_n^8 (Q_n + R_n - 5A_n) - \frac{J_x^2 J_y^2}{16|J_z|^3}\sum_n^4 (Z_n + 5Y_n) - \frac{5}{16|J_z|^3}\left(J_x^4 \sum_{n=1}^{2} X_n + J_y^4 \sum_{n=3}^{4} X_n\right)$$

Numerical Case Study

- Taking a 24-spin (6i,4j) toroidal configuration in the A_z phase.
- Effective Hamiltonian given by:



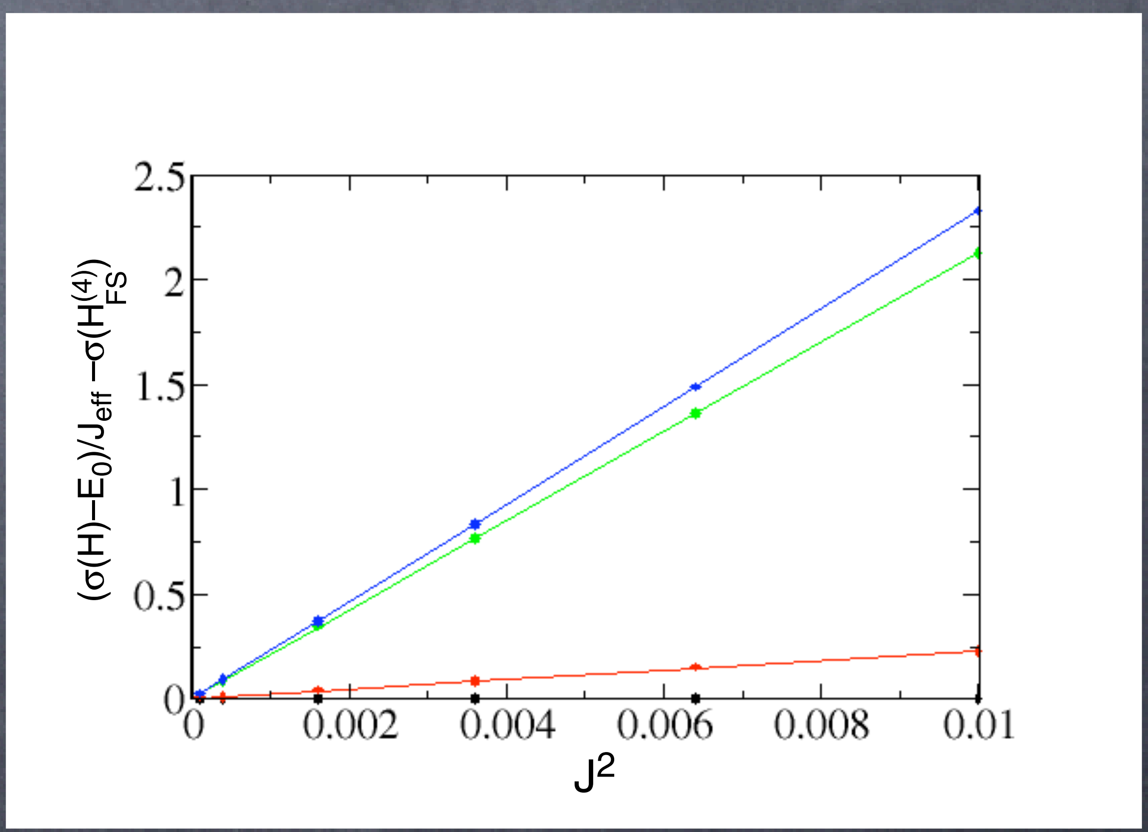
$$H_{\text{eff}} = cI + J_{\text{eff}}(H_{\text{K}} + H^{(4)}_{\text{FS}}) + \text{O}(J^6) + \ldots$$

H_{K} is toric code Hamiltonian on effective spins

$$H^{(4)}_{FS} = 5(\qquad\qquad)$$

$$J_{eff} = \frac{J^4}{16|J_z|^3} \quad \text{with} \quad J = J_x = J_y$$

- For small values of J diagonalise the full Hamiltonian with everything raised by dividing by J_{eff}.
- Diagonalised effective fourth order finite size terms and subtracted.
- Expect to be left with toric code spectrum and sixth order effects.
- σ(M) is defined as appropriately ordered spectrum of operator M.



$$\frac{\sigma(H) - E_0}{J_{\text{eff}}} - \sigma(H^{(4)}_{\text{FS}}) = \sigma(H_{\text{K}}) + \text{O}(J^2).$$

Numerical Case Study

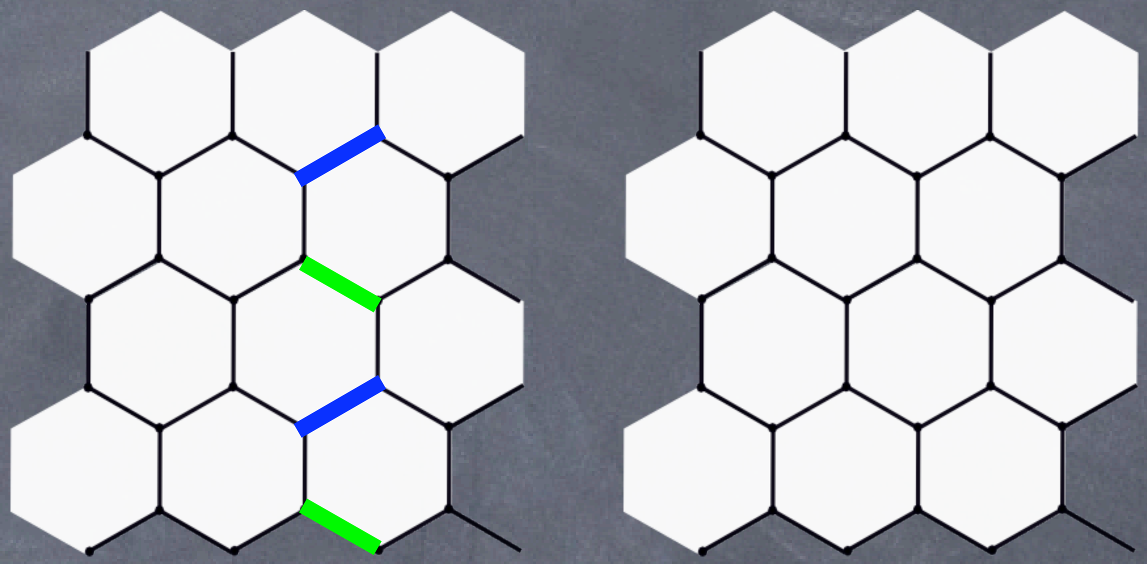
- Taking a 24-spin (6i,4j) toroidal configuration in the A_z phase.
- Effective Hamiltonian given by:

$$H_{\text{eff}} = cI + J_{\text{eff}}(H_{\text{K}} + H_{\text{FS}}^{(4)}) + \text{O}(J^6) + \ldots$$

H_k is toric code Hamiltonian on effective spins

$$H_{FS}^{(4)} = 5(\sum_n^6 Y_n \qquad\qquad)$$

$$J_{eff} = \frac{J^4}{16|J_z|^3} \quad \text{with} \quad J = J_x = J_y$$



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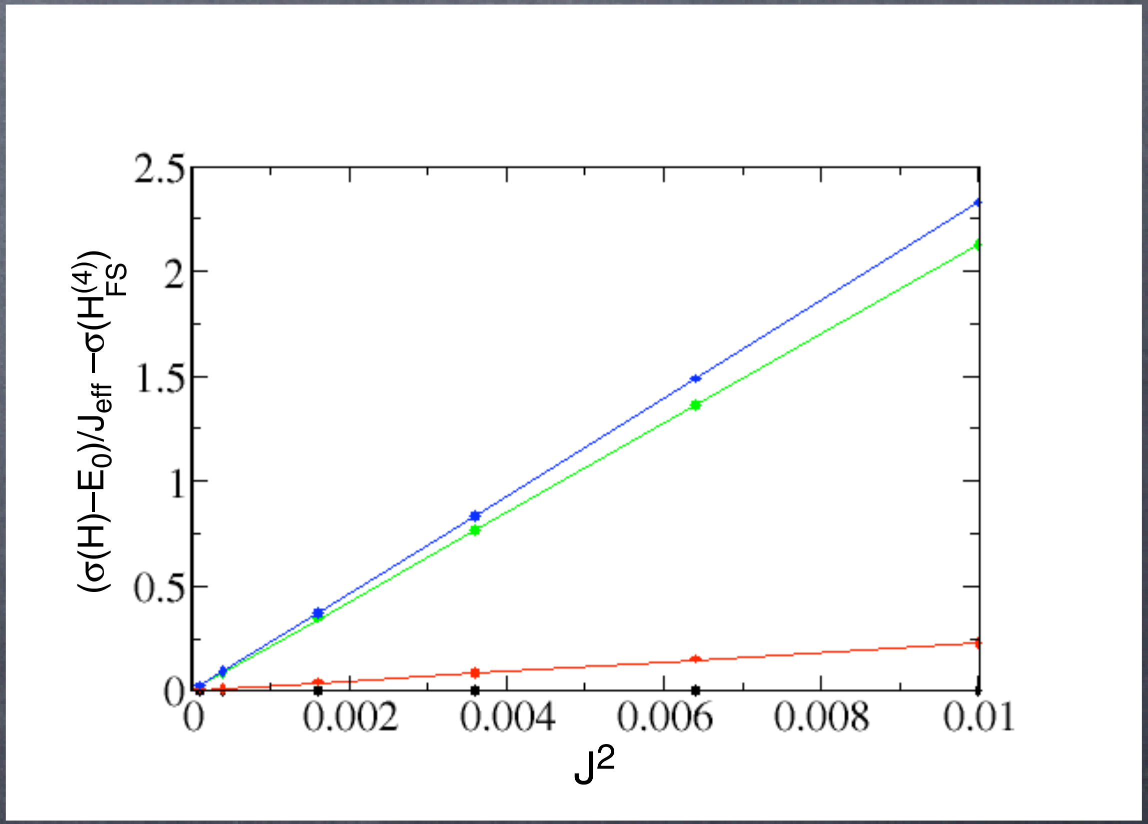


$$\frac{\sigma(H) - E_0}{J_{\text{eff}}} - \sigma(H_{\text{FS}}^{(4)}) = \sigma(H_{\text{K}}) + \text{O}(J^2).$$

Numerical Case Study

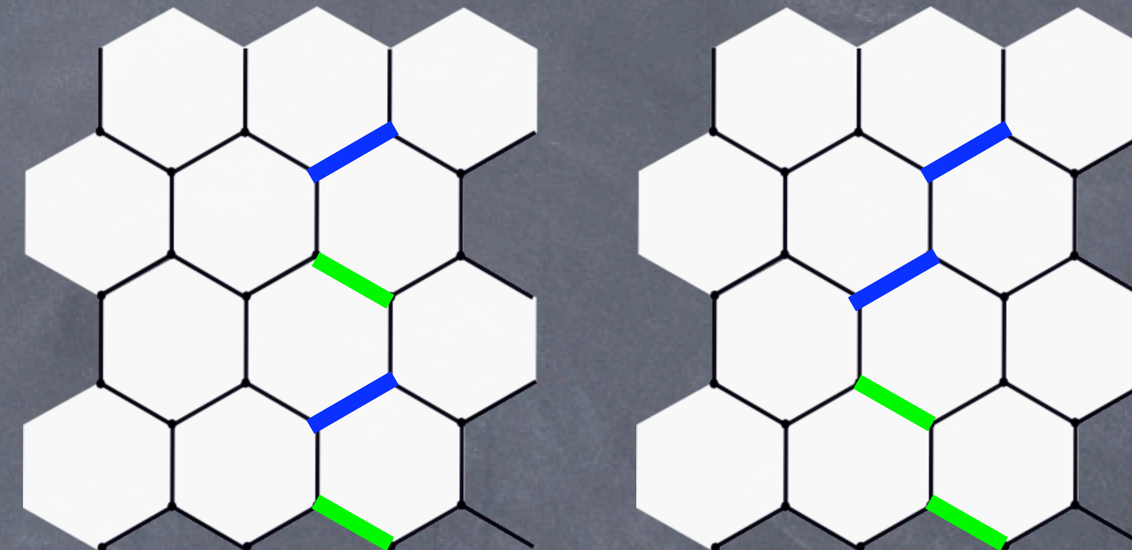
- Taking a 24-spin (6i,4j) toroidal configuration in the A_z phase.
- Effective Hamiltonian given by:

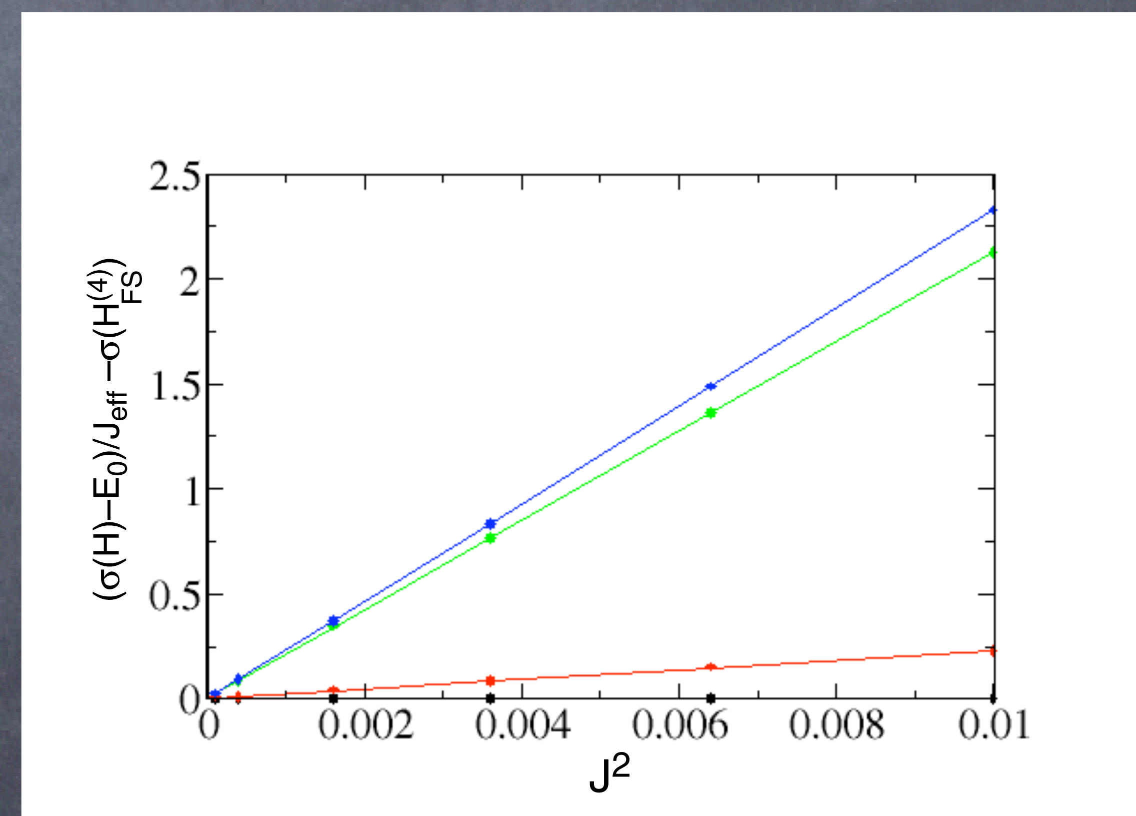
$$H_{\text{eff}} = cI + J_{\text{eff}}(H_{\text{K}} + H_{\text{FS}}^{(4)}) + \text{O}(J^6) + \ldots$$

H_k is toric code Hamiltonian on effective spins

$$H_{FS}^{(4)} = 5\left(\sum_n^6 Y_n - \sum_n^{12} A_n\right)$$

$$J_{eff} = \frac{J^4}{16|J_z|^3} \quad \text{with} \quad J = J_x = J_y$$





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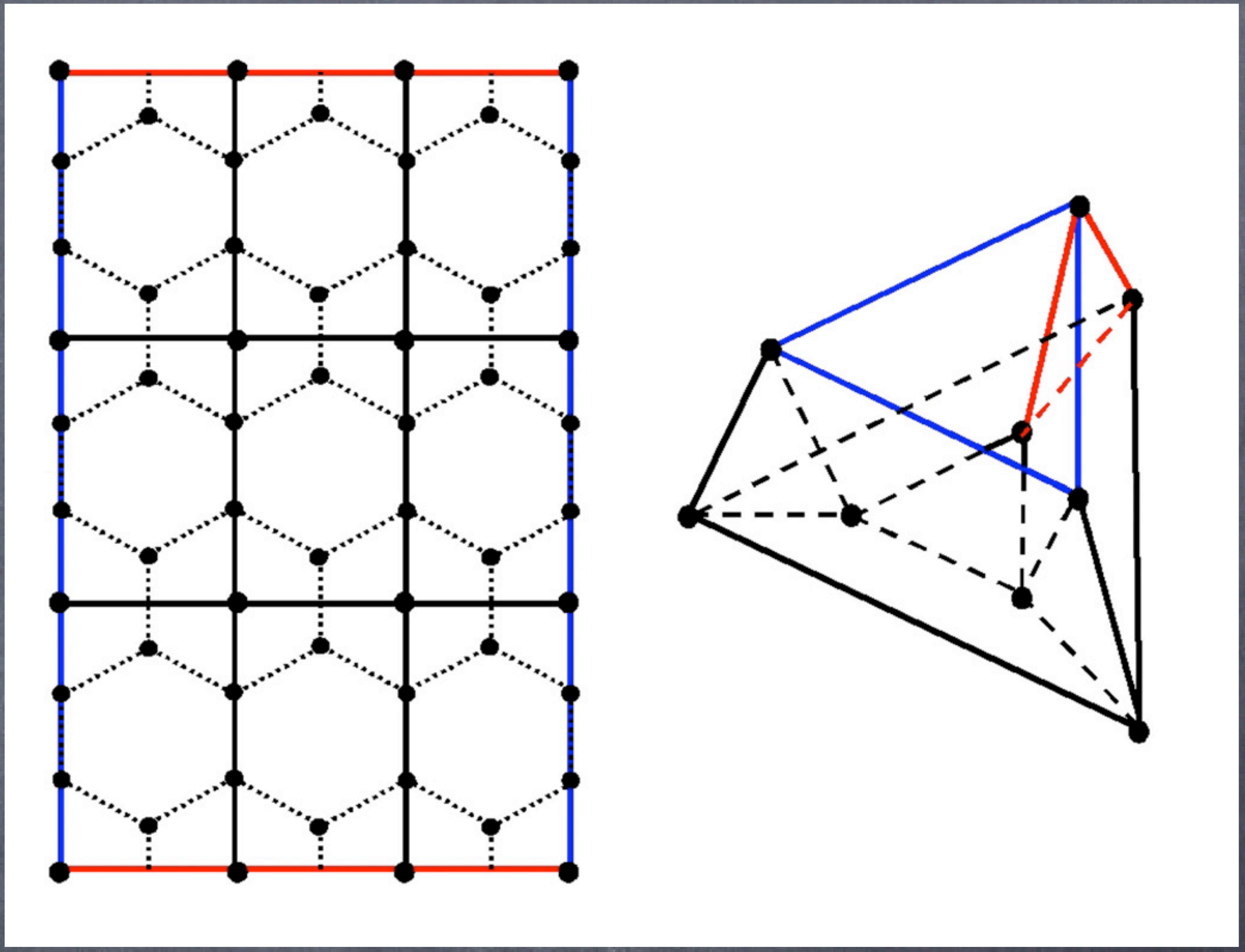
$$\frac{\sigma(H) - E_0}{J_{\text{eff}}} - \sigma(H_{\text{FS}}^{(4)}) = \sigma(H_{\text{K}}) + \text{O}(J^2).$$

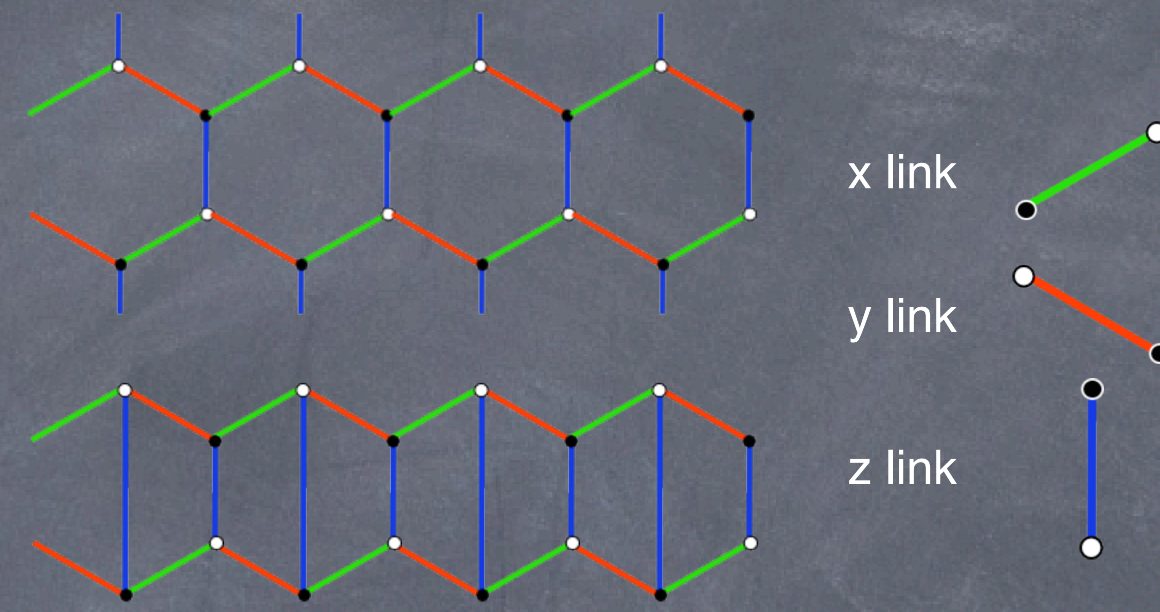
Avoiding Finite Size Effects

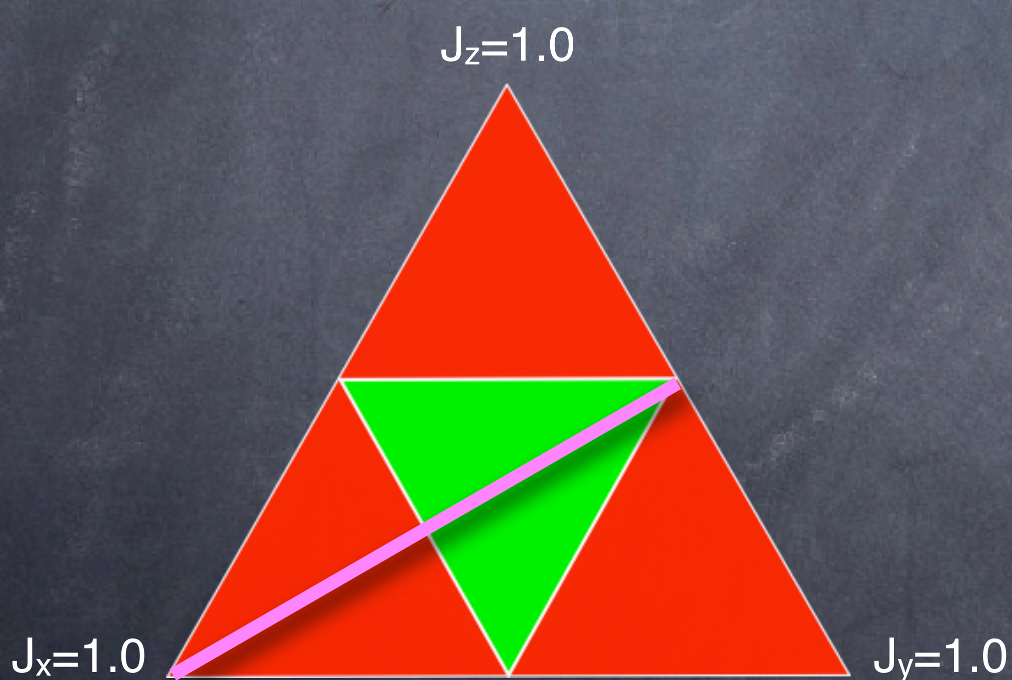
- The smallest finite configuration where the fourth-order perturbative expansion is equivalent to the toric code Hamiltonian with no lower order terms is the 36-spin (3i,6j) configuration.
- An interesting observation is that this is also the smallest configuration satisfying the requirement that plaquette terms share only one effective spin.

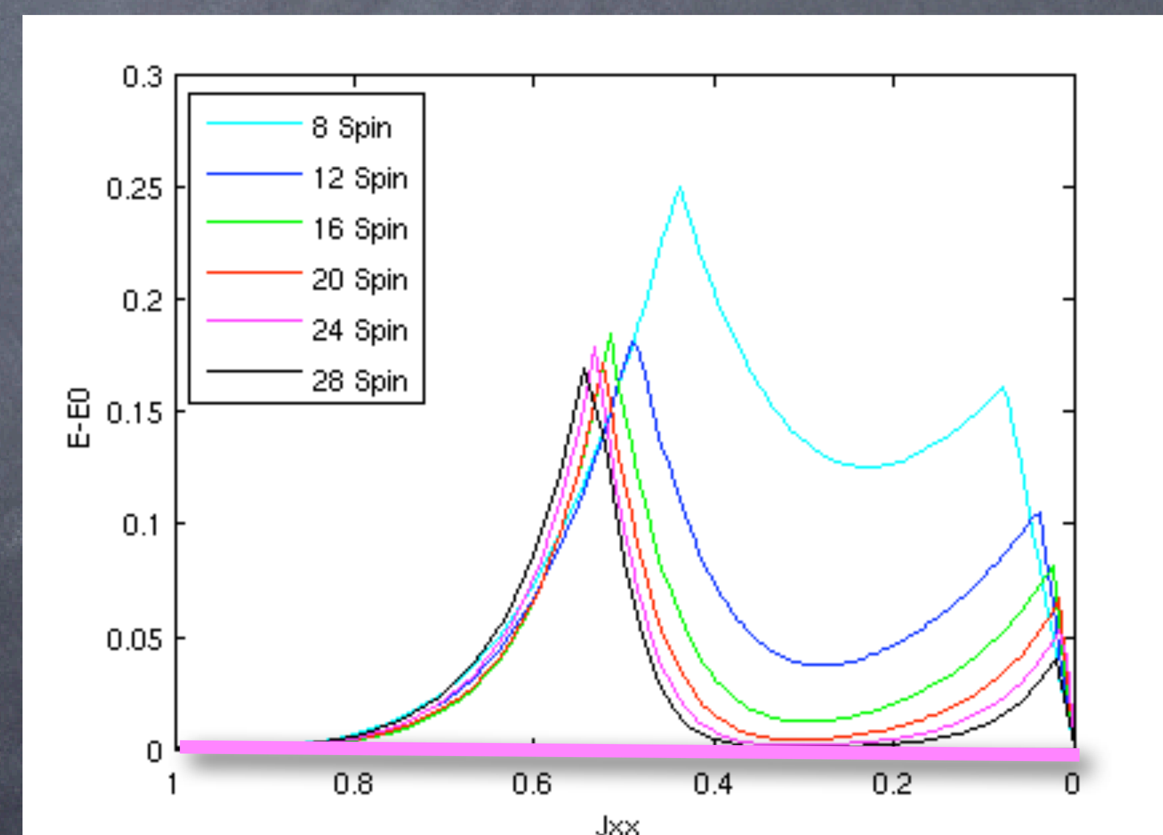


Thin Torus Limit

- Model compressed in z-link direction until only two plaquettes wide.
- Resulting model is Quasi 1D.
- Initial exact diagonalisation calculations were interesting.
- Can be treated more effectively numerically (DMRG) and analytically (CFT).





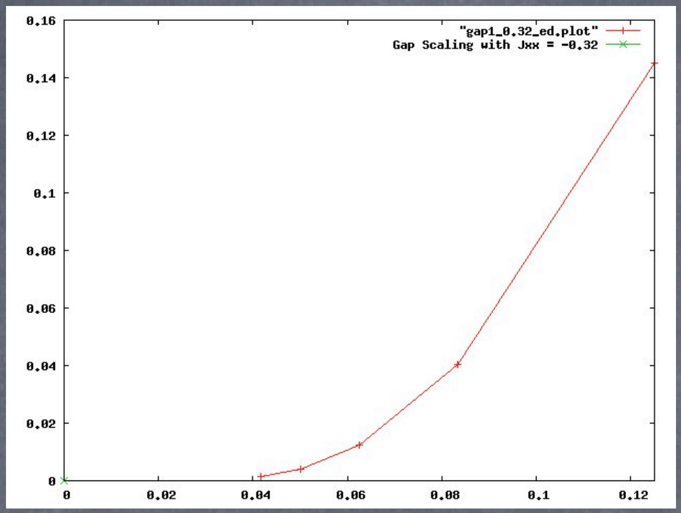


Thin Torus Limit

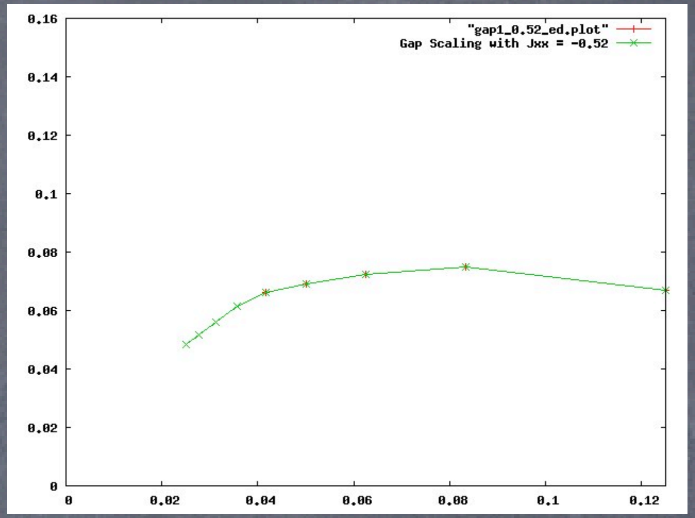
- There is a quantum phase transition at $J_{xx} = 0.5$, $J_{yy} = J_{zz} = 0.25$.

  Xiao-Yong Feng et al , 2007 PRL 98 087204

- At this critical point using DMRG (from ALPS) it was found that the energy gap goes to zero linearly with one over the system size.
- This relation indicates that the critical point can be described by a conformal field theory.
- It was discovered that the central charge is 0.5 indicating that the critical point is described by an Ising conformal field theory.

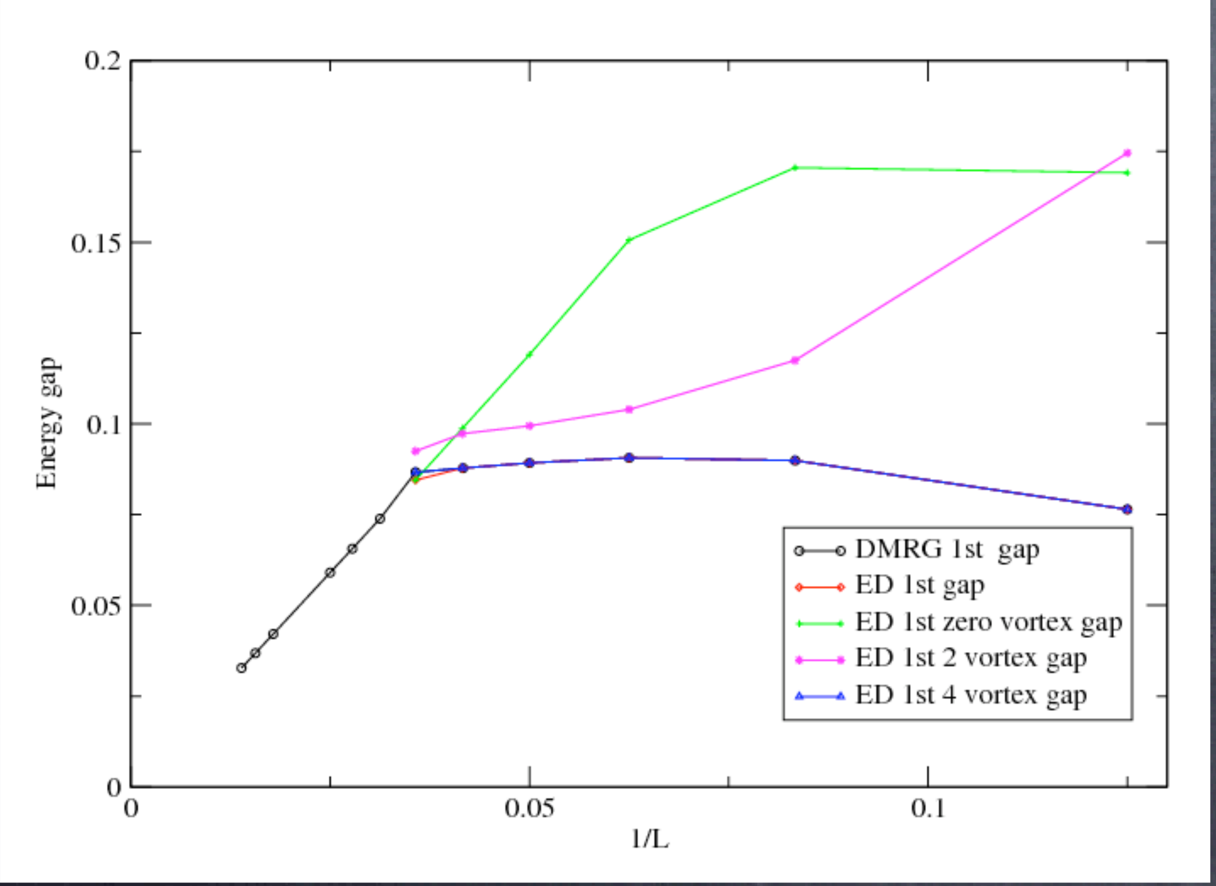


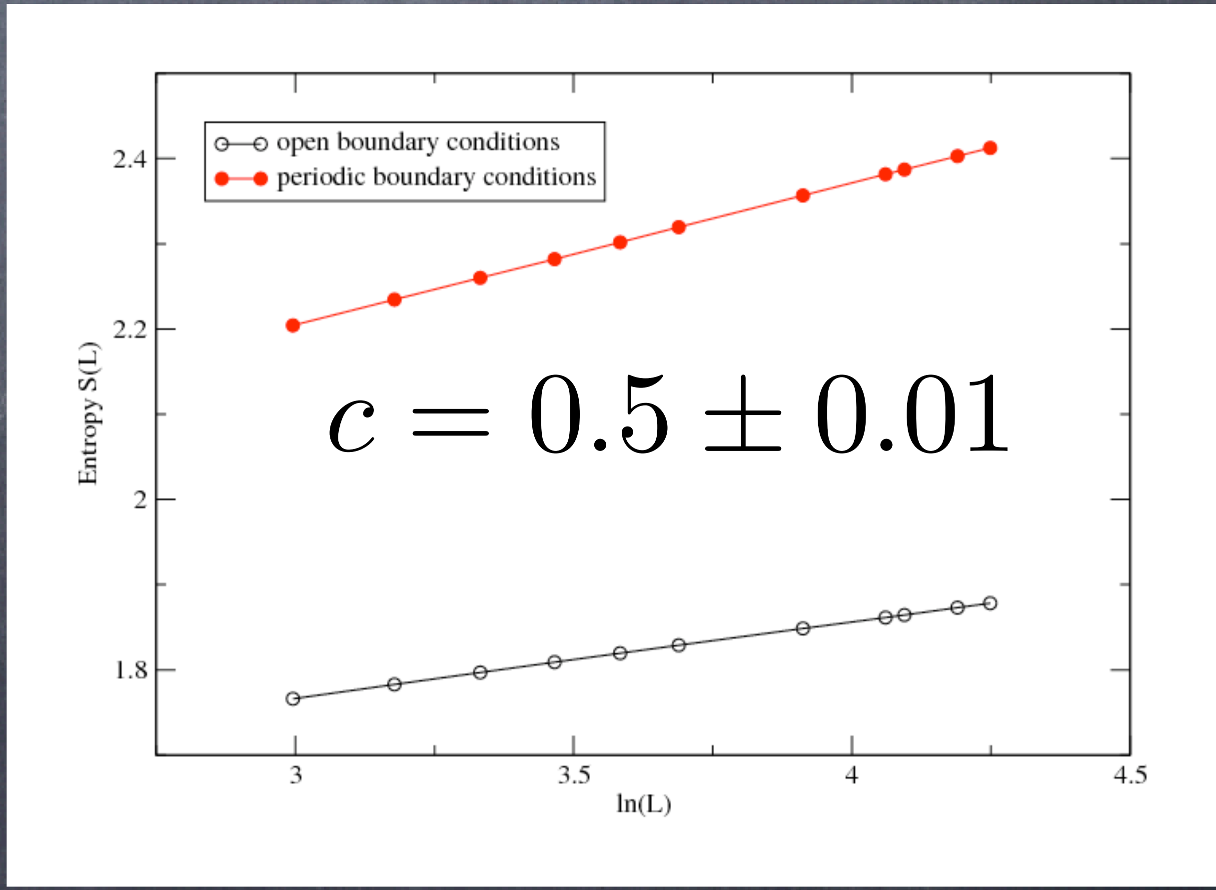
Jx = -0.32



Jx = -0.52

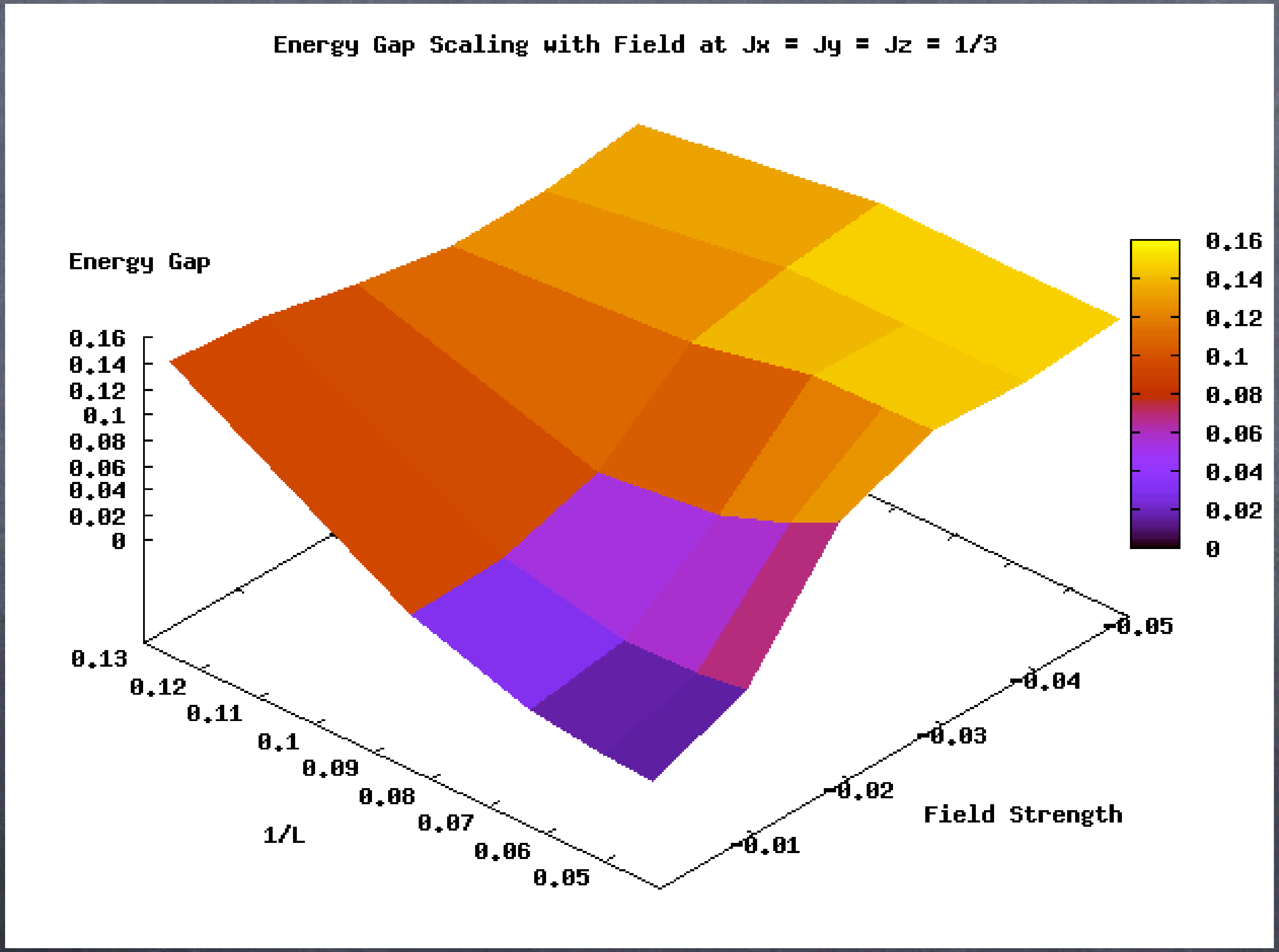
$$S(L) \propto \frac{c}{3} log(L)$$





Thin Torus Limit

- Early investigations indicate that when an external magnetic field is switched on a gap opens in the centre of the phase diagram.



Fendley Quantum Loop Gas Models

- Freedman loop gas models have a $d = \sqrt{2}$ barrier.
- Prevents realization of topological phases with k > 2 (e.g. Fibonacci anyons).
- Barrier can be overcome by using choosing a different inner product.
- Perturbed toric code.

$$H = H_{toric} + uH_u$$

$$\begin{pmatrix} <1|1> & <1|\hat{1}> \\ <\hat{1}|1> & <\hat{1}|\hat{1}> \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}, \lambda = \pm\frac{1}{d}$$

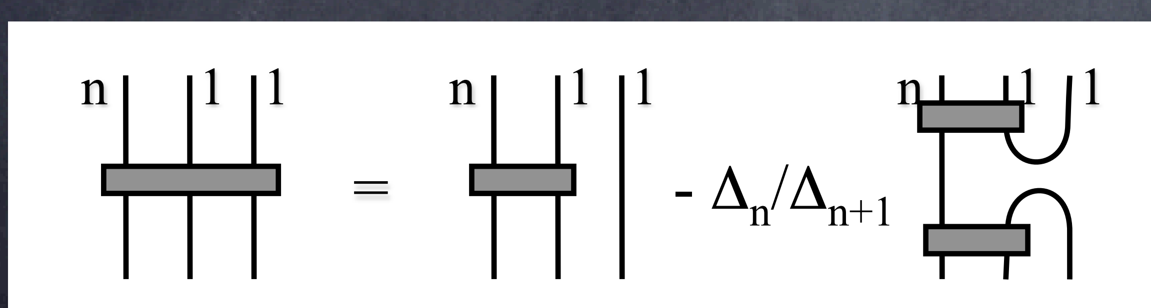
$$H_{toric} = \sum_V W_V + \sum_F \overline{W}_F$$

$$W_V = \frac{1}{2}(1 - \sigma^z_{V1}\sigma^z_{V2}\sigma^z_{V3}\sigma^z_{V4})$$

$$\overline{W}_F = \frac{1}{2}(1 - \sigma^x_{F1}\sigma^x_{F2}\sigma^x_{F3}\sigma^x_{F4})$$

$$H_u = \sum_V W_V \sum_{a=1}^{4} \sigma^z_{V_a} + \sum_F \overline{W}_F \sum_{a=1}^{4} \sigma^x_{F_a}$$

Can add Jones-Wenzl projectors provides $SO(3)_k$ theory at arbitrary level of theory k.



$d = 2\cos(\pi/(k+2))$

$\Delta_{-1} = 0$

$\Delta_0 = 1$

$\Delta_{n+1} = d\,\Delta_n - \Delta_{n-1}$

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Fendley Quantum Loop Gas Models

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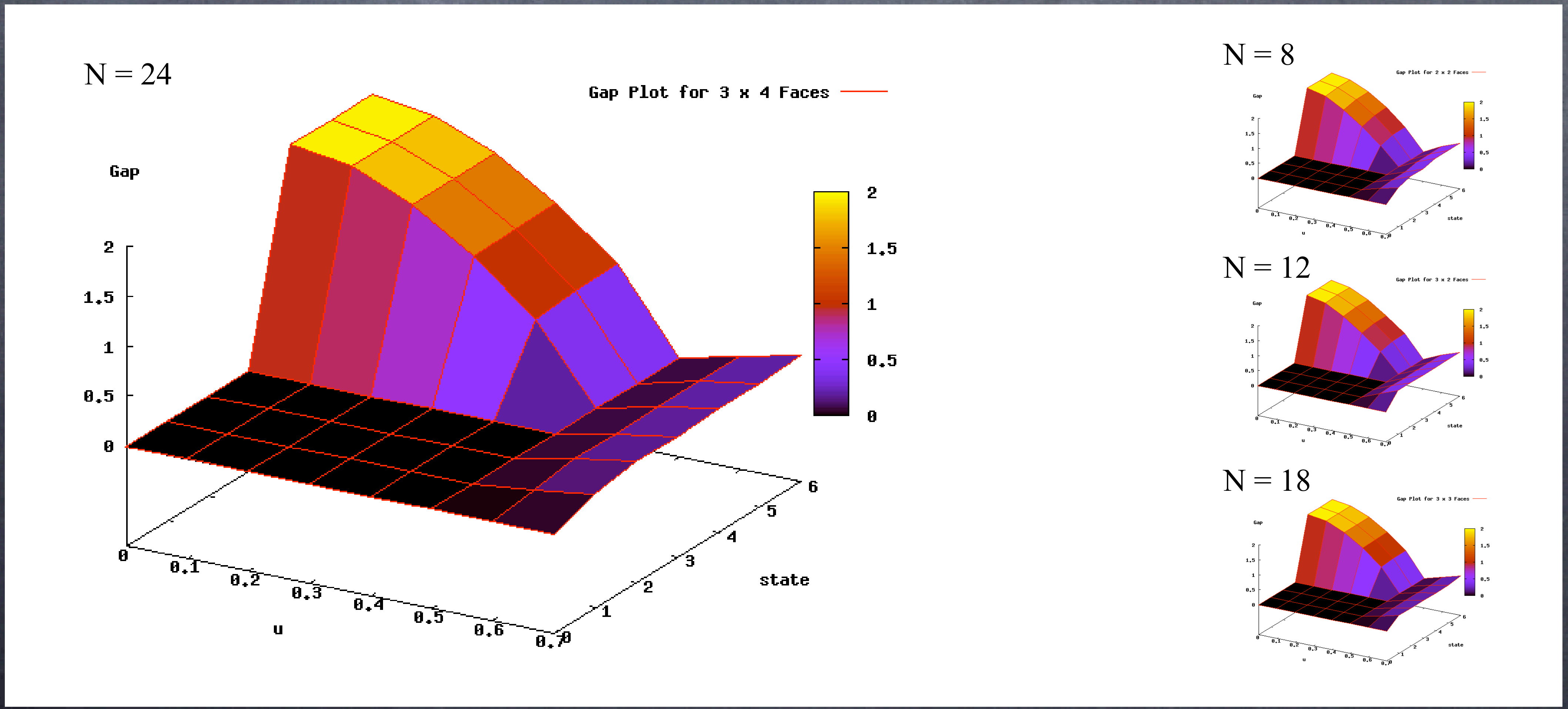

Numerics

- Table summarising numerical tools.

	Method is exact	1D systems	2D systems	Fermionic systems (sign)	Parallel	Momentum sectors	Sz sectors	multi Site interactions	Established technique	Periodic bondary conditions	Scaling	No. spins possible
In-House Exact Diagonalisation Code	✓	✓	✓	✓	✓	UD	✓	✓	✓	✓	exp(N)	~28
ALPS Exact Diagonalisation	✓	✓	✓	✓	✗	✓	✗	✗	✓	✓	exp(N)	~20
ALPS DMRG	✗	✓	✗	✓	✗	✗	✗	✗	✓	✓	S	80+
PEPS	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗	S	-
Quantum Monte Carlo	✗	✓	✓	✗	✓	✗	✗	✗	✓	✓	poly(N)	-

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PEPS	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗	S	-
Quantum Monte Carlo	✗	✓	✓	✗	✓	✗	✗	✗	✓	✓	poly(N)	-

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PEPS	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗	S	-
Quantum Monte Carlo	✗	✓	✓	✗	✓	✗	✗	✗	✓	✓	poly(N)	-

Numerics

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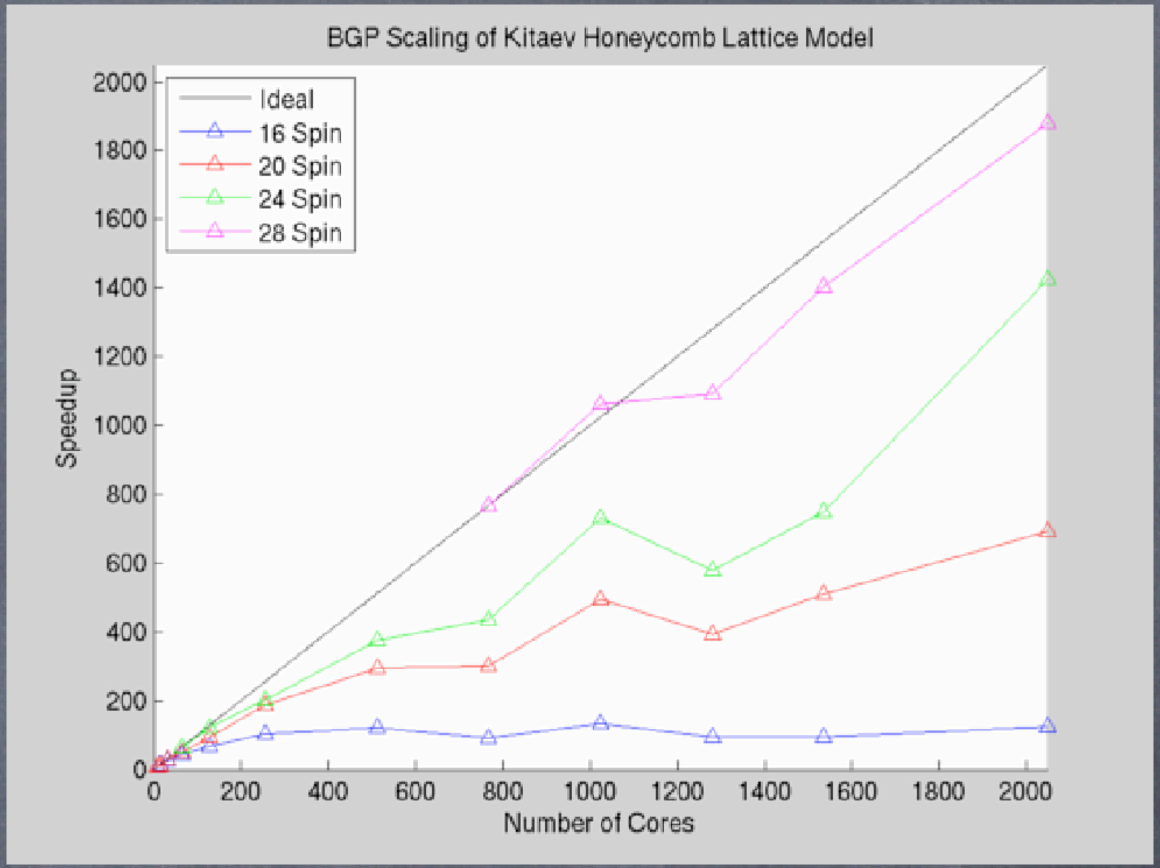
PEPS	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗	S	-
Quantum Monte Carlo	✗	✓	✓	✗	✓	✗	✗	✗	✓	✓	poly(N)	-

Numerics

Exact diagonalisation code has been developed.

- The code is written in C.
- It makes use of the PETSc and SLEPc libraries.
- It is capable of running efficiently on massively parallel distributed memory machines.
- Can be used for large range of spin 1/2 systems without modification.

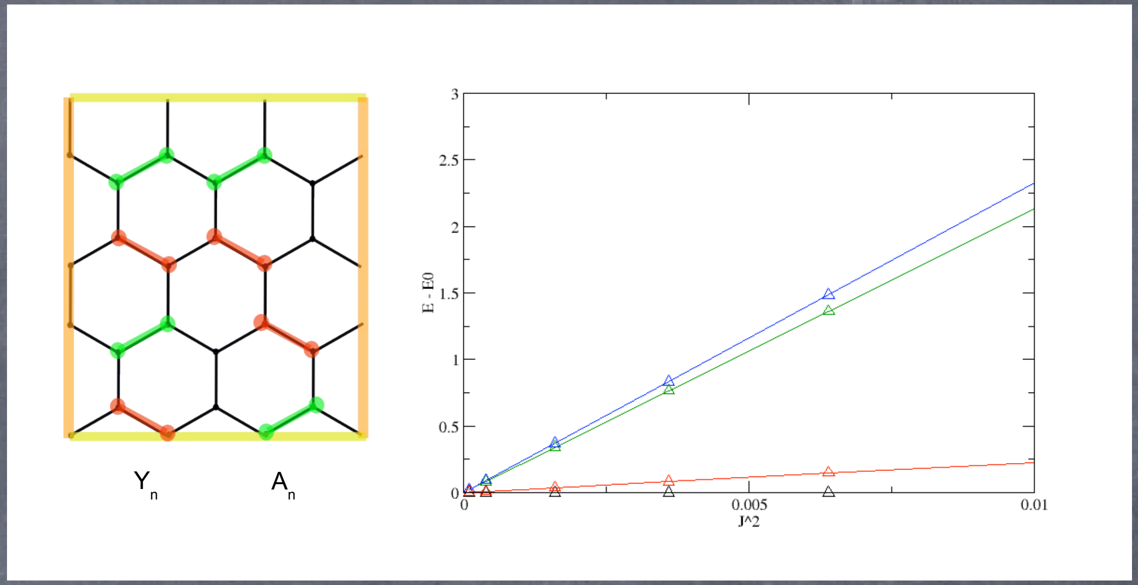


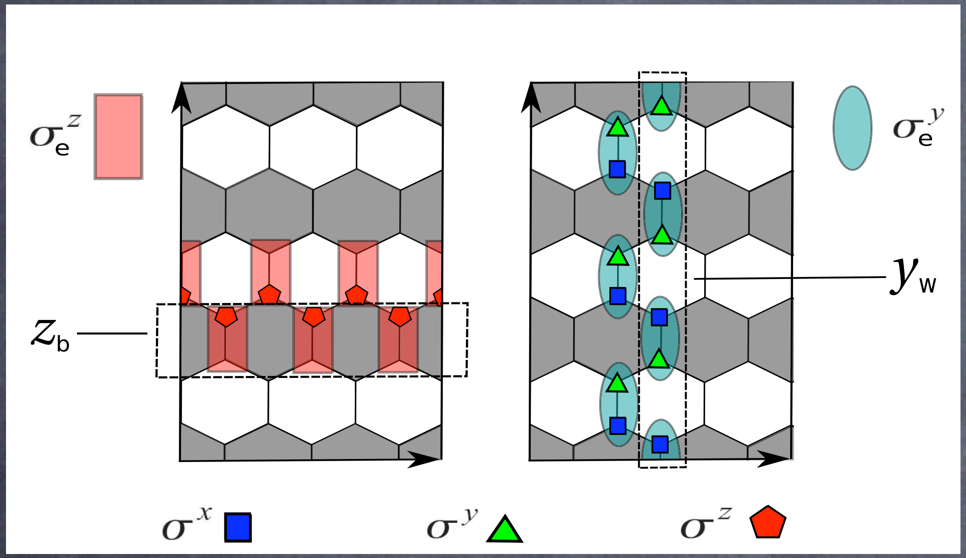
Conclusions

- Showed non-constant finite size corrections appearing up to the order of the toric code mapping in Kitaev honeycomb model.
- Numerical case study demonstrates accuracy of numerics and demonstrates how corrections can be applied.
- Can be used to aid in understanding of fermionization approaches. (arXiv:0903.5211)
- Interesting critical point in thin torus limit described by Ising conformal field theory.
- Evidence of gap opening with magnetic field.

Recent Publications

G. Kells, N. Moran and J. Vala,
Finite size effects in the Kitaev honeycomb lattice model on a torus,
J. Stat. Mech. (2009) P03006

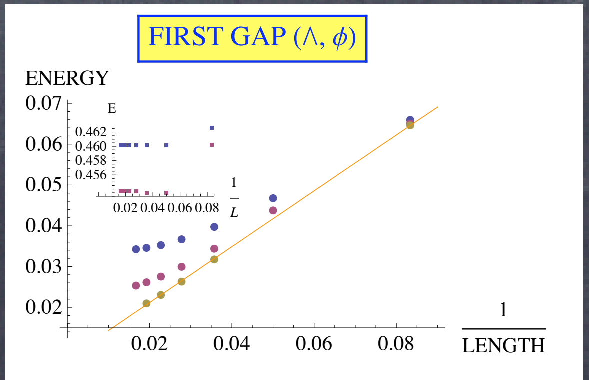




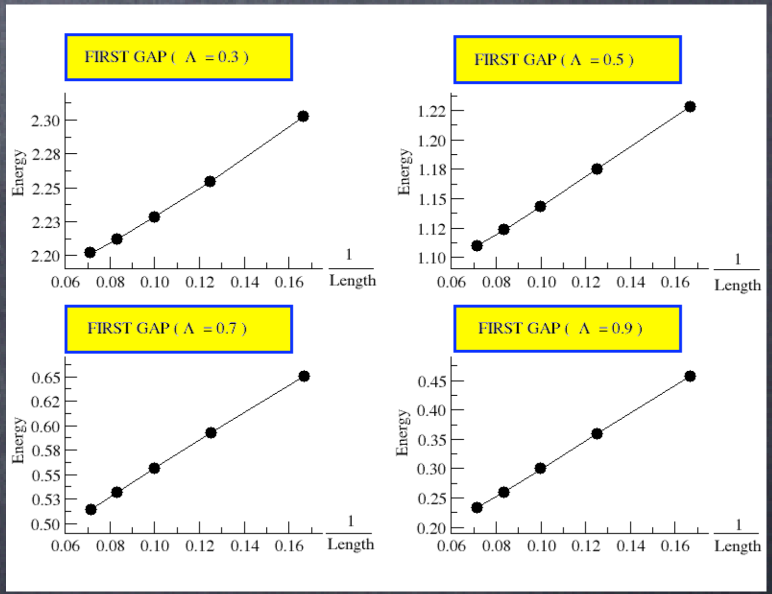
G. Kells, A. T. Bolukbasi, V. Lahtinen, J. K. Slingerland, J. K. Pachos, J. Vala,
Topological degeneracy and vortex manipulation in Kitaev's honeycomb model.
Phys. Rev. Lett. 101, 240404 (2008).

E. Rico, R. Hübener, S. Montangero, N. Moran, B. Pirvu, J. Vala, H.J. Briegel,
Valence Bond States: Link models,
Submitted to Ann. of Phys.
arXiv:0811.1049

Exact diagonalisation on Blue Gene/P was used to verify approximate DMRG calculations.



Gap Scaling using DMRG



Gap Scaling using ED

Collaborators

Advisor:

- Jiri Vala (NUIM)

Postdoctoral fellow:

- Graham Kells (NUIM)

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- Adrian Feiguin (StationQ)
- Paul Fendley (Univ. of Virginia)
- Simon Trebst (StationQ)

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Thank You!

Questions?