Quantum critical spin chains and entanglement entropy in 2D Rokhsar-Kivelson wave functions

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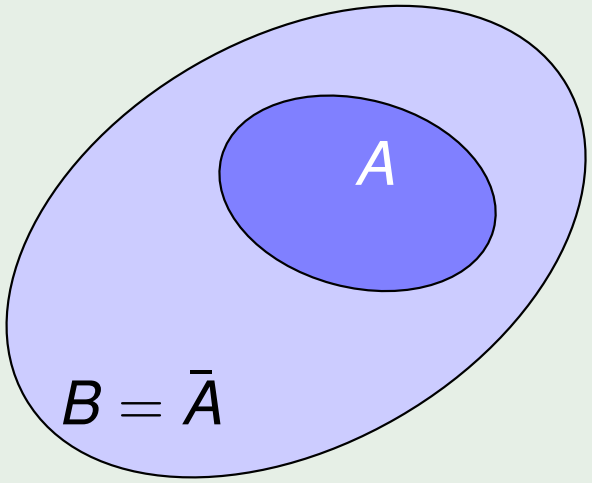
Outline

What is entanglement entropy?

System subdivision



Entanglement for a wave function $|\Psi\rangle$

- reduced density matrix :

$$\rho_A = \mathrm{Tr}_B \, |\Psi\rangle\langle\Psi|$$

- Entanglement entropy :

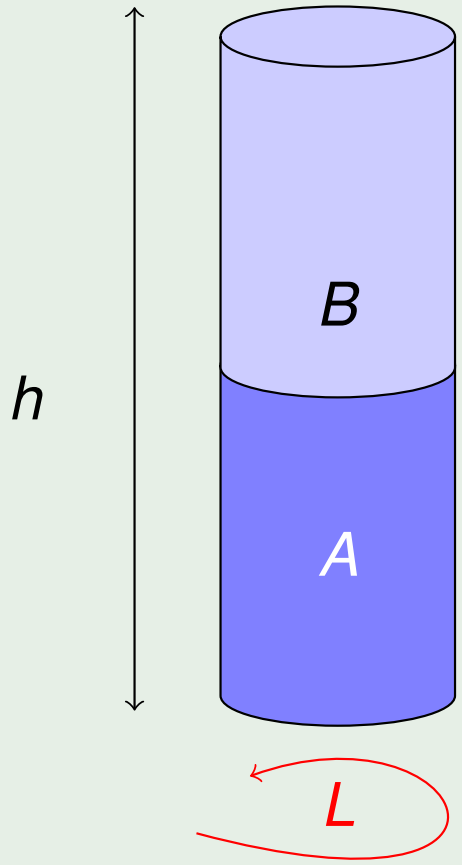
$$S^{VN} \stackrel{def}{=} -\mathrm{Tr}\, \rho_A \log \rho_A$$

Useful applications

- 1D Quantum spin chains : long segment of length L : $S^{VN} \simeq \frac{c}{3} \log L$ where c is the central charge of the corresponding CFT (see Calabrese & Cardy, JSM 2004)
- Topological order in gapped systems : $S^{VN} = \alpha L + \gamma_{topo}$ (see Levin & Wen, and Kitaev & Preskill, PRL 2006)
- Fractionnal quantum Hall effect (see Haque, Zozulya & Schoutens, PRL 2007 and Friedman & Levine, PRB 2008)

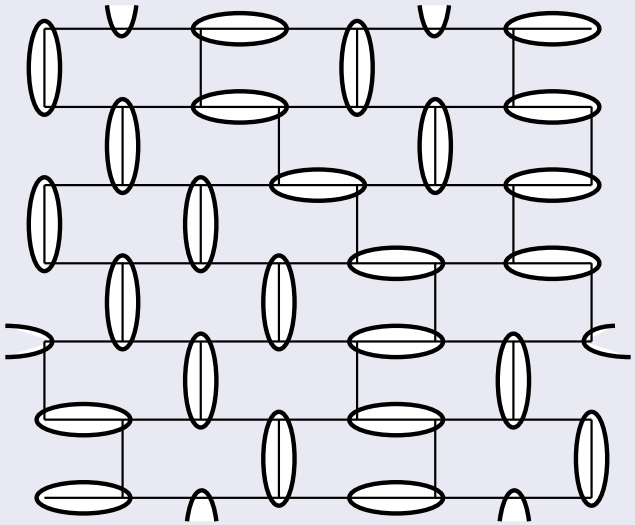
Choice of geometry and Wave function

Geometry : long cylinder $h \to \infty$



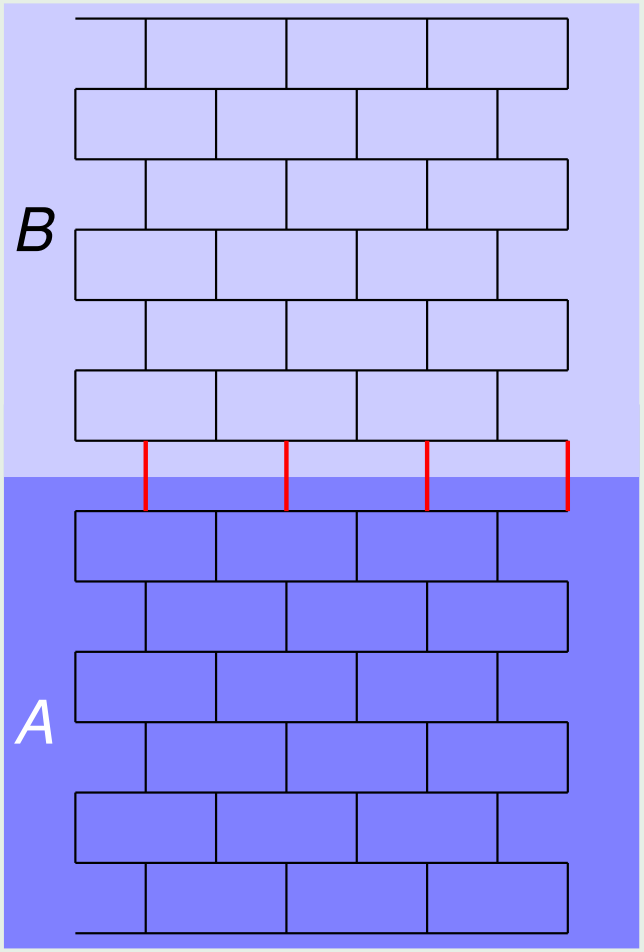
Quantum dimers on the honeycomb

- RK wave function : $|\Psi\rangle = \sum_c |c\rangle$
- Configurations : fully-packed dimers on the honeycomb lattice
- Example :



Entanglement entropy as a Shannon entropy

Geometry : long cylinder



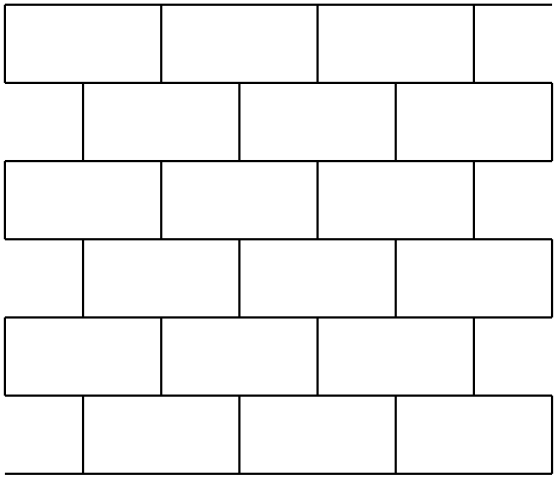
Schmidt decomposition

$$
\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{\mathcal{Z}}} \sum_{c_A} |c_A\rangle \otimes \sum_{c_B | c_A \cup c_B = c} |c_B\rangle \\
&= \frac{1}{\sqrt{\mathcal{Z}}} \sum_{i} \sum_{c_A(i)} |c_A(i)\rangle \otimes \sum_{\substack{c_B(i) \\ c_A(i) \cup c_B(i) = c}} |c_B(i)\rangle \\
&= \sum_{i} \sqrt{\frac{\mathcal{Z}_A(i)\mathcal{Z}_B(i)}{\mathcal{Z}}} |\Psi_A(i)\rangle |\Psi_B(i)\rangle
\end{aligned}
$$

Result

- $p(i) = \frac{\mathcal{Z}_A(i)\mathcal{Z}_B(i)}{\mathcal{Z}}$
- $S^{VN} = -\sum_i p(i) \log p(i)$

Mapping onto free fermions



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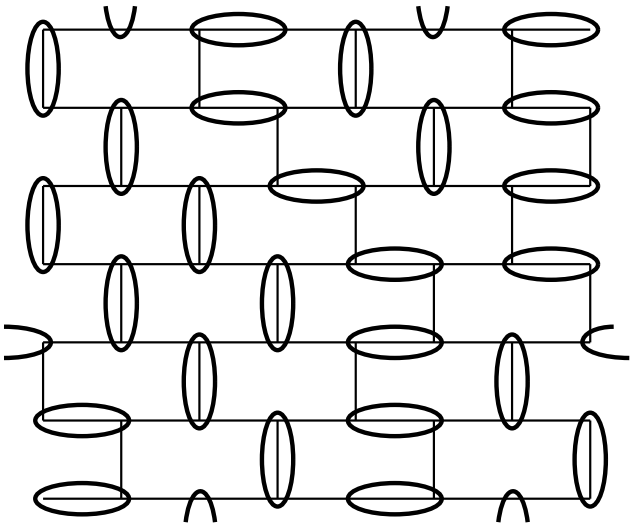

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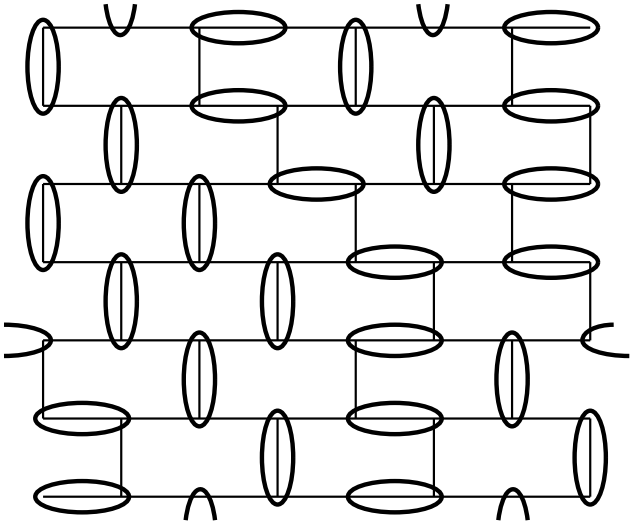

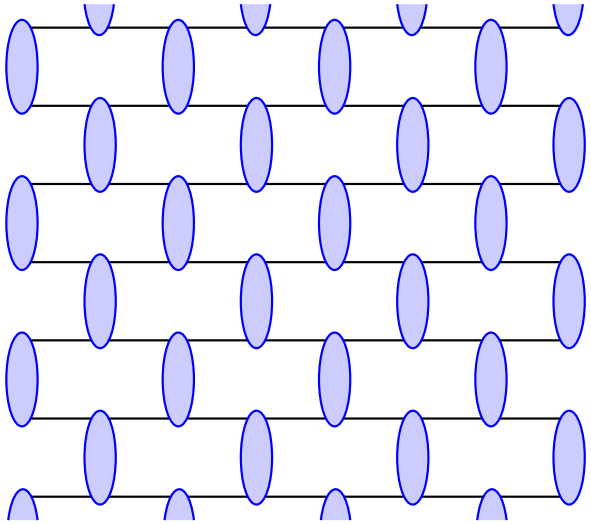

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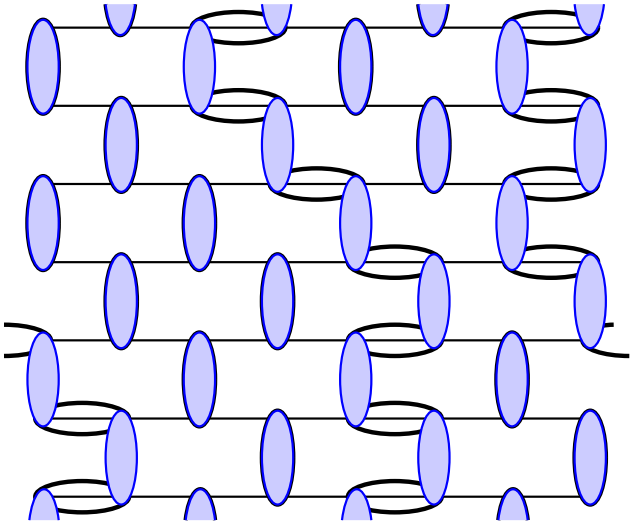

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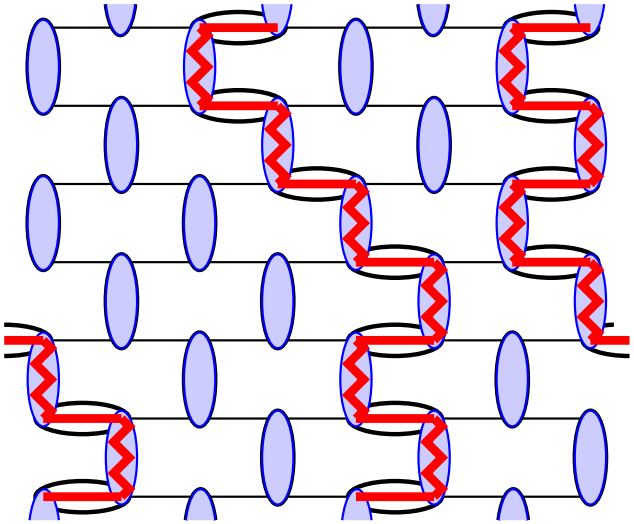

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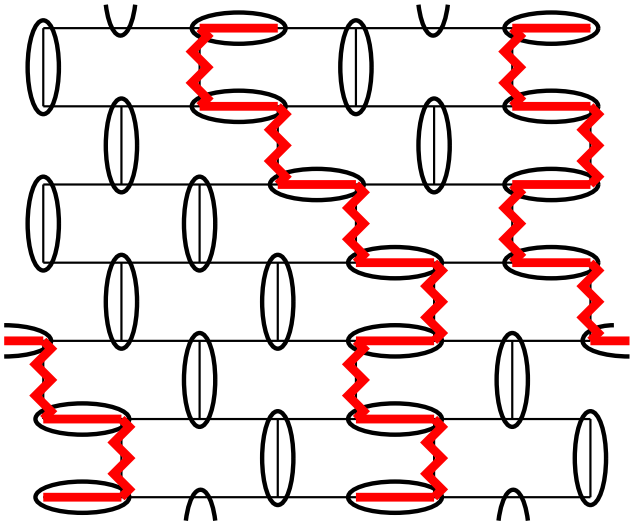
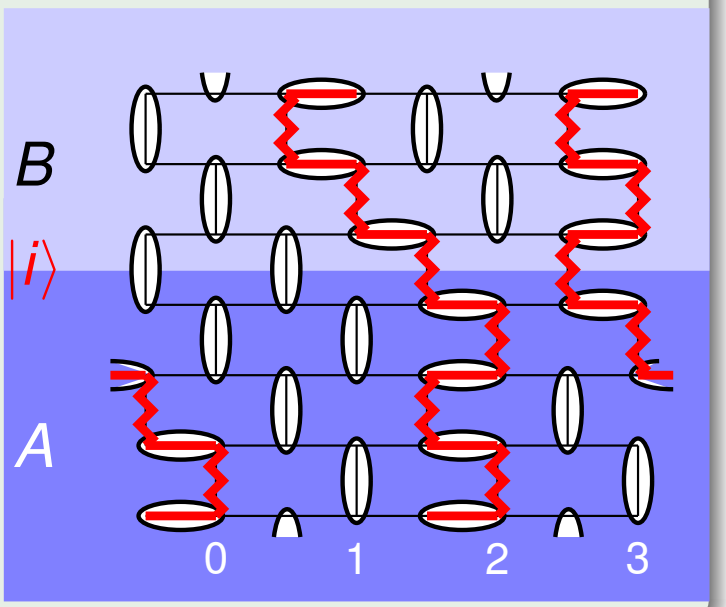


Transfert matrix $\mathcal{T}$

- fermionic representation
- conserved number of fermions
- $\mathcal{T}|0\rangle = |0\rangle$
- $\mathcal{T} c_j^\dagger \mathcal{T}^{-1} = c_j^\dagger + c_{j+1}^\dagger$
- $\mathcal{T}$ can be diagonalized exactly
- long cylinder => only the dominant eigenvector of $\mathcal{T}$ will matter.

Boundary configurations

Example : $|i\rangle = c_2^\dagger c_3^\dagger |0\rangle$

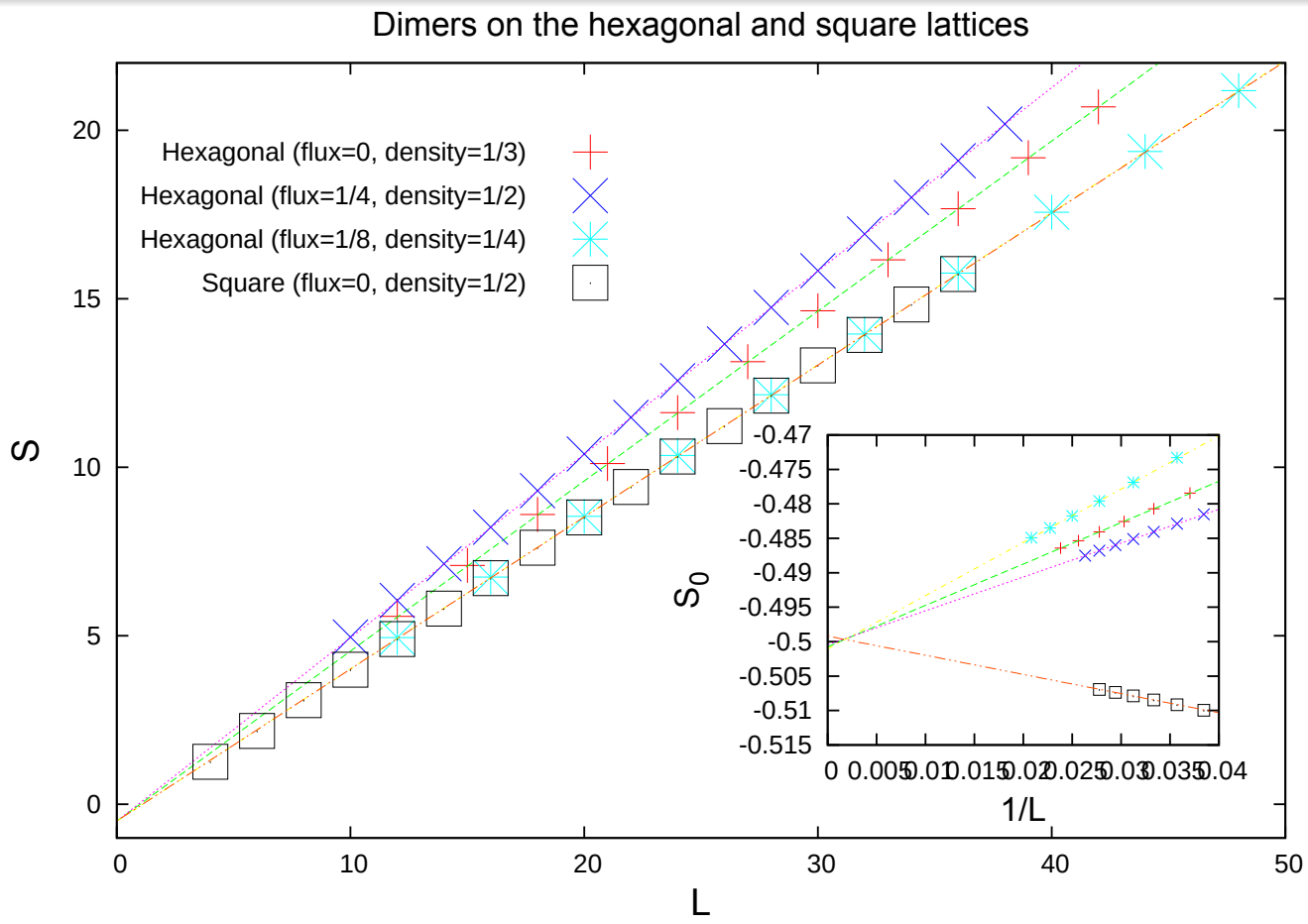


- Boundary configuration : $|i\rangle = c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger \dots c_{\alpha_n}^\dagger |0\rangle$
- $p(i)$ given by a Vandermonde determinant :

$$p(i) = \frac{1}{L^n} \prod_{1 \le j < j' \le n} \left| e^{2i\pi\alpha_j/L} - e^{2i\pi\alpha_{j'}/L} \right|^2$$

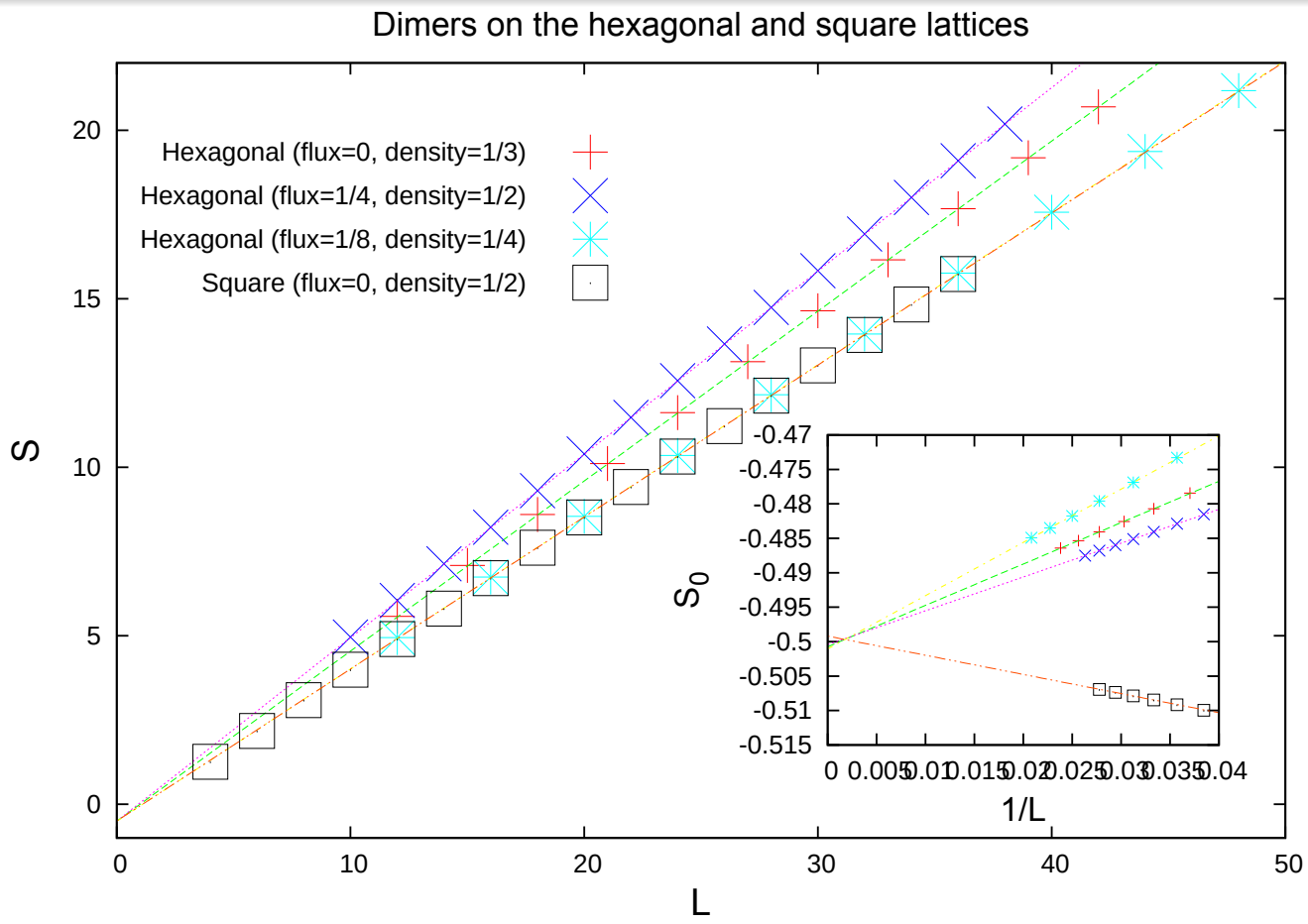
- $S^{VN} = S_{\text{Shannon}} = -\sum_i p(i) \log p(i)$

Scaling behaviour for the entropy



- Very good agreement with $S^{VN} = \mu L - 1/2$

Scaling behaviour for the entropy



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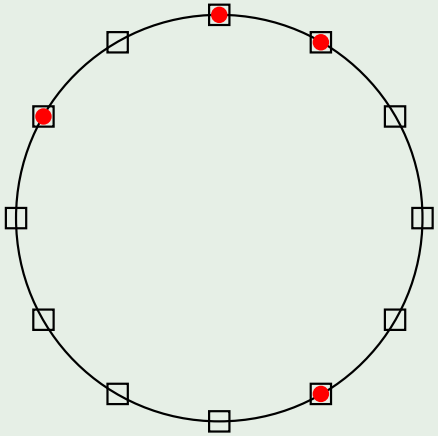
Discretized Dyson Gas on the annulus

Lattice gas



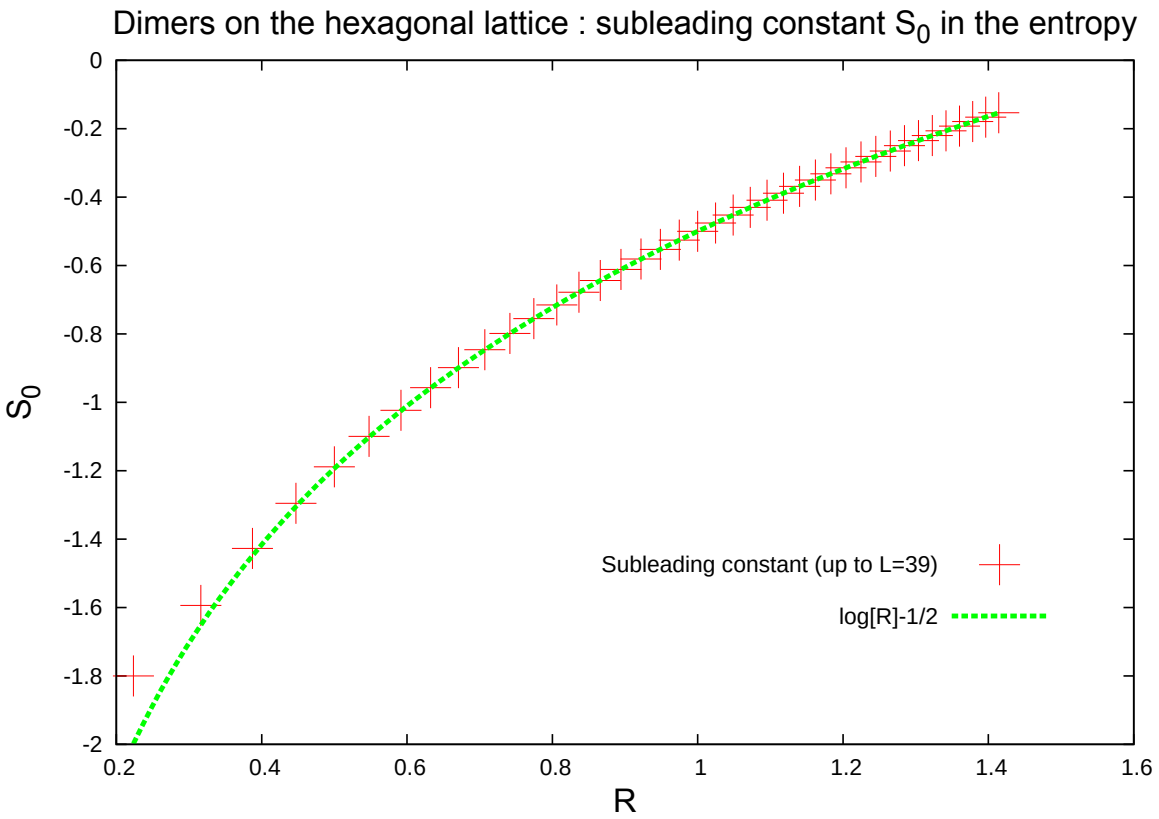
- $n = 4$ charges
- $L = 12$ sites

- n charges interacting via 2D a Coulomb potential on an annulus with L sites:

$$E(i) = -\sum_{j<j'} \log \left| e^{2i\pi\alpha_j/L} - e^{2i\pi\alpha_{j'}/L} \right| + \frac{n}{2} \log L$$

- $Z(\beta) = \sum_i e^{-\beta E(i)}$
- $\beta = 2$: dimers on the honeycomb
- $\beta = 4$: Haldane-Shastry chain

Subleading constant for general β



Results

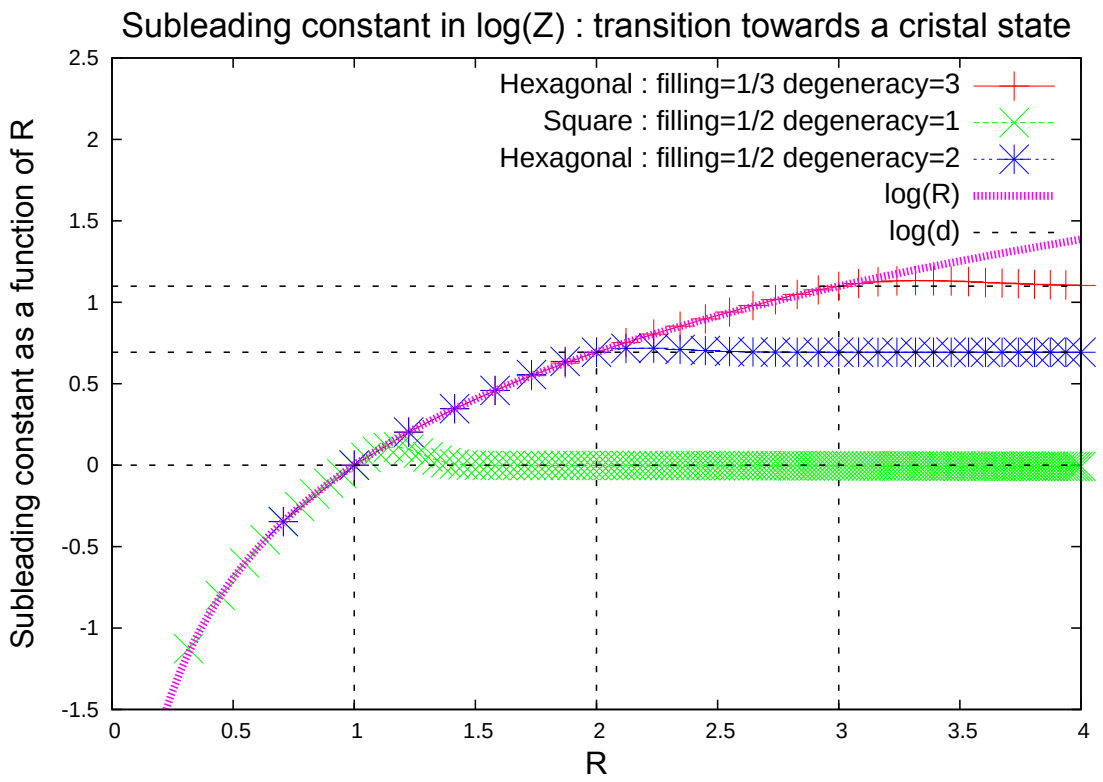
- $\beta = 2R^2$
- Subleading constant :

$$S_0(R) = \log R - 1/2$$

- R : compactification radius of the corresponding free bosonic field (CFT with $c = 1$)

Phase transition towards a crystal state

- d : degeneracy of the ground state
- $\beta \to \infty$: $Z(\beta) \simeq d e^{-\beta E_0} \Rightarrow S_0 = \log d$



Phase transition at $R = d$

- $(logZ)_0 = \begin{cases} \log R & , \quad R \leq d \\ \log d & , \quad R > d \end{cases}$
- $S_0 = \begin{cases} \log R - 1/2 & , \quad R \leq d \\ \log d & , \quad R > d \end{cases}$

Free field approach

- Dimers on a bipartite lattice $\leftrightarrow$ Free compactified boson.
- Continuous electrostatic energy on the annulus :

$$E[\rho] = -\frac{1}{2}\int_0^{2\pi} d\theta \rho(\theta) \int_0^{2\pi} d\theta' \rho(\theta') \log \left| 2\sin(\frac{\theta-\theta'}{2}) \right|$$

- Partition function :

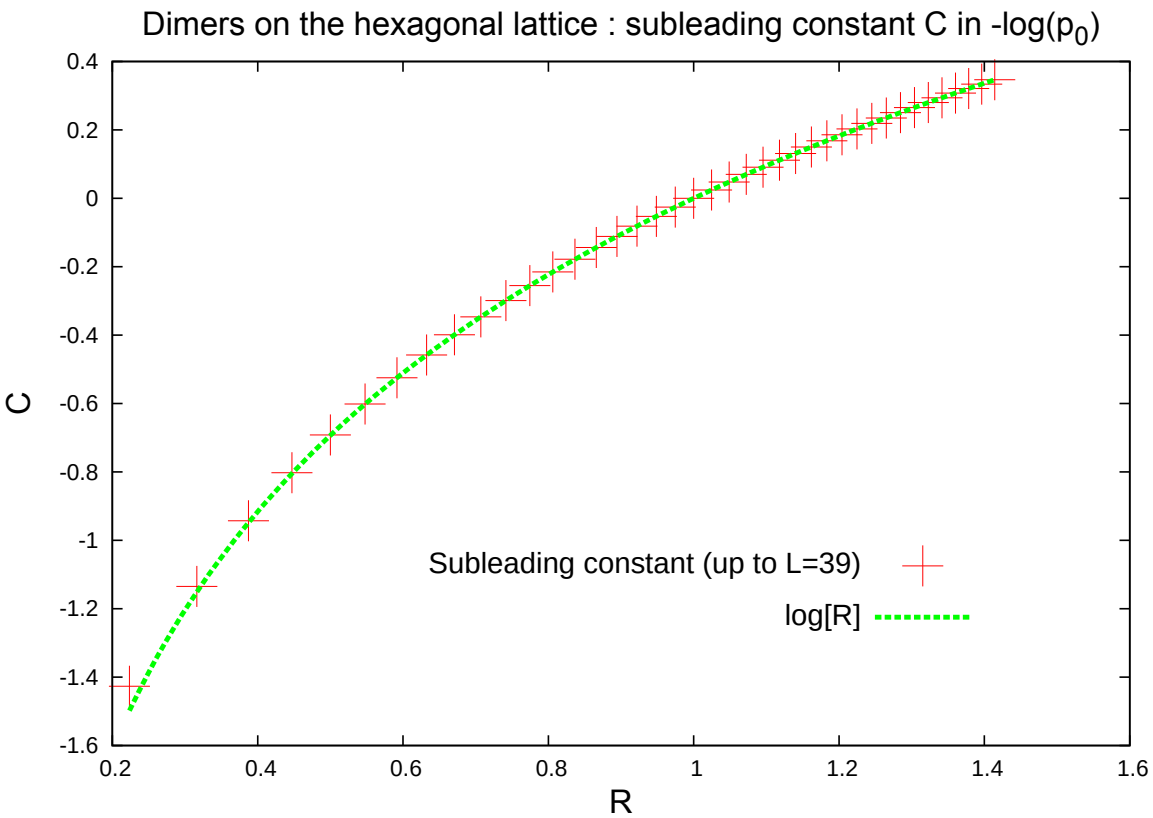
$$Z(\beta) = \int [\mathcal{D}\rho] e^{-\beta E[\rho]} = \prod_{n\geq 1} \frac{1}{\pi\beta n} \overbrace{=}^{\text{Zeta reg.}} \sqrt{\frac{\beta}{2}} = R$$

- Entropy :

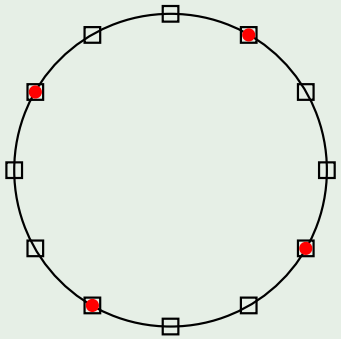
$$S(\beta) = (1-\beta\partial_\beta)\log Z(\beta) = \log R - 1/2$$

Probability of the most likely configuration

- $p_0(\beta) \leftrightarrow$ most likely configuration
- scaling of $-\log p_0$



$p_0(\beta)$ for $n = 4$, $L = 12$



Subleading constant γ

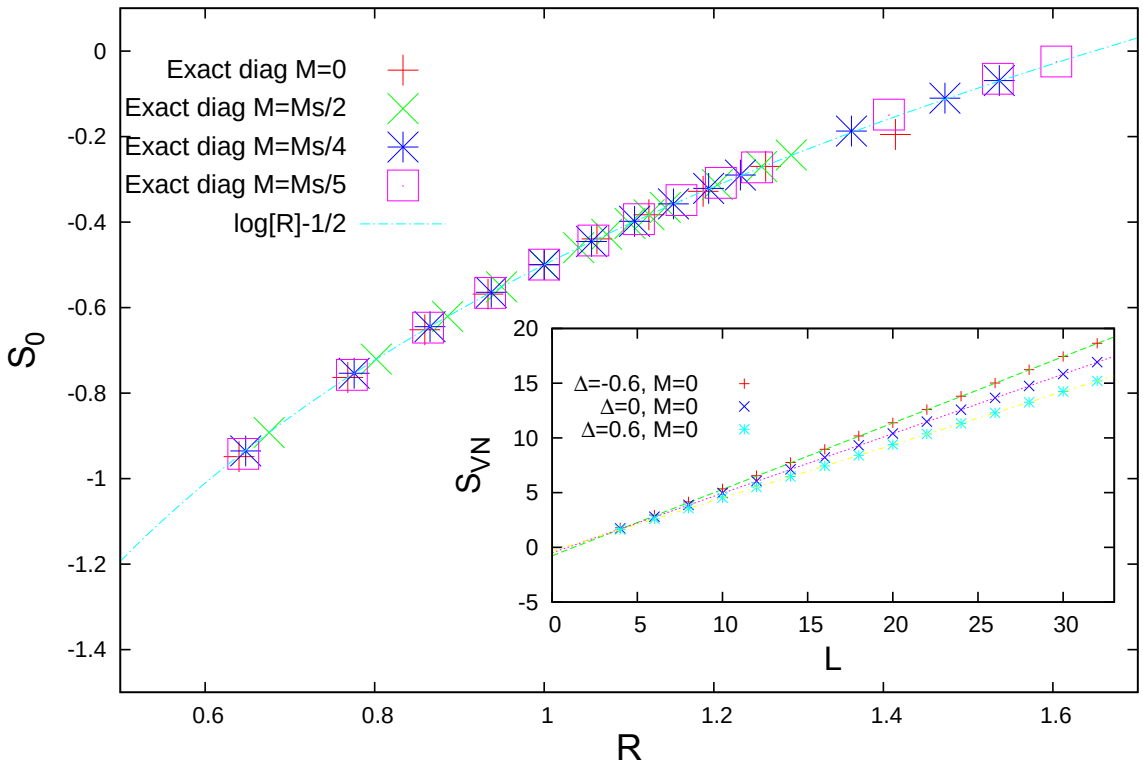
- $\gamma = \log R$
- Analytic proof in the case $\beta \in 2\mathbb{N}$, $2\beta < L/n$

Quantum spin chain : XXZ

XXZ chain

$$\mathcal{H} = \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^z$$

Spin 1/2 XXZ chain : subleading constant in the entropy



Subleading constant in the entropy

- $R = f(\Delta, h)$ in the critical area
- $S_0 = \log R - 1/2$

Summary

- Entanglement entropy of critical 2D wave functions
- Choice of geometry and wave functions => Calculations simplify considerably
- Universal subleading constant $S_0 = \log R - 1/2$
- (See also related work by Hsu & al, PRB 2009)