8.9.4

Champ vectoriel massif:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} V_\mu V^\mu \ ;$$

$$\text{e.o.m.} \quad \partial^\mu F_{\mu\nu} - m^2 V_\nu = 0 \ ;$$

tenseur energie-impulsion canonique:

$$T'^{\mu\nu} = -\frac{\delta \mathcal{L}}{\delta \partial_\mu V^\rho} \partial^\nu V^\rho + \eta^{\mu\nu} \mathcal{L} = F^\mu{}_\rho \, \partial^\nu V^\rho + \eta^{\mu\nu} \mathcal{L} \ ;$$

$$(J_{\mu\nu})^{\ell m} V_m = -i \left(\delta_\mu{}^\ell V_\nu - \delta_\nu{}^\ell V_\mu \right) \ ;$$

$$\frac{i}{2} \omega^{\mu\nu} (J_{\mu\nu})^{\ell m} V_m = \frac{1}{2} \left(\omega^{\ell\nu} V_\nu - \omega^{\mu\ell} V_\mu \right) = \omega^\ell{}_\nu V^\nu \ ;$$

$$\tilde{S}^{\rho\mu\nu} = -\frac{i}{2} \frac{\delta \mathcal{L}}{\delta \partial_\rho V^\ell} (J^{\mu\nu})^\ell{}_m V^m = \left(\frac{-i}{2}\right) \left(-F^\rho{}_\ell \right) (-i) \left(\delta_\mu{}^\ell V_\nu - \delta_\nu{}^\ell V_\mu \right) =$$

$$= \frac{1}{2} \left(F^{\rho\mu} V^\nu - F^{\rho\nu} V^\mu \right)$$

$$S^{\rho\mu\nu} = \frac{1}{2} \Big(F^{\rho\mu} V^\nu - F^{\rho\nu} V^\mu - F^{\mu\rho} V^\nu + F^{\mu\nu} V^\rho - F^{\nu\rho} V^\mu + F^{\nu\mu} V^\rho \Big) = F^{\rho\mu} V^\nu \ ;$$

$$\Theta^{\mu\nu} = T'^{\mu\nu} + \partial_\rho S^{\rho\mu\nu} = F^\mu{}_\rho \, \partial^\nu V^\rho + \eta^{\mu\nu} \left(-\frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} - \frac{m^2}{2} V_\rho V^\rho \right)$$

$$- F^\mu{}_\rho \, \partial^\rho V^\nu + \partial_\rho F^{\rho\mu} V^\nu = \left(\begin{matrix} \text{utiliser} \\ \text{e.o.m.} \end{matrix} \right)$$

$$= F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + m^2 \left(V^\mu V^\nu - \frac{1}{2} \eta^{\mu\nu} V^2 \right)$$

$m=0$: $\quad \Theta_\mu{}^\mu = F_{\mu\nu} F^{\mu\nu} - \frac{1}{4}\delta_\mu{}^\mu F_{\rho\sigma}F^{\rho\sigma} = 0.$

$D^\mu = x^\nu T_\nu{}^\mu$ est conservé :

$$\partial_\mu D^\mu = \delta_\mu{}^\nu T_\nu{}^\mu + x^\nu \partial_\mu T_\nu{}^\mu = 0.$$

D^μ est le générateur des dilatations : invariance d'échelle.

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & & 0 & B_1 \\ -E_3 & & & 0 \end{pmatrix} \quad ; \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & & 0 & B_1 \\ E_3 & & & 0 \end{pmatrix}$$

$$F_{\mu\nu}F^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2);$$

$$\theta^{00} = F^{0\lambda}F^0{}_\lambda + \frac{1}{4}F\cdot F = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

$$\theta^{0i} = F^{0j}F_j{}^i = E^j \,\varepsilon_j{}^i{}_k B^k = -(\vec{E}\wedge\vec{B})^i \,;$$

En quantisation canonique, $\Pi^i = F^{i0} = -E^i$

dans la jauge de Coulomb, $\vec{\nabla}\cdot\vec{A} = 0$, et $A^0 = 0$ en absence de champs de matière.

$$\int [x^\mu T_\mu{}^0(x), A_j(y)]\, d^3x =$$

$$= \int x_0 \left[\frac{1}{2}\vec{E}^2(x), A_j(y)\right] + x^k [F^{0l}F_{lk}, A_j(y)] =$$

$$= \int -x_0 E^j(x)\,\delta(x-y) + x^k F_l{}^k \delta^l_j\, \delta(x-y) =$$

$$= \int \left(-x^\mu \partial_\mu A_j(x) + x^k \partial_j A_k(x)\right)\delta(x-y)$$

Invariance d'échelle :

$$x'^\mu = \lambda x^\mu$$

$$\Phi'(x') = \lambda^{\Delta} \Phi(x) \qquad \Delta = \text{scaling dim.}$$

$\lambda = 1+\varepsilon$: $\delta x^\mu = \varepsilon x^\mu$; $\quad \Phi'(x+\varepsilon x) = \Phi'(x) + \varepsilon x^\mu \partial_\mu \Phi = \Phi(x) + \varepsilon \Delta \Phi(x)$;

$$\delta\Phi(x) = \Phi'(x) - \Phi(x) = \cancel{\Phi'(x) - \Phi(x)} - \varepsilon x^\mu \partial_\mu \Phi + \varepsilon \Delta \Phi \, ;$$

$$(\partial_\mu \Phi)'(x') = \frac{\partial}{\partial x'^\mu} \Phi'(x') = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \left(\lambda^\Delta \Phi(x) \right) =$$

$$= \left(\delta_\mu^\nu (1-\varepsilon) - \partial_\mu \varepsilon^\nu \right) \partial_\nu \left(\Phi + \varepsilon \Delta \Phi \right) =$$

$$= \left(\delta_\mu^\nu - \varepsilon \delta_\mu^\nu - \partial_\mu \varepsilon x^\nu \right) \left(\partial_\nu \Phi + \varepsilon \Delta \partial_\nu \Phi + \partial_\nu \varepsilon \Delta \Phi \right) =$$

$$= \partial_\mu \Phi + \varepsilon \Delta \partial_\mu \Phi + \partial_\mu \varepsilon \Delta \Phi - \varepsilon \partial_\mu \Phi - \partial_\mu \varepsilon \, x^\nu \partial_\nu \Phi \, ;$$

mesure : $d^D x' = \det\left(\frac{\partial x'}{\partial x} \right) d^D x = \det\left(\delta^\mu_\nu + \varepsilon \delta^\mu_\nu + \partial_\nu \varepsilon x^\mu \right) d^D x =$

$\det(1+M) = 1 + \mathrm{tr}\, M$: $\qquad d^D x' = \left(1 + \varepsilon D + x^\mu \partial_\mu \varepsilon \right) d^D x$;

Si S invariante pour ε constant, on retient les termes $\sim \partial\varepsilon$:

$$\delta S = \int d^D x \left(x^\mu \partial_\mu \varepsilon \, \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \left(\partial_\mu \varepsilon \Delta \Phi - \partial_\mu \varepsilon \, x^\nu \partial_\nu \Phi \right) \right) =$$

$$= \int d^D x \; \partial_\mu \varepsilon \left(x^\mu \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \left(-x^\nu \partial_\nu \Phi + \Delta \Phi \right) \right)$$

$$= + \int d^D x \; \partial_\mu \varepsilon \, j^\mu_D \, , \qquad j^\mu_D = x^\nu T_{c\,\nu}{}^{\mu} + \Delta \Phi \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \, ,$$

en general, $j^\mu_D \neq D^\mu$

Champ scalaire: $\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{\lambda}{4}\varphi^4$; $\Pi = \frac{\delta\mathcal{L}}{\delta\partial_0\varphi} = \partial_0\varphi$;

$$T_c^{\mu\nu} = \partial^\mu\varphi\,\partial^\nu\varphi - \frac{1}{2}\eta^{\mu\nu}(\partial\varphi)^2 + \frac{\lambda}{4}\eta^{\mu\nu}\varphi^4 ;$$

e.o.m. $\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} - \frac{\delta\mathcal{L}}{\delta\varphi} = \partial_\mu\left(-\partial^\mu\varphi\right) + \lambda\varphi^3 = -\partial^2\varphi + \lambda\varphi^3 = 0$

$$\eta_{\mu\nu}T_c^{\mu\nu} = \left(1-\frac{4}{2}\right)(\partial\varphi)^2 - \lambda\varphi^4 = -(\partial\varphi)^2 - \lambda\varphi^4 \neq 0$$

$$D^\mu = x^\nu T_{c\nu}{}^{\mu} = x^0 T_0{}^0 + x^i T_i{}^0 =$$

$$= x^0\left(\partial_0\varphi\,\partial^0\varphi\left(1-\frac{1}{2}\right)\right) + x^i\partial_i\varphi\,\partial^0\varphi + \text{termes indep. de } \Pi$$

$$= -\frac{1}{2}x^0\Pi^2 - x^i\partial_i\varphi\,\Pi$$

$$\int d^3x\ \{D^0(x), \varphi(y)\} = -\int d^3x\left(x^0\,\Pi\,\delta(x-y) + x^i\partial_i\varphi\,\delta(x-y)\right) =$$

$$= -y^\mu\partial_\mu\varphi(y)$$ n'est pas la transf. conforme !

Tenseur modifié : $T^{\mu\nu} = T^{\mu\nu}_c + \alpha\left(-\eta^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu\right)\varphi^2$

$$\eta_{\mu\nu}T^{\mu\nu} = -(\partial\varphi)^2 - \lambda\varphi^4 + \alpha\left(-3\,\partial^2\right)\varphi^2 =$$

$$= -(\partial\varphi)^2 - \lambda\varphi^4 - 3\alpha\,\partial^\mu\left(2\varphi\,\partial_\mu\varphi\right) =$$

$$= -(\partial\varphi)^2 - \lambda\varphi^4 - 6\alpha\left((\partial\varphi)^2 + \varphi\,\partial^2\varphi\right)$$

$\alpha = -\frac{1}{6}$: $\eta_{\mu\nu}T^{\mu\nu} = -\lambda\varphi^4 + \varphi\,\partial^2\varphi = 0$ on-shell.

$$\int \alpha \left\{ x^\nu \left(-\delta_\nu^{\,0}\, \partial^2 + \partial_\nu \partial^0 \right) \varphi^2 ,\ \varphi(y) \right\} =$$

$$= \int \alpha \left\{ x^0 \left(-\partial^2 + \partial_0 \partial^0 \right) \varphi^2 + x^i \partial_i \partial^0 \varphi^2 ,\ \varphi(y) \right\} =$$

$$= \int 2\alpha\ x^i \left\{ \partial_i \left(\varphi\, \partial^0 \varphi \right),\ \varphi(y) \right\} = -2\alpha \int x^i \partial_i \left(\varphi\, \delta(x-y) \right)$$

$$= 2\alpha \int \partial_i x^i \varphi\, \delta(x-y) = 6\alpha\, \varphi(y)$$

$$\alpha = -\frac{1}{6} : \int dx \left\{ D^0(x),\ \varphi(y) \right\} = -\varphi(y) - \cancel{y} \cdot \partial \varphi(y)$$