Excitations at fixed momentum in a 1D Bose gas

Exam of the "Quantum Fluids" course, M1 ICFP 2015-2016

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In the whole exam we shall consider a homogeneous one dimensional gas of spinless non-relativistic bosons of mass m. We will study the gas excitations using a classical field model with contact interactions in the grand canonical ensemble. That is equivalent to take the (grand canonical) hamiltonian :

$$H = \int_{\mathbb{R}} dx \left\{ \frac{\hbar^2}{2m} \left| \frac{d\psi}{dx} \right|^2 + \frac{g}{2} |\psi|^4 - \mu |\psi|^2 \right\} \tag{1}$$

and the equation of motion

$$i\hbar \partial_t \psi(x,t) = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + g|\psi|^2 - \mu \right] \psi \tag{2}$$

which is formally identical to the Gross-Pitaevskii equation that we introduced in the lectures. In the whole exam we limit ourselves to the $g > 0$ case.

1 Ground state

1. We wish to find the field $\psi(x)$ that minimizes the energy H. Find a differential equation imposing that $\delta H = 0$ for any variation $\delta\psi$, $\delta\psi^*$

$$\delta H = \int \delta\psi^* \frac{\partial H}{\partial \psi^*} + \delta\psi \frac{\partial H}{\partial \psi} = 0 \qquad \forall \quad \delta\psi, \ \delta\psi^* \tag{3}$$

   You will recognize the time-independent Gross-Pitaevskii equation.

2. Show that the uniform field $\psi_0 = \sqrt{\rho}$, where ρ is the gas density, is a solution of the equation and calculate μ as a function of ρ (equation of state).
3. Calculate the momentum $P_0 = -i\hbar \int \psi_0^* \frac{d\psi_0}{dx}$ of such solution.

2 Field that minimizes the energy at fixed momentum

We now wish to find the field $\psi(x)$ that minimizes the energy $(H - E_0)$ for a fixed average momentum, where $E_0 = H[\psi_0]$ is the energy of the ground state. To this end we use the method of Lagrange multipliers and look for the extrema of

$$F = (H - E_0) - vP \qquad \text{with} \qquad v \in \mathbb{R} \qquad \text{and} \quad P = -i\hbar \int \psi^* \frac{d\psi}{dx} \tag{4}$$

1. Find the differential equation satisfied by the minimizer ψ by imposing $\delta F = \int \delta\psi^* \frac{\partial F}{\partial \psi^*} + \delta\psi \frac{\partial F}{\partial \psi} = 0$ for any variation $\delta\psi$ and $\delta\psi^*$
2. Make the equation adimensional by choosing the healing length $\xi = \sqrt{\frac{\hbar^2}{m\mu}}$ as length unit, $\bar{x} = \frac{x}{\xi}$, and by introducing the normalized field $\bar{\psi} = \frac{\psi}{\sqrt{\rho}}$. We look for localized excitations such that, in our infinite system in the thermodynamic limit, we have

$$\lim_{x\to\pm\infty} |\bar{\psi}| = 1 \qquad \text{and} \qquad \lim_{x\to\pm\infty} \frac{d\bar{\psi}}{dx} = 0\,. \tag{5}$$

   Show that the obtained non-linear Schrödinger equation (NLS) for $\bar{\psi}$ depends only on a single parameter $s \equiv \frac{v}{c}$ where $c = \sqrt{\frac{\mu}{m}}$ is the speed of sound. In all the following we shall consider $|s| \leq 1$.

3. In the following questions we will integrate the NLS. The idea is to recognize a formal equivalence with a Newton equation for a fictitious particle that moves in a 1D potential, an equation that one can integrate using conservation of energy for the fictitious particle.
   (a) Introduce in the previously found NLS the modulus-phase decomposition of the field
   $$\bar{\psi}(\bar{x}) = \phi(\bar{x})e^{i\theta(\bar{x})} \qquad \text{with} \qquad \phi(\bar{x}), \theta(\bar{x}) \in \mathbb{R} \tag{6}$$
   (b) Take the imaginary part of the NLS, recognize a total derivative and obtain
   $$\phi^2(\theta' - s) = \text{constant} \tag{7}$$
   where $\theta' \equiv \frac{d\theta}{d\bar{x}}$.
   (c) By imposing $\lim_{x\to\pm\infty} \theta' = 0$ and $\lim_{x\to\pm\infty} \phi = 1$ (localized excitation) find the value of the constant and express θ' as a function of s and ϕ.
   (d) Take the real part of the NLS and show that one obtains for ϕ the following Newton equation (where our $\bar{x}$ plays the role of a time t and our $\phi(\bar{x})$ plays the role of $x_{\text{Newt}}(t)$)
   $$\phi'' = -\frac{dU_{\text{Newt}}(\phi)}{d\phi} \quad \text{with} \quad U_{\text{Newt}}(\phi) = \frac{s^2}{2\phi^2} + \frac{s^2\phi^2}{2} - \frac{\phi^4}{2} + \phi^2 \tag{8}$$
   (e) Energy conservation for the Newton problem reads
   $$\frac{1}{2}\phi'^2 + U_{\text{Newt}}(\phi) = E_{\text{Newt}} \tag{9}$$
   By using the condition of a localized perturbation : $\lim_{\bar{x}\to\pm\infty} \phi' = 0$ and $\lim_{\bar{x}\to\pm\infty} \phi = 1$ express E_{Newt} as a function of s and write explicitly the equation
   $$\frac{d\phi}{\sqrt{2(E_{\text{Newt}} - U_{\text{New}}(\phi))}} = d\bar{x} \tag{10}$$
   by replacing in it E_{Newt} and $U_{\text{New}}(\phi)$ by their values.
   (f) Show that after the change of variable $F = \phi^2$ (we suppose $\phi^2 < 1$) one obtains the equation
   $$\frac{dF}{2(1-F)\sqrt{F - s^2}} = d\bar{x} \tag{11}$$
   (g) We give the primitive
   $$\int \frac{dF}{2(1-F)\sqrt{F-s^2}} = -\frac{1}{2}\frac{1}{\sqrt{1-s^2}} \ln\left(\frac{\sqrt{1-s^2} - \sqrt{F-s^2}}{\sqrt{1-s^2} + \sqrt{F-s^2}}\right) \tag{12}$$
   Deduce that
   $$\phi_s^2(\bar{x}) = s^2 + (1-s^2)\,\text{th}^2(\sqrt{1-s^2}\,\bar{x}) \tag{13}$$
   is a solution of the equation for the modulus square of $\bar{\psi}$. One finds in fact an infinite number of degenerate solutions that differ by a translation $\bar{x} \to \bar{x} - \bar{x}_0$.
   (h) Show that
   $$\theta_s = \frac{\pi}{2} - \arctan\left(\frac{\sqrt{1-s^2}\,\text{th}(\sqrt{1-s^2}\,\bar{x})}{s}\right) \tag{14}$$
   is a solution of the equation (7) for the phase of $\bar{\psi}$. One will use the equation (13). In the following we shall name $\bar{\psi}_s(x)$ the solution field.
4. Make a figure of the modulus square $\phi_s^2(x)$ and the phase $\theta_s(x)$ of the solution field as a function of $\bar{x}$ in the particular case $s = 0^+$. Show that the density perturbation extends over a characteristic length that one will specify and that the phase varies abruptly over the same length.
5. Give the value of the phase jump for any s ($|s| \leq 1$),
   $$\Delta\theta_s \equiv \lim_{x\to-\infty} \theta_s(x) - \lim_{x\to+\infty} \theta_s(x) \tag{15}$$
6. Show that for $s = 1$, $\psi_s(x)$ reduces to the ground state ψ_0 within a constant phase factor.

3 Time evolution and interpretation of v (hence of s)

1. Show that if $\psi(x)$ is a solution of the non-linear Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + g|\psi|^2 - \mu + i\hbar v\frac{d}{dx}\right)\psi(x) = 0 \tag{16}$$

then $\tilde{\psi}(x,t) \equiv \psi(x - vt)$ is a solution of the time-dependent non-linear Schrödinger equation (2).

2. Give a physical interpretation of this fact.
3. Deduce that the solution found in section 2 for $\psi(x)$ describes a phase and density perturbation that propagates without deformation (it's in fact a grey soliton) and give the physical interpretation of v.

4 Energy and momentum of the soliton

1. We want to calculate the energy of the field $\psi_s(x)$ of section 2 relative to the energy of the ground state. By using the previously obtained results, show that we are led to the simple integral

$$\epsilon_s \equiv H[\psi_s] - E_0 = \frac{c\,p_F}{\pi}\int_{\mathbb{R}} d\bar{x}(\phi_s(\bar{x})^2 - 1)^2 \tag{17}$$

where $p_F = \hbar\pi\rho$ is the Fermi momentum of a 1D Fermi gas with the same density.

2. Calculate the integral in (17) and show that

$$\epsilon_s = \frac{c\,p_F}{\pi}\frac{4}{3}(1 - s^2)^{3/2} \tag{18}$$

3. We now calculate the momentum $p_s \equiv P[\psi_s]$ of the field $\psi_s(x)$. Knowing that ψ_s minimizes $H[\psi] - E_0 - vP[\psi]$, show that

$$\frac{d}{ds}\left[H[\psi_s] - E_0 - csP[\psi_s]\right] = -c\,P[\psi_s] \tag{19}$$

Deduce that

$$\frac{dp_s}{ds} = \frac{1}{c\,s}\frac{d\epsilon_s}{ds} \tag{20}$$

4. What is the value of $p_{s=1}$? By integrating the previous equation between 1 and s,

$$p_s - p_{s=1} = -\frac{4p_F}{\pi}\int_1^s \sqrt{1 - t^2}dt \tag{21}$$

show that one finally has

$$p_s = \frac{2p_F}{\pi}\left(\frac{\pi}{2} - s\sqrt{1 - s^2} - \text{arcsin}s\right) \tag{22}$$

Using the equations (22) and (18) one can give a parametric representation of the curve $\epsilon_s(p_s)$ as in figure 1.

5. Starting from equations (22) and (18) show analytically that the curve $\epsilon_s(p_s)$ is symmetric with respect to $p_s = p_F$.
6. Give the values of the quantities ϵ_s, p_s and $\frac{d\epsilon_s}{dp_s}$ for $s = 0, s = 1, s = -1$.

5 Landau critical velocity

1. Recall the definition of Landau critical velocity for an excitation spectrum with a dispersion relation $\epsilon(k)$.
2. Show that the excitations of the branch ψ_s shown on the figure 1 lead to a Landau critical velocity that is lower than the one of the phononic excitation branch, and of which you shall specify the value.
3. Contrarily to what we have done up to now, in the following group of questions we will first consider a finite size system of density $\rho = N/L$, with periodic boundary conditions, and we will take the thermodynamic limit $N \to \infty$, $L \to \infty$ with $N/L = \rho$, only in a second step.
   (a) Let us consider, for the finite size system, the field ψ^L that minimizes the energy at fixed momentum $p_s = 2p_F$. What is the momentum per particle ?

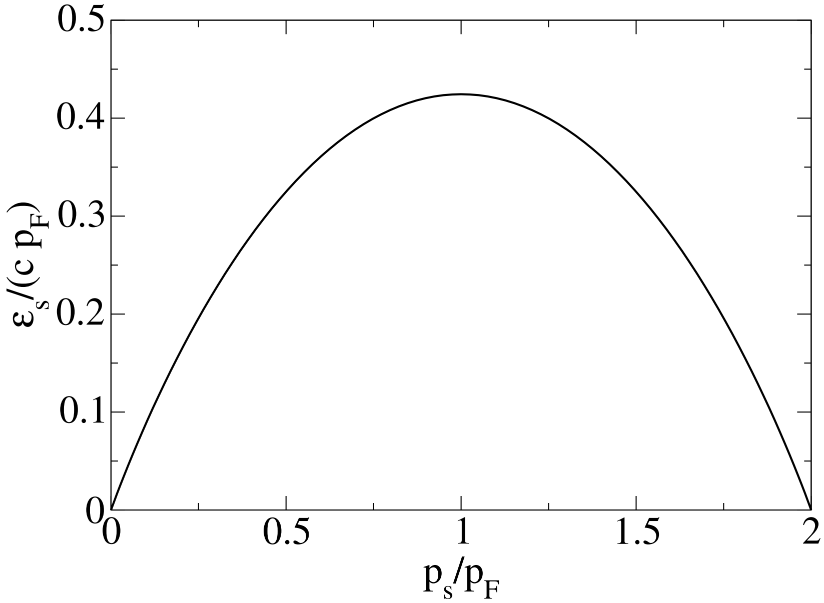

FIGURE 1 – Dispersion relation $\epsilon_s(p_s)$ for the minimal energy state at fixed momentum in a 1D Bose gas. This excitation branch, that we studied here within the classical field approximation, is also obtained in the exact solution of the quantum problem. In that context it is called the "Lieb second branch".

(b) By which transformation the field ψ^L can be obtained starting from the ground ψ_0^L ? What is its excitation energy ϵ^L (energy difference between ψ^L and ψ_0^L) ?

(c) One will assume that the critical Landau velocity for a large enough finite size system is imposed by the state ψ^L. Give its value (you don't need to perform calculations). What is its value in the thermodynamic limit ?

(d) In our 1D system, the excitation energy and the momentum of the field ψ^L remain non-infinite in the thermodynamic limit. Is it the same in 3D ?

6 Potential barrier for the decay of the supercurrent state

1. Imagine that we have prepared the system in the state ψ_s with $p_s = 2p_F$. Estimate the height of the potential barrier that separates the supercurrent state ψ_s from the ground ψ_0 for our 1D system in the thermodynamic limit.