“Quantum Fluids" exam, M1 ICFP 2014-2015

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Total time : 3 hours, calculators and cell phones are not allowed

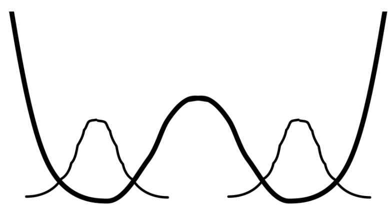
1 Part I : Phase dynamics in a bosonic Josephson junction

1.1 The relative phase distribution

In the course we have introduced so-called “phase states" for which the relative phase of two superfluids is well-defined. The state $|\varphi\rangle$ defined in the following way has a relative phase of 2φ :

$$|\varphi\rangle = |N : \frac{1}{\sqrt{2}}\left(e^{i\varphi}\phi_a + e^{-i\varphi}\phi_b\right)\rangle \tag{1}$$

where we have put N particles in the same superposition of the two wave functions ϕ_a et ϕ_b with a relative phase equal to 2φ. We assume N to be even in the whole exercise.



ϕ_a ϕ_b

FIGURE 1 – Doubl-well potential

The wave functions ϕ_a and ϕ_b respectively represent de ground state of a Bose-Einstein condensate in the left and right well of a double-well atomic trapping potential. To describe the system at zero temperature (which we assume throughout this exercise), we will restrict ourselves to these two modes. The Fock states $|N_a, N_b\rangle$ having exactly N_a particles in the wave function ϕ_a and N_b particles in the wave function ϕ_b thus form a complete basis for the considered Hilbert space.
By passing through second quantisation notation ($a^\dagger$ and $b^\dagger$ create a particle in the wave function ϕ_a and ϕ_b) one can express the phase state as a superposition of the Fock states

$$|\varphi\rangle = \frac{1}{\sqrt{N!}}\left(\frac{e^{i\varphi}a^\dagger + e^{-i\varphi}b^\dagger}{\sqrt{2}}\right)^N |0\rangle = \sum_{N_a=0}^{N} c_{N_a} e^{i(N_a-N_b)\varphi}|N_a, N_b\rangle \tag{2}$$

where $N_b = N - N_a$ and the coefficients $c_{N_a} = \sqrt{\frac{1}{2^N}\binom{N}{N_a}}$ involve the binomial coefficients.

1. Let $\hat{n} = \hat{N}_a - \hat{N}_b$ be the difference operator of the numbers of particles in the two wells. Show that

$$e^{i\alpha\hat{n}}|\varphi\rangle = |\varphi + \alpha\rangle \tag{3}$$

   What can one conclude from this relation ?

2. In contrast to what we did in the course, we will not define a phase operator here. Instead we will introduce a phase distribution for the different states of the two condensates. Show first that

$$\langle\varphi|\varphi'\rangle = [\cos(\varphi - \varphi')]^N \tag{4}$$

   What can one conclude for the “width of the phase state" in the limit of large N ? In other words, what is the dependence of $\langle\varphi|\varphi'\rangle$ on $(\varphi - \varphi')$ for large N ?

3. Each state with N particles of the two condensates can be expressed in the over-complete basis of phase states

$$|\psi\rangle = \mathcal{A} \int_{-\pi/2}^{\pi/2} \frac{d\varphi}{\pi} c(\varphi) |\varphi\rangle \tag{5}$$

where $\mathcal{A}$ is a normalisation constant.

Show that in the limit of large N the coefficients $c(\varphi)$ are given by the formula

$$c(\varphi) = \mathcal{A}^{-1} \sum_{N_a=0}^{N} (c_{N_a})^{-1} e^{-i(N_a - N_b)\varphi} \langle N_a, N_b | \psi \rangle \tag{6}$$

Use the relation

$$\sum_{m\in\mathbb{Z}} e^{imx} = 2\pi\, \delta(x) \quad \text{pour} \quad x \in [-\pi, \pi] \tag{7}$$

4. By approximating $\langle \varphi | \varphi' \rangle \simeq \sqrt{2\pi/N} \delta(\varphi - \varphi')$ for large N, show that in order to have

$$\int_{-\pi/2}^{\pi/2} \frac{d\varphi}{\pi} |c(\varphi)|^2 = 1 \tag{8}$$

the normalisation constant $\mathcal{A}$ has to be

$$\mathcal{A} \simeq \left(\frac{N\pi}{2} \right)^{1/4} \tag{9}$$

We will interpret $|c(\varphi)|^2$ as a probability distribution for the relative phase between the two condensates.

5. By using the asymptotic expression of the binomial coefficients for large N

$$\binom{N}{N_a} \simeq \frac{2^N}{\sqrt{\pi N/2}} e^{-\frac{(N_a - N_b)^2}{2N}} \tag{10}$$

calculate the distribution $P(n)$ of the relative number of particles $n = (N_a - N_b)$ in the phase state (2). Calculate the mean value of n and the standard deviation.

1.2 Blurring of the phase after interruption of the tunnel coupling

Suppose that, thanks to the tunneling effect, the two condensates are initially prepared in a state $|\psi(t = 0)\rangle$ where the distribution of the relative phase is given by a Gaussian with width $\Delta\varphi$

$$c(\varphi, t = 0) = \mathcal{G}_0 \exp\left(-\frac{\varphi^2}{4\,\Delta\varphi^2} \right) \qquad \text{with} \quad \frac{1}{\sqrt{N}} \ll \Delta\varphi \ll 1 \tag{11}$$

and $\mathcal{G}_0$ a constant.

1. By neglecting the width of the phase state with respect to $\Delta\varphi$ since $\Delta\varphi \gg 1/\sqrt{N}$, show that the choice (11) corresponds to taking a Gaussian distribution for the relative number of particles

$$\langle N_a, N_b | \psi(0) \rangle = \mathcal{N} \exp\left(-\frac{(N_a - N_b)^2}{4\,\Delta n^2} \right) \tag{12}$$

with $\Delta n \Delta\varphi = 1/2$. One has to change the sum into an integral for which one takes the boundaries to infinity. You can use

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \qquad \text{for} \quad \mathbb{R}e(a) > 0 \tag{13}$$

2. At $t = 0$ one interrupts the tunnel coupling between the two condensates by, for example, increasing the barrier between the two wells. As we will show later in section 1.4, the system initially in the state $|\psi(0)\rangle$ will then evolve according to the Hamiltonian

$$H = \frac{\hbar\chi}{4} (\hat{N}_a - \hat{N}_b)^2 \tag{14}$$

where one has assumed that the interactions between the atoms are repulsive.

Express the coefficient $\langle N_a, N_b | \psi(t) \rangle$ as a function of $\langle N_a, N_b | \psi(0) \rangle$ and calculate $|c(\varphi, t)|^2$ up to a normalisation constant by changing the summing into an integral and taking the integration limits to infinity.

3. Show that the phase distribution $|c(\varphi,t)|^2$ remains gaussian and that the variance $\Delta\varphi^2$ as a function of time takes the form
$$\Delta\varphi^2(t) = \Delta\varphi^2(0) + \alpha\,\Delta n^2 t^2 \tag{15}$$
Give the expression for the coefficient a.
4. Point out the correspondence between our problem and the problem of spreading of a gaussian wave packet for a free particle. What corresponds in our case to "effective mass" of the particule ?
Evaluate the blurring time of the relative phase $t_{\rm br}$ after which
$$\Delta\varphi^2(t_{\rm br}) \simeq 1 \tag{16}$$
as a function of χ and Δn^2.

1.3 Evolution at long times : Schrödinger cats

In this section we take as initial state a phase state, defined in section 1.1
$$|\psi(0)\rangle = |\varphi\rangle \tag{17}$$
Suppose that immediately after the preparation of the state (17) at $t = 0$ one interrupts the tunnel coupling such that the system evolves according to the Hamiltonian (14) :
$$|\psi(t)\rangle = e^{-i\frac{\chi}{4}\hat{n}^2 t}|\psi(0)\rangle \qquad \text{with} \quad \hat{n} = \hat{N}_a - \hat{N}_b \tag{18}$$

1. Show that for n even one has the identity
$$e^{-i\frac{\pi}{8}n^2} = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{2}(n+\frac{1}{2})}\right) \tag{19}$$
2. Show that at the time t_c
$$t_c = \frac{\pi}{2\chi} \tag{20}$$
the initial phase state evolves into a Schrödinger cat state which is a superposition of two states with different phase. Remember that N is even.
3. Represent schematically the distribution of the phase $|c(\varphi)|^2$ with $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}[$ at three times : $t = 0$, $t = t_c$ et $t = 2t_c$. Which state does one obtain at $t = 2t_c$?

1.4 Non-linear Hamiltonian and numerical application

In this section we derive the expression (14) for the Hamiltonian in case of two condensates without tunnel coupling and in the presence of repulsive interaction between the atoms. One supposes that the two wells a and b are identical and one calls $H_a(\hat{N}_a)$ the Hamiltonian of a pure condensate with N_a atoms in well a and similarly for b.

1. Explain why in absence of the tunnel coupling one can write the following for the system with the two condensates
$$H(\hat{N}_a, \hat{N}_b) = H_a(\hat{N}_a) + H_a(\hat{N}_b) \tag{21}$$
2. One considers the states $|\psi\rangle$ of the two condensates with the same average number of particles in each well and with weak relative fluctuations in the particle number
$$\frac{\Delta N_a}{\langle N_a\rangle} = \frac{\Delta N_b}{\langle N_b\rangle} \ll 1 \qquad \text{and} \qquad \langle N_a\rangle = \langle N_b\rangle = \frac{N}{2} \tag{22}$$
Carefully show that the two initial states respectively considered in section 1.2 and in section 1.3 satisfy this constraint, and explain why this remains true at each time when the system evolves with the Hamiltonian (14).

3. One expands the Hamiltonian (21) around $N_a = \langle N_a \rangle = N/2$ and $N_b = \langle N_b \rangle = N/2$ up to order two. Explain how one gets

$$H(\hat{N}_a, \hat{N}_b) = E_0 + \frac{\hbar\chi}{4}\hat{n}^2 \qquad \text{avec} \quad \hat{n} = \hat{N}_a - \hat{N}_b \tag{23}$$

and relate the coefficient χ to a derivative of the chemical potential μ_a (here $\mu_a = \mu_b$) of a condensate in the well a or b.

4. In the Thomas-Fermi limit one has for a harmonic isotropic well with frequency a

$$\mu_a(N_a) = \frac{\hbar\omega}{2}\left(15 N_a \frac{a_s}{a_{\text{ho}}}\right)^{2/5} \tag{24}$$

where a_s is the s-wave scattering length characterising the interaction between two atoms, a_{ho} is the extension of the ground state of the harmonic oscillator in the well, and N_a is the number of atoms.

Estimate the time t_c (20) at which one obtains a cat for for $\langle N_a \rangle = \langle N_b \rangle = 500$, $a_s = 5$ nm considering rubidium atoms and $\omega/(2\pi)$=50 s^{-1}. Remember that $a_{\text{ho}} = \sqrt{\frac{\hbar}{m\omega}}$ with m the mass of a rubidium atom, $m = 87 \times 1.66 \times 10^{-27}$kg et $\hbar \simeq 10^{-34}$ m^2 kg s^{-1}.
Estimate also the blurring time t_{br} (16) by taking $\Delta n(0) = \sqrt{N}$.

2 Part II : Fluctuations in a gas of bosons cut into two pieces

One considers an ideal gas of bosons with mass m and trapped in an isotropic harmonic potential in three dimensions. The one-particle Hamiltonian takes the form

$$\hat{h}_1 = -\frac{\hbar^2\Delta}{2m} + \frac{1}{2}m\omega^2\hat{r}^2\,. \tag{25}$$

One cuts the trapped gas into two (to the left and to the right of the plane $x = 0$) and one measures the fluctuations of the difference of particle number $N_G - N_D$ (see Figure 2). One describes the system in the grand-canonical ensemble.

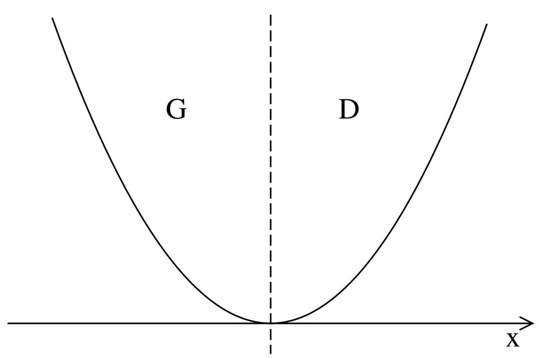


FIGURE 2 – *One cuts the trapped gas into two (to the left (*gauche*) and to the right (*droite*) of the plane $x = 0$) and one measures the fluctuations of the difference of particle number $N_G - N_D$.*

2.1 Exact solution

1. The one-particle operator for which the mean value gives the spatial density of the gas can be written as

$$\hat{\mathcal{A}}(\mathbf{r}) = \sum_i |i : \mathbf{r}\rangle\langle i : \mathbf{r}| \tag{26}$$

where the index i sums over all the particles of the system. It is implied that $|i:\mathbf{r}\rangle\langle i:\mathbf{r}|$ is the operator $|\mathbf{r}\rangle\langle\mathbf{r}|$ for the particle i and the identity for all the other particles.

Express the operators $\hat{N}_G$ et $\hat{N}_D$ that count the number of particles in the half-space to the left and to the right of the plane $x=0$ as integrals of $\hat{\mathcal{A}}(\mathbf{r})$.

2. Explain why $\langle N_G - N_D\rangle = 0$.
3. The correlation functions $g^{(1)}(\mathbf{r},\mathbf{r}')$ and $g^{(2)}(\mathbf{r},\mathbf{r}')$, respectively involving one-particle and two-particle operators, are defined by

$$
\begin{aligned}
g^{(1)}(\mathbf{r},\mathbf{r}') &= \left\langle \sum_{i=1} |i:\mathbf{r}\rangle\langle i:\mathbf{r}'| \right\rangle && (27)\\
g^{(2)}(\mathbf{r},\mathbf{r}') &= \left\langle \sum_{i=1}\sum_{j\neq i} |i:\mathbf{r}\rangle\langle i:\mathbf{r}| \otimes |j:\mathbf{r}'\rangle\langle j:\mathbf{r}'| \right\rangle . && (28)
\end{aligned}
$$

Physically $g^{(1)}(\mathbf{r},\mathbf{r}')$ represents the spatial coherence between the points $\mathbf{r}$ and $\mathbf{r}'$ in the gas, while $g^{(2)}(\mathbf{r},\mathbf{r}')$ gives the probability to find a particle at $\mathbf{r}$ and one at $\mathbf{r}'$.
Show that

$$\mathrm{Var}(N_G - N_D) = \langle N\rangle + 2\left[\int_G d^3\mathbf{r}\int_G d^3\mathbf{r}'\, g^{(2)}(\mathbf{r},\mathbf{r}') - \int_G d^3\mathbf{r}\int_D d^3\mathbf{r}'\, g^{(2)}(\mathbf{r},\mathbf{r}')\right] \tag{29}$$

where $\int_G$ and $\int_D$ denote the integrals over the half-space on the left and on the right, and where $\hat{N} = \hat{N}_G + \hat{N}_D$ is the total number operator.

4. Let $\{|\lambda\rangle\}$ be the eigen basis of $\hat{h}_1$ with $\hat{h}_1|\lambda\rangle = \epsilon_\lambda|\lambda\rangle$. One can show that in the grand-canonical ensemble the mean value of a one-particle operator $\hat{\mathcal{A}} = \sum_i A(i)$ is given by

$$\langle\mathcal{A}\rangle = \sum_\lambda \langle N_\lambda\rangle\langle\lambda|A|\lambda\rangle \tag{30}$$

with $\langle N_\lambda\rangle$ the average occupation number of the state λ.
Show that the spatial density $\rho(\mathbf{r})$ and the function $g^{(1)}(\mathbf{r},\mathbf{r}')$ can be obtained by taking the matrix elements of the one-body density operator $\hat{\rho}^{(1)}$ which was introduced in the course

$$\hat{\rho}^{(1)} = \frac{1}{z^{-1}e^{\beta\hat{h}_1} - 1} \tag{31}$$

with $z = e^{\beta\mu}$ the fugacity.

5. In the grand-canonical ensemble one can show that

$$g^{(2)}(\mathbf{r},\mathbf{r}') = g^{(1)}(\mathbf{r},\mathbf{r}')g^{(1)}(\mathbf{r},\mathbf{r}') + g^{(1)}(\mathbf{r},\mathbf{r})g^{(1)}(\mathbf{r}',\mathbf{r}') \tag{32}$$

We will then calculate the function $g^{(1)}(\mathbf{r},\mathbf{r}')$.
Express $g^{(1)}(\mathbf{r},\mathbf{r}')$ as a power series in the fugacity z.

6. The expression

$$g^{(1)}_{1P}(\mathbf{r},\mathbf{r}') = \langle\mathbf{r}'|\frac{e^{-\beta\hat{h}_1}}{\mathcal{Z}}|\mathbf{r}\rangle \tag{33}$$

represents the function $g^{(1)}$ for a single particle in a harmonic potential with $\mathcal{Z}$ the partition function for one particle.

   (a) Calculate $\mathcal{Z}$ for a particle in an isotropic harmonic potential in 3D. Express the result as a function of

$$\eta \equiv \beta\hbar\omega = \frac{\hbar\omega}{k_B T} \tag{34}$$

(b) We give the expression for the function $g_{1P}^{(1)}$ for one particle in a harmonic potential (see Landau vol. V, § 30)

$$g_{1P}^{(1)}(\mathbf{r},\mathbf{r}') = \left(\frac{1}{\sqrt{2\pi}\Delta x}\right)^3 e^{-\frac{1}{2\Delta x^2}\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)^2} e^{-\frac{\Delta p^2}{2\hbar^2}(\mathbf{r}-\mathbf{r}')^2} \tag{35}$$

with

$$\Delta x = \sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2\mathrm{th}\frac{\eta}{2}}}\,; \qquad \Delta p = \sqrt{\hbar m\omega}\frac{1}{\sqrt{2\mathrm{th}\frac{\eta}{2}}} \tag{36}$$

Use these results to express the function $g^{(1)}(\mathbf{r},\mathbf{r}')$ of the system with $\langle N\rangle$ particles as a power series in the fugacity z.

7. After redefining the fugacity $z \to \tilde{z}$ in a convenient way, show that

$$\langle N\rangle = \sum_{l=1}^{\infty} \tilde{z}^l \left(\frac{1+\coth\frac{l\eta}{2}}{2}\right)^3 \tag{37}$$

Explain why the new definition of the fugacity is useful.
In a similary way (you are not asked to do this) one finds :

$$\mathrm{Var}(N_G - N_D) = \langle N\rangle + \sum_{s=1}^{\infty} c_s \tilde{z}^s \tag{38}$$

with

$$c_s = \sum_{l=1}^{s-1} \frac{1-\frac{4}{\pi}\arctan\left[\mathrm{th}\frac{l\eta}{2}\,\mathrm{th}\frac{(s-l)\eta}{2}\right]^{1/2}}{\left(1-e^{-\eta s}\right)^3} \tag{39}$$

et $c_1 = 0$.

The equations (37), (38) and (39) constitute the exact solution of our problem. We will use these results later in section 2.3.

2.2 Pure condensate regime $T = 0$

At zero temperature all the particles are in the condensate. We thus have $\langle N_0\rangle = \langle N\rangle$ et $\langle N_{\lambda\neq 0}\rangle = 0$.

1. Starting from the equations (30) and (32) calculate the functions $g^{(1)}(\mathbf{r},\mathbf{r}')$ and $g^{(2)}(\mathbf{r},\mathbf{r}')$ in the grand-canonical ensemble in this regime as a function of the ground-state wave function ϕ_0.
2. Calculate $\mathrm{Var}(N_G - N_D)$ in this regime.

2.3 Approximate formulas for $k_BT \gg \hbar\omega$

In this part we consider the limit of large number of particles and a temperature $k_BT \gg \hbar\omega$, which is typically the case experimentally. We can thus derive simple approximate formulas which will allow us to make analytic predictions and understand the physics of the problem.

2.4 Non-condensed regime $T > T_c$

In this subsection we will see that for $T > T_c$ the variance of the difference of particle numbers on the left and on the right is only slightly super-poissonian.

1. In absence of the condensate with $N \gg 1$ and $k_BT \gg \hbar\omega$, one can expand the expressions for $\langle N\rangle$ and $\mathrm{Var}(N_G - N_D)$ by taking the limit $\eta \to 0$ in each term of the sum. Show that to lowest order one obtains

$$\langle N\rangle = \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(\tilde{z}) \tag{40}$$

with $g_\alpha(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^\alpha}$ the Bose function with index α.

2. Derive the expression for the critical temperature T_c for condensation as a function of $\langle N \rangle$ and ω.
3. Take the same limit $\eta \to 0$ in the expression (38) for $\mathrm{Var}(N_G - N_D)$. Keep the dominant order and express $\mathrm{Var}(N_G - N_D)/\langle N \rangle$ as a function of $\tilde{z}$ and T/T_c.
4. One has $\zeta(2)/\zeta(3) \simeq 1.37$ with $\zeta(\alpha) = g_\alpha(1)$ the Riemann function with argument α. Show that for $T = T_c$ $\mathrm{Var}(N_G - N_D)/\langle N \rangle$ is slightly bigger than one, so that the variance of the difference of particle numbers is only slightly super-poissonian.

2.5 Condensed regime $T < T_c$

In this subsection we will see that for $T < T_c$ the variance of the difference of particle numbers on the left and on the right is strongly super-poissonian.

1. The exact results (37)-(39) cannot be used easily in the condensed regime because the sums converge very slowly for $\tilde{z} \simeq 1$. Show that one can rewrite these results in following equivalent form

$$\langle N \rangle = \langle N_0 \rangle + \sum_{l=1}^{\infty} \tilde{z}^l \left[-1 + \left(\frac{1 + \coth(l\eta/2)}{2} \right)^3 \right], \tag{41}$$

and

$$\mathrm{Var}(N_L - N_R) = \langle N \rangle + c_\infty \langle N_0 \rangle + \sum_{s=1}^{\infty} (c_s - c_\infty) \tilde{z}^s \tag{42}$$

where $\langle N_0 \rangle$ is the average number of particles in the condensate and where we have introduced $c_\infty \equiv \lim_{s\to\infty} c_s$. One has (you are not asked to show this)

$$c_\infty \simeq \frac{2 \ln 2}{\eta} . \tag{43}$$

2. Show that the term between brackets that appears in the sum in (41) goes to zero for $l \to \infty$.
3. In the equation (41) we can now put $\tilde{z} = 1$ and take the limit $\eta \to 0$ in each term of the sum. Keep the leading term in $1/\eta$ and show that one finds the usual expression for the condensed fraction

$$\frac{\langle N_0 \rangle}{\langle N \rangle} \simeq 1 - \frac{T^3}{T_c^3}. \tag{44}$$

4. The same procedure can be used to treat the sum (last term) in (42). The calculation is analogous to the one in question 3 of the section 2.4. In fact the term containing c_∞ is sub-leading in $1/\eta$ with respect to c_s. Show that for $\eta \to 0$ one gets

$$\sum_{s=1}^{\infty} (c_s - c_\infty) \sim \langle N \rangle \frac{\zeta(2) - \zeta(3)}{\zeta(3)} \frac{T^3}{T_c^3} \tag{45}$$

5. The second term in (42) remains to be rewritten by eliminating $\langle N_0 \rangle$ in favour of $\langle N \rangle$ and T/T_c. Show that one thus obtains the following simple formula (valid for $T < T_c$)

$$\mathrm{Var}(N_L - N_R) \simeq \langle N \rangle \left[1 + \frac{\zeta(2) - \zeta(3)}{\zeta(3)} \frac{T^3}{T_c^3} + 2 \ln 2 \left(\frac{k_B T_c}{\hbar\omega} \right) \left(1 - \frac{T^3}{T_c^3} \right) \frac{T}{T_c} \right]. \tag{46}$$

6. Indicate whether the following statements about the three terms inside the brackets in the expression (46) and $\mathrm{Var}(N_L - N_R)$ are correct or not :
   (a) The first term (i.e. the "1") is of quantum origin
   (b) The first term is present even at $T = 0$.
   (c) The second term is a contribution only from the non-condensed particles.
   (d) The second term is smaller than one.
   (e) The third term is zero in absence of the condensate.
   (f) The third term diverges for $T = T_c$.

(g) The third term is zero for $T = 0$.

(h) The third term involves at the same time the condensed fraction and the non-condensed particles.

(i) For $T < T_c$ the expression for $\mathrm{Var}(N_L - N_R)$ given in (46) is positive.

(j) For fermions one would have the same expression for $\mathrm{Var}(N_L - N_R)$ but with an opposite sign in the first term.

7. In Figure 3 we show $\mathrm{Var}(N_L - N_R)$ divided by $\langle N \rangle$ as a function of T/T_c. By keeping only the third term between the brackets in (46), estimate for which value of T/T_c $\mathrm{Var}(N_L - N_R)$ is maximum. Compare with the figure. One has $2^{-2/3} \simeq 0.63$.
8. Still keeping only the third term between the brackets in (46), express the maximum value of $\mathrm{Var}(N_L - N_R)$ as a function of $\hbar\omega$ and $k_B T_c$.
9. How does this maximum value depend on $\langle N \rangle$?

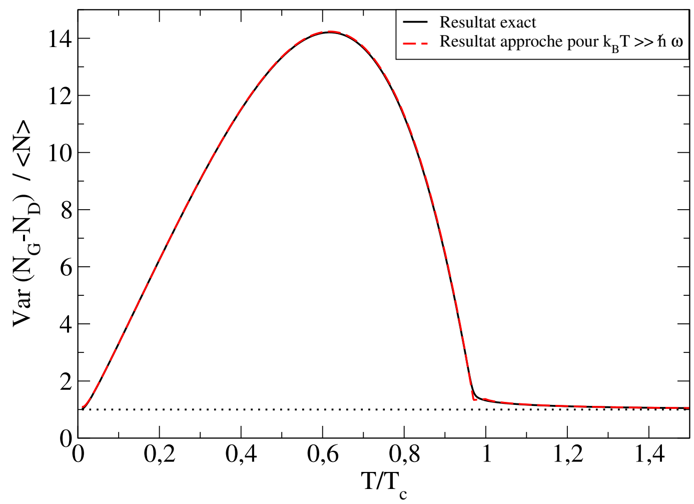


FIGURE 3 – Variance of $N_L - N_R$ (divided by $\langle N \rangle$) as a function of the temperature (divided by T_c). $\langle N \rangle = 13000$. Solid line : exact result. Dashed line : approximation for $k_B T \gg \hbar\omega$. (The two lines lie almost on top of each other).