Hydrodynamics of a Bose-Einstein condensate

Homework for the course “Quantum liquids" M1 ICFP 2014-2015

to be handed in on Friday January 9th 2015

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1 Derivation of the hydrodynamic equations

We have derived the time-dependent Gross-Pitaevskii equation, which describes a Bose-condensed atomic gas. Using $\Psi = \sqrt{N}\phi$, one has

$$i\,\hbar\,\partial_t \Psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\Delta + U(\vec{r}) + g|\Psi(\vec{r},t)|^2 \right] \Psi(\vec{r},t)\,. \tag{1}$$

with ϕ the condensate wave function (which is supposed to be pure), N the number of atoms, g the coupling constant characterising the interactions between the atoms, and U an external trapping potential.
We will rewrite this equation in its hydrodynamic form by taking

$$\Psi = \sqrt{\rho}\, e^{iS/\hbar} \tag{2}$$

with ρ the density and

$$\vec{v} = \frac{\vec{\mathrm{grad}} S}{m} \tag{3}$$

the velocity field of the superfluid.

1.1 Continuity equation

In quantum mechanics the probability current is defined as

$$\vec{j} = \frac{\hbar}{2im}\left[\Psi^* \vec{\mathrm{grad}} \Psi - c.c. \right]. \tag{4}$$

The Schrödinger equation is such that it conserves the probability :

$$\partial_t |\Psi|^2 + \mathrm{div}[\,\vec{j}\,] = 0\,. \tag{5}$$

1. Show that $\vec{j} = \rho\,\vec{v}$ by using Eqs. (**??**),(**??**) and (**??**).
2. Start from Eq. (**??**) and derive the continuity equation for the superfluid.

1.2 Euler equation

We insert Eq. (**??**) in the non-linear Schrödinger equation.

1. Show that the imaginary part of Eq. (**??**) gives the continuity equation. To do so, express first the kinetic energy term $-\hbar^2 \Delta\Psi/(2m)$ as a function of :
   $\Delta\sqrt{\rho}\,,\ (\vec{\mathrm{grad}}\,\rho)\cdot(\vec{\mathrm{grad}} S)\,,\ \Delta S\,,\ (\vec{\mathrm{grad}}\, S)^2$
   and use the formulas given at the end.
2. Show that the real part of Eq. (**??**) reads

$$m\partial_t \vec{v} + \vec{\mathrm{grad}} \left[\frac{1}{2} m v^2 + U + \rho g - \frac{\hbar^2}{2m} \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right] = 0 \tag{6}$$

3. Show that

$$\vec{\mathrm{grad}} \left[\frac{1}{2} m v^2 \right] = m(\vec{v}\cdot\vec{\mathrm{grad}})\vec{v} \tag{7}$$

   and hence one gets the convective derivative of $\vec{v}$.

4. The term in $\Delta\sqrt{\rho}/\sqrt{\rho}$ is negligible for a condensate in the Thomas-Fermi limit, and so one gets

$$m\left[\partial_t + (\vec{v}\cdot\vec{\mathrm{grad}})\right]\vec{v} = -\vec{\mathrm{grad}}\,[U+\rho g] \tag{8}$$

5. Give an interpretation to each of the terms of this equation. In particular, calculate the free energy of the gas in the Thomas-Fermi approximation for $U = 0$ (and $T = 0$), and rewrite the Euler equation such that the pressure term of the gas appears explicitly.

2 Linearisation of the equations and the speed of sound

1. Suppose that the trapping potential is perturbed weakly compared to its stationary value :

$$U(\vec{r},t) = U_0(\vec{r}) + \delta U(\vec{r},t). \tag{9}$$

This perturbation causes small deviations of the density and velocity field of the gas with respect to their stationary values :

$$\rho(\vec{r},t) = \rho_0(\vec{r}) + \delta\rho(\vec{r},t) \tag{10}$$

$$\vec{v}(\vec{r},t) = \vec{0} + \delta\vec{v}(\vec{r},t). \tag{11}$$

By neglecting the non-linear terms in $\delta\rho$ and $\delta\vec{v}$ in the Euler equation and the continuity equation, derive two linear evolution equations for $\delta\rho$ and $\delta\vec{v}$.

2. Take the time derivative of the linear equation for $\delta\rho$. Show that one can eliminate $\delta\vec{v}$ in the resulting equation. On should find a closed relation for $\delta\rho$.
3. We now look at times after the perturbation has been applied to the gas such that $\delta U(\vec{r},t) = 0$ around the position $\vec{r}$. Show that the density perturbation induced in the gas propagates according to

$$\frac{\partial^2\delta\rho}{\partial t^2} - \mathrm{div}\left[\frac{g\rho_0(\vec{r})}{m}\vec{\mathrm{grad}}\,\delta\rho\right] = 0. \tag{12}$$

4. If the condensate at rest would have a uniform density ρ_0, what would be the type of waves that propagate through the gas because of the perturbation δU ? Specify the dispersion relation of these waves. Express the velocity c of these waves in terms of ρ_0 and g.
5. Numerical application : calculate the velocity c for a gas of ^{23}Na atoms with density $4\times10^{20}/\mathrm{m}^3$. One has $g = 1.1\times10^{-50}$ in SI units. (1 u.m.a. $= 1,56\times10^{-27}$kg).

3 Hydrodynamic modes in an isotropic harmonic trap.

1. Consider the case of an isotropic harmonic potential

$$U_0(\vec{r}) = \frac{1}{2}m\omega^2 r^2. \tag{13}$$

The density ρ_0 of the gas at rest satisfies $\rho_0(\vec{r})g + U_0(\vec{r}) = \mu$ with μ the chemical potential of the gas (one finds the Thomas-Fermi limit again). Write the position variables in units of λ_i, $r_i = \lambda_i u_i$, $i = 1,2,3$, and derive that propagation equation (**??**) becomes

$$\frac{\partial^2 f(\vec{u},t)}{\partial t^2} - \frac{\omega^2}{2}\sum_{i=1,2,3}\frac{\partial}{\partial u_i}\left[(1-u^2)\frac{\partial}{\partial u_i}f(\vec{u},t)\right] = 0 \tag{14}$$

with $u^2 = u_1^2 + u_2^2 + u_3^2$.

2. Since the gas has a finite volume the eigenfrequencies Ω of the modes of the propagation equation (**??**) form a discrete set. Which eigenvalue equation must be satisfied by all eigenfrequencies Ω ? Use the reduced form (**??**) of the propagation equation.

3. We look for a solution of the eigenvalue equation of the form

$$f(\vec{u}\,) = F_l(u) Y_l^m(\theta, \phi) \tag{15}$$

where u is the modulus of $\vec{u}$, the angles θ, ϕ are the polar and azimuthal angles of the spherical coordinates, and where $Y_l^m(\theta, \phi)$ are the spherical harmonics. Which property of the system allows us to use this form ? Get the eigenvalue equation for $F_l(u)$. We remind that the action of the laplacian on a function f of the form (**??**) is given by :

$$\Delta f(\vec{u}\,) = \frac{\partial^2 f(u,\theta,\phi)}{\partial u^2} + \frac{2}{u}\frac{\partial f(u,\theta,\phi)}{\partial u} - \frac{l(l+1)}{u^2} f(\vec{u}\,). \tag{16}$$

4. We look for the solution $F_l(u)$ by writing it as an expansion in the variable u :

$$F_l(u) = u^s (a_0 + a_2\, u^2 + \ldots + a_{2k}\, u^{2k} + \ldots) \tag{17}$$

with s a positive integer. When one inserts this form into the eigenvalue equation for $F_l(u)$, one should ensure that a term in u^{s-2} should not appear. Derive from this that $s = l$.

5. Establish following recurrence relation for the coefficients a_{2k} by enforcing the disappearance of the coefficient of the term in u^{l+2k} :

$$\begin{aligned} a_{2k+2}[(l+2k+2)(l+2k+1) + 2(l+2k+2) - l(l+1)] &= \\ a_{2k}\left[(l+2k)(l+2k-1) + 4(l+2k) - l(l+1) - 2\frac{\Omega^2}{\omega^2}\right] & . \end{aligned} \tag{18}$$

6. One can show that the series diverges for $u = 1$ if there are an infinite number of non-zero coefficients a_{2k}. Explain why this is unacceptable physically. One must thus have a finite number of coefficients that are non-zero. Let's call n the smallest of the indices k such that $a_{2k+2} = 0$. Derive that

$$\Omega^2 = \omega^2 (2n^2 + 2n\,l + 3n + l). \tag{19}$$

7. Calculate the frequency of the breathing mode $n = 1$, $l = 0$ and the dipolar mode $n = 0$, $l = 1$ as a function of ω.

4 Formulas

$$\begin{aligned} \mathrm{div}\,(b\,\vec{a}) &= \vec{a}\cdot \vec{\mathrm{grad}}\, b + b\, \mathrm{div}\, \vec{a} & (20) \\ \vec{\mathrm{grad}}\,(\vec{a}\cdot\vec{a}) &= 2\,(\vec{a}\cdot\vec{\mathrm{grad}})\,\vec{a} + 2\,\vec{a}\times(\vec{\mathrm{rot}}\,\vec{a}) & (21) \\ \vec{\mathrm{rot}}\,(\vec{\mathrm{grad}}\, b) &= 0 & (22) \end{aligned}$$