Scalar electrodynamics:

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

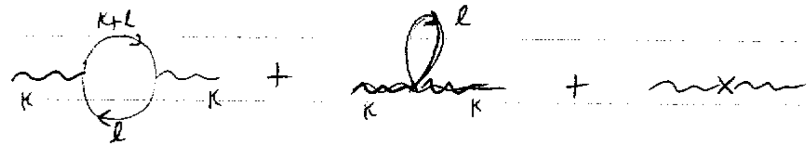
$$\mathcal{L}_0 = -\partial^\mu \varphi^+ \partial_\mu \varphi - m^2 \varphi^+ \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\alpha}{2} (\partial \cdot A)^2$$

$$\mathcal{L}_1 = i Z_1 e \left(\varphi^+ \partial_\mu \varphi - (\partial_\mu \varphi^+)\varphi\right) A^\mu - Z_4 e^2 \varphi^+ \varphi A_\mu A^\mu - \frac{1}{4} Z_\lambda \lambda (\varphi^+\varphi)^2 +$$

$$- (Z_2 - 1)\partial^\mu \varphi^+ \partial_\mu \varphi - (Z_m - 1) m^2 \varphi^+ \varphi - \frac{1}{4}(Z_3 - 1) F^{\mu\nu} F_{\mu\nu}$$

$\mathcal{L}_1$ contient toutes les interactions et le counterterms.

Photon self-energy:



$$(2\pi)^4 i \Pi^{\mu\nu}(k) = Z_1^2 e^2 \int d^4 l \, \frac{(k+2l)^\mu (k+2l)^\nu}{\left((k+l)^2 + m^2\right)(l^2 + m^2)}$$ (m^2 devrait être $m^2 - i\varepsilon$)

$$+ 2 i Z_4 e^2 g^{\mu\nu} \int \frac{d^4 l (-i)}{(l^2 + m^2)}$$

$$- i (2\pi)^4 (Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

$$\Rightarrow i \Pi^{\mu\nu} = e^2 \int \frac{d^4 l}{(2\pi)^4} \frac{N^{\mu\nu}}{\left((l+k)^2 + m^2\right)(l^2 + m^2)} \qquad - i (Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

à cet ordre, on peut prendre $Z_1 = Z_4 = 1$:

$$N^{\mu\nu}(l,k) = (2l+k)^\mu (2l+k)^\nu - 2\left[(l+k)^2 + m^2\right] g^{\mu\nu}$$

regularisation dimensionale: $e \to e\,\mu^{\varepsilon/2}$, $\varepsilon = 4-d$

Feynman trick: $$\frac{1}{D_1 D_2} = \int_0^1 dx \, \frac{1}{\left[x D_1 + (1-x) D_2\right]^2}$$

$$x D_1 + (1-x) D_2 = x\left((\ell+K)^2 + m^2\right) + (1-x)\left(\ell^2+m^2\right) =$$

$$= \ell^2 + m^2 + 2x\,\ell\cdot K + x K^2 = (\ell + xK)^2 + x(1-x)K^2 + m^2$$

$$\equiv q^2 + D \quad , \quad q = \ell + xK, \quad D = x(1-x)K^2 + m^2,$$

$$N^{\mu\nu} = \left(2q + (1-2x)K\right)^\mu \left(2q + (1-2x)K\right)^\nu - 2\left[\left(q + (1-x)K\right)^2 + m^2\right] g^{\mu\nu}$$

$$= 4 q^\mu q^\nu + (1-2x)^2 K^\mu K^\nu - 2\left[q^2 + (1-x)^2 K^2 + m^2\right] g^{\mu\nu} + (\text{linear in } q)$$

on a: $\int d^dq\, q^\mu q^\nu f(q^2) = g^{\mu\nu} A, \quad A = \frac{1}{d}\int d^dq\, q^2 f(q^2)$

$$\Rightarrow N^{\mu\nu} = 2\left(\frac{2}{d} - 1\right) q^2 g^{\mu\nu} + (1-2x)^2 K^\mu K^\nu - 2\left[(1-x)^2 K^2 + m^2\right] g^{\mu\nu}$$

(valide sous l'intégral)

~~Sinon~~ $\cancel{\int \frac{d^d q_E}{(2\pi)^d} \frac{q_E^2}{(q_E^2+D)^2} = \Omega_d \int dr \frac{r^{d+1}}{(r^2+D)^2} = \frac{\Omega_d}{2}\int dr\, r^d \frac{d}{dr}\left(-\frac{1}{r^2+D}\right) =}$

$\cancel{= \frac{\Omega_d}{2}}$

Formula (12.38) : $\int \frac{d^d k\,(k^2)^s}{(k^2+D)^N} = i\pi^{d/2} \frac{\Gamma(\frac{d}{2}+s)\Gamma(N-\frac{d}{2}-s)}{\Gamma(\frac{d}{2})\,\Gamma(N)} D^{\frac{d}{2}+s-N}$

$s=1,\ N=2: \quad i2\pi^{d/2} \frac{\Gamma(\frac{d}{2}+1)\,\Gamma(1-\frac{d}{2})}{\Gamma(\frac{d}{2})\,\Gamma(2)} D^{\frac{d}{2}-1} = I_{1,2}$

$s=0,\ N=2: \quad i2\pi^{d/2} \frac{\Gamma(\frac{d}{2})\,\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d}{2})\,\Gamma(2)} D^{\frac{d}{2}-2} =$

$$I_{1,2} = D\cdot\frac{2}{2-d}\cdot\frac{d}{2}\, I_{0,2}\,; \quad \frac{2-d}{d} I_{1,2} = D\, I_{0,2}$$

$\Rightarrow N^{\mu\nu} \simeq 2D g^{\mu\nu} + (1-2x)^2 k^\mu k^\nu - 2\left[(1-x)^2 k^2 + m^2\right] g^{\mu\nu}$

$= \left(2x - 2x^2 - 2 + 4x - 2x^2\right) k^2 g^{\mu\nu} + (1-2x)^2 k^\mu k^\nu$

$= -2(1-x)(1-2x)\, k^2 g^{\mu\nu} + (1-2x)^2 k^\mu k^\nu$

$x = y + \frac{1}{2}$:

$2(1-x) = 1-2y$

$1-2x = -2y$

: $N^{\mu\nu} = 2y(1-2y)\, k^2 g^{\mu\nu} + 4y^2 k^\mu k^\nu$

$\int_{-1/2}^{1/2} dy$: $N^{\mu\nu} \sim -4y^2 (k^2 g^{\mu\nu} - k^\mu k^\nu)$ transverse.

$\int_{-\frac{1}{2}}^{\frac{1}{2}} N\, dy = -\frac{1}{3}\,(k^2 g^{\mu\nu} - \dots)$

$$\mu^{\varepsilon}\int \frac{d^d q}{(2\pi)^d}\,\frac{1}{(q^2+D)^2} = \frac{i\mu^{\varepsilon}\,2\pi^{2-\frac{\varepsilon}{2}}}{2(2\pi)^{4-\varepsilon}}\left(\frac{2}{\varepsilon}-\gamma+O(\varepsilon)\right) D^{-\frac{\varepsilon}{2}}$$

$$= \frac{i}{16\pi^2}\left(\frac{D}{\pi\mu^2}\right)^{-\frac{\varepsilon}{2}}\left(\frac{2}{\varepsilon}-\gamma+O(\varepsilon)\right) = \frac{i}{8\pi^2\varepsilon} - \frac{i}{16\pi^2}\left(\gamma+\log\frac{D}{\pi\mu^2}\right) + O(\varepsilon)$$

$$= \frac{i}{8\pi^2\varepsilon} + \frac{i}{16\pi^2}\log\left(\frac{\tilde{\mu}^2}{D}\right), \quad \tilde{\mu} = \mu\sqrt{\pi}\,e^{-\frac{\gamma}{2}}$$

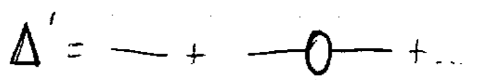
$$\Pi^{\mu\nu} = \Pi(k^2)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \Rightarrow \Pi(k^2) = -\frac{e^2}{24\pi^2\varepsilon} - \left(Z_3 - 1\right) + \text{finite}$$

$\overline{MS}$ scheme: on garde la partie divergente en ε $\Rightarrow$ $Z_3 = 1 - \dfrac{e^2}{24\pi^2\varepsilon}$.

OS scheme: $\Pi(0) = 0$:

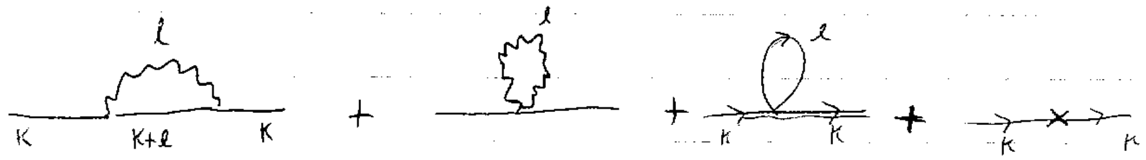
$$Z_3 = 1 - \frac{e^2}{24\pi^2\varepsilon} - \frac{e^2}{24\pi^2}\log\frac{\tilde{\mu}}{m}$$

In OS, $\Pi(q^2) = \Pi_{loop}(q^2) - \Pi_{loop}(0)$, mais pas dans $\overline{MS}$.



$$= \Delta\left(1 - \Pi^*\Delta\right)^{-1} = \left(1 - \Delta\Pi^*\right)^{-1}\Delta$$

Correction à la self-energie scalaire : Π_φ



$$i\Pi_\varphi(K^2) = Z_1^2\, e^2 \int \frac{d^4 l}{(2\pi)^4} \, \frac{P_{\mu\nu}(l)(2K+l)^\mu (2K+l)^\nu}{l^2\left((l+K)^2+m^2\right)}$$

$$+\, 2i Z_4\, e^2 g^{\mu\nu} \int \frac{d^4 l}{(2\pi)^4}\, \frac{-i P_{\mu\nu}(l)}{l^2+m_\gamma^2} \longleftarrow \text{IR regulator}$$

$$+\, Z_\lambda \lambda \int \frac{d^4 l}{(2\pi)^4}\, \frac{1}{l^2+m^2} \;-\; i\left(Z_2-1\right)K^2 - i\left(Z_m-1\right)m^2 \;;$$

$P_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$, Landau gauge : $\Delta_{\mu\nu} = \frac{P_{\mu\nu}}{l^2 - i\varepsilon}$;

$l^\mu P_{\mu\nu}(l) = 0$, $g^{\mu\nu}P_{\mu\nu} = d-1$:

$$i\Pi_\varphi = 4e^2\mu^\varepsilon \int \frac{d^d l}{(2\pi)^d}\, \frac{P_{\mu\nu}(l) K^\mu K^\nu}{l^2\left((l+K)^2+m^2\right)} - 2(d-1)\, e^2 \mu^\varepsilon \int \frac{d^d l}{(2\pi)^d}\, \frac{1}{l^2+m_\gamma^2}$$

$$-\lambda \mu^\varepsilon \int \frac{d^4 l}{(\;)}\, \frac{1}{l^2+m^2} - i(Z_2-1)K^2 - i(Z_m-1)m^2 \;;$$

utilisant : $\mu^\varepsilon \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2+m^2} = -\frac{i}{8\pi^2\varepsilon} m^2$, le deuxième terme disparaît quand $m_\gamma \to 0$

$$\int \frac{P_{\mu\nu} K^\mu K^\nu}{\ell^2((\ell+K)^2+m^2)} = \int \frac{\ell^2 K^2 - (\ell\cdot K)^2}{(\ell^2)^2\left((\ell+K)^2+m^2\right)} =$$

$$= \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int dx_1 dx_2 \, \frac{\delta(1-x_1-x_2)\; x_1 \; (\ell^2K^2-(\ell\cdot K)^2)}{\left(x_1\ell^2 + \underbrace{\cancel{(1-x)}}_{x_2}\left((\ell+K)^2+m^2\right)\right)^3} = 2\int_0^1 x\,dx \int dq \, \frac{N}{(q^2+D)^3}$$

$$q = \ell + xK \quad , \quad D = x(1-x)K^2 + xm^2$$

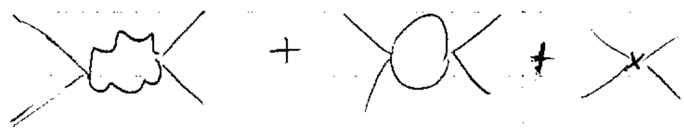
$$N = \ell^2K^2 - (\ell\cdot K)^2 = (q-xK)^2K^2 - (q\cdot K - xK^2)^2 =$$

$$= q^2K^2 - (q\cdot K)^2 = \left(1-\frac{1}{d}\right) q^2K^2$$

$$\Rightarrow \; Z_2 = 1 + \frac{3e^2}{8\pi^2\varepsilon}$$

$$Z_m = 1 + \frac{\lambda}{8\pi^2\varepsilon}$$

Z_λ peut être calculé à impulsion $=0$:



$$\Rightarrow \; Z_\lambda = 1 + \frac{3e^4}{8\pi^2\lambda\varepsilon} + \frac{5\lambda}{16\pi^2\varepsilon}$$

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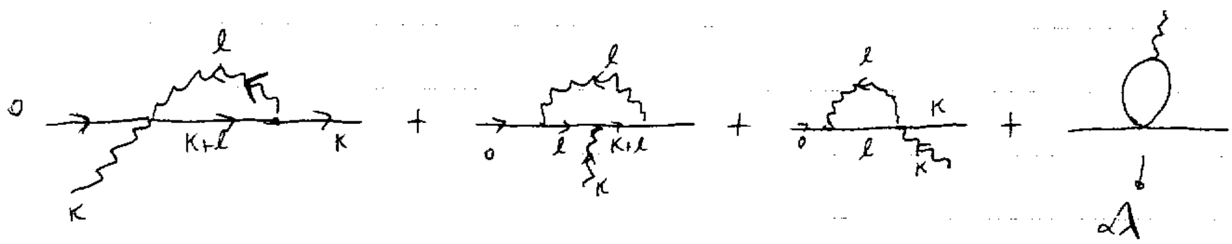
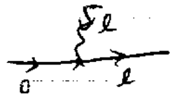
invariance de jauge : $A_\mu \to A_\mu + \partial_\mu \Lambda \;\Rightarrow\; Z_4 = \frac{Z_1^2}{Z_2} \;\Rightarrow\; Z_1 = 1 + \frac{3e^2}{8\pi^2\varepsilon}$

en fait, $Z_1 = Z_2 = Z_4$.

Photon-scalar-scalar diffusion:



On prends $\vec{K} = 0$, alors  $\sim \ell^\mu$, mais $\ell^\mu P_{\mu\nu}(\ell) = 0 \Rightarrow$

seulement le premier diagramme contribue

$$i V_3^\mu(K,0) = -i e Z_1 K^\mu + (-ieZ_1)(-2ie^2 Z_4)\,\eta^{\mu\nu}(-i)^2 \int \frac{d^4\ell}{(2\pi)^4}\frac{P_{\nu\rho}(\ell)\,(K+\ell)^\rho}{\ell^2((\ell+K)^2+m^2)}$$

$$\simeq -ieZ_1 K^\mu + 2e^3\,\eta^{\mu\nu}\,2K^\rho \int \frac{d^4\ell}{(2\pi)^4}\frac{P_{\nu\rho}(\ell)}{\ell^2((\ell+K)^2+m^2)}$$

à ordre K^μ: $\quad -ieZ_1K^\mu + 4e^3K^\rho\eta^{\mu\nu}\int\frac{d^4\ell}{(2\pi)^4}\frac{P_{\nu\rho}(\ell)}{\ell^2(\ell^2+m^2)}$

$$= -ieZ_1K^\mu + 4e^3K^\rho\eta^{\mu\nu}\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2(\ell^2+m^2)}\,(\eta_{\nu\rho})\left(1-\frac{1}{d}\right)$$

$$= -ieZ_1K^\mu + 4e^3K^\mu\left(1-\frac{1}{d}\right)\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2+xm^2)^2}$$

$$= ieK^\mu\left(-Z_1+\frac{3e^2}{8\pi^2\varepsilon}\right) \quad\Rightarrow\quad Z_1 = 1+\frac{3e^2}{8\pi^2\varepsilon}$$

Transformation de gauge: $\varphi \to \varphi e^{iq\alpha}$, $A_\mu \to A_\mu - \partial_\mu \alpha$

$$\delta \int \left(-Z_2 \partial_\mu \varphi^+ \partial^\mu \varphi + i Z_1 e (\varphi^+ \partial_\mu \varphi - \partial_\mu \varphi^+ \varphi) A^\mu - Z_4 e^2 \varphi^+ \varphi A_\mu A^\mu \right) =$$

$$= \int \left(-Z_2 q \left(-i \varphi^+ \partial_\mu \alpha \partial^\mu \varphi + i \partial_\mu \varphi^+ \partial^\mu \alpha \varphi \right) \right) +$$

$$+ i Z_1 e (\varphi^+ \partial_\mu \varphi - \partial_\mu \varphi^+ \varphi)(-\partial^\mu \alpha) + i Z_1 e q \times (2 i \partial_\mu \alpha \varphi^+ \varphi) - Z_4 e^2 \varphi^+ \varphi (-2 A_\mu \partial^\mu \alpha)$$

$$= \int i \partial^\mu \alpha (\varphi^+ \partial_\mu \varphi - \partial_\mu \varphi^+ \varphi)(Z_2 q - Z_1 e) + 2 \varphi^+ \varphi A_\mu \partial^\mu \alpha (Z_4 e^2 - Z_1 e q)$$

$$\Rightarrow \frac{Z_2}{Z_1} = \frac{e}{q}, \quad \frac{Z_4}{Z_1} = \frac{q}{e} = \frac{Z_1}{Z_2}; \quad \boxed{Z_4 = \frac{Z_1^2}{Z_2}}$$

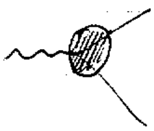
Nöther current : $J^\mu = +ie Z_2 \left(\varphi^+ \partial^\mu \varphi - \partial^\mu \varphi^+ \varphi\right) - 2e^2 Z_1\, \varphi^+ \varphi A_\mu$

(vient de $\delta S = \int J^\mu \partial_\mu \alpha$, sans considérer la variation du champ de jauge, et avec $q = e$: on veut que e soit la charge physique).

Considérons : $\langle 0 | T\left(J^\mu(x)\varphi(y)\varphi^+(z)\right) | 0 \rangle$

$$\partial^\mu \langle T\left(J^\mu(x)\varphi(y)\varphi^+(z)\right)\rangle = \langle T\left(\partial_\mu J^\mu \varphi\varphi^+\right)\rangle + \delta(x^0-y^0)\langle T[J^0(x),\varphi(y)]\varphi^+(z)\rangle$$
$$+\delta(x^0-z^0)\langle T\varphi(y)[J^0(x),\varphi^+(z)]\rangle$$

$$= \delta(x^0-y^0)\left(-ie\ \langle T\varphi(y)\varphi^+(z)\rangle\right) + \delta(x^0-z^0)\, ie \langle T\varphi(y)\varphi^+(z)\rangle$$

Vertex function :  couplage au photon : $\frac{\delta \mathcal{L}}{\delta A_\mu} = ie Z_1 \left(\varphi^+ \overleftrightarrow{\partial}^\mu \varphi\right) - 2e^2 Z_4 \varphi^+\varphi A_\mu$

$$= \frac{Z_1}{Z_2} J^\mu \quad \left(\text{en fait } Z_4 = \frac{Z_1^2}{Z_2}\right)$$

$$\Rightarrow \int dx\,dy\,dz\, e^{-ipx-iqy+iq'z}\ \frac{Z_1}{Z_2}\langle T J^\mu(x)\varphi(y)\varphi^+(z)\rangle := -ie(2\pi)^4\delta^4(p+q-q')\, V_3^\mu(q,q')\,\Delta(q)\Delta(q')$$

Fourier transform : $e\frac{Z_1}{Z_2}(q-q')_\mu\, V_3^\mu(q,q')\,\Delta(q)\Delta(q')(2\pi)^4\delta^{(4)}(p+q-q') =$

$$= -ie\int dy\,dz\, e^{-i(p+q)y} e^{iq'z}\langle T\varphi(y)\varphi^+(z)\rangle + ie\int dy\,dz\, e^{-iqy} e^{i(q'-p)z}\langle T\varphi(y)\varphi^+(z)\rangle$$

$$= -ie\int dy\,dz\, e^{-i(p+q)y} e^{iq'z}\ (-i)\int\frac{d^4l}{(2\pi)^4} e^{il(y-z)}\Delta(l)$$

$$+ ie\int dy\,dz\, e^{-iqy} e^{i(q'-p)z}\ (-i)\int\frac{d^4l}{(2\pi)^4} e^{il(y-z)}\Delta(l) =$$

$$= -e \int \frac{dl}{(2\pi)^4} \int dy\, e^{iy(l-p-q)} \int dz\, e^{iz(q'-l)}\, \Delta(l)$$

$$+ e \int \frac{dl}{(2\pi)^4}\, \Delta(l) \int dy\, e^{iy(l-q)} \int dz\, e^{iz(q'-p-l)}$$

$$= e\left(\Delta(q) - \Delta(q')\right) (2\pi)^4\, \delta(p+q-q')$$

$$\Rightarrow \quad (q-q')_\mu\, \Gamma^\mu(q,q') = \frac{Z_1}{Z_2}\left[\frac{1}{\Delta(q')} - \frac{1}{\Delta(q)}\right]$$

In $\overline{MS}$, $Z_1 = 1 + \frac{c_1}{\varepsilon} + \dots$, $Z_2 = 1 + \frac{c_2}{\varepsilon} + \dots$ $\frac{Z_1}{Z_2}$ finie $\Rightarrow Z_1 = Z_2$.

OS: $\Gamma^\mu(q,q') = (q-q')^\mu$ pour $q^2 = q'^2 = -m^2$, $(q-q')^2 \approx 0$

$$\Rightarrow (q^2 - q'^2) = 0 = \frac{Z_1}{Z_2}\left[\left(\frac{1}{q'^2+m^2}\right)^{-1} - \left(\frac{1}{q^2+m^2}\right)^{-1}\right] = \frac{Z_1}{Z_2}(q^2 - q'^2)$$

$\Rightarrow Z_1 = Z_2$.

Théorie originelle (bare): $\mathcal{L}_0 = \mathcal{L}(m_0, g_0)$

$$= \mathcal{L}(m, g, \mu) \quad \text{en terme de champs renormalisés.}$$

Mais les paramètres m_0, g_0 sont invariants pour changement de μ.

$$e_0 = Z_3^{-\frac{1}{2}} Z_2^{-1} Z_1 \mu^{\varepsilon/2} e \qquad \text{(du couplage cubique)}$$

$$e_0^2 = Z_3^{-1} Z_1^{-1} Z_4 \mu^{\varepsilon} e^2 \qquad \text{(quartique)} \quad \Rightarrow \quad e_0 = Z_3^{-1/2} \mu^{\varepsilon/2} e$$

$$\lambda_0 = Z_2^{-2} Z_\lambda \mu^{\varepsilon} \lambda$$

$$\log\left(Z_3^{-\frac{1}{2}}\right) = + \frac{1}{2} \frac{e^2}{24\pi^2 \varepsilon} \, ; \qquad \log e_0 = \log e + \frac{\varepsilon}{2} \log\mu - \frac{1}{2} \log Z_3$$

$$0 = \frac{d \log e_0}{d \log\mu} = \frac{d \log e}{d \log\mu}\left(1 + e \frac{d \log\left(Z_3^{-\frac{1}{2}}\right)}{de}\right) + \frac{\varepsilon}{2}$$

$$0 = \frac{de}{d\log\mu}\left(1 + e \frac{d \log Z_3^{-\frac{1}{2}}}{de}\right) + \frac{\varepsilon}{2} e$$

On peut résoudre itérativement: $\dfrac{de}{d\log\mu} = -\dfrac{\varepsilon}{2} e \left(1 - e \dfrac{d \log Z_3^{-\frac{1}{2}}}{de} + \ldots\right) = \varepsilon e + \beta(e) + O(\varepsilon)$

$$\beta = \frac{de}{d\log\mu} = \frac{1}{2} e^2 \frac{d \log Z_3^{-\frac{1}{2}}}{de} = \frac{e^3}{48\pi^2} + \ldots$$

$\beta > 0$; $e(\mu)$ augmente avec l'énergie.