5.7.4.

a) Propagateurs retardés/avancés:

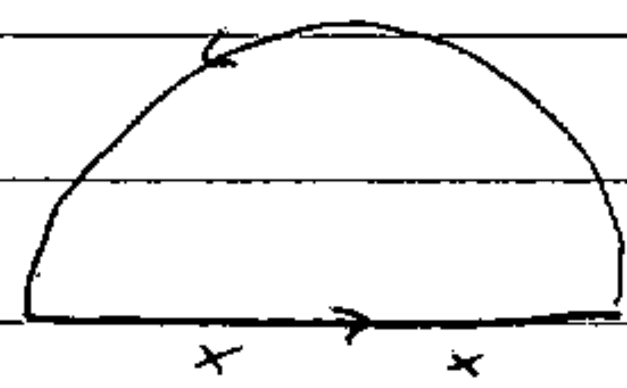
$$\Delta_{\substack{ret\\adv}} = \int \frac{d^3p}{(2\pi)^4}\,\frac{d\omega}{2\pi}\,\frac{e^{ipx}}{-(p^0 \pm i\varepsilon)^2+\vec{p}^{\,2}+m^2} = \int \frac{-e^{-i\omega t+i\vec{p}\cdot\vec{x}}}{(\omega\pm i\varepsilon+E)(\omega\pm i\varepsilon-E)} \quad \omega = p^0$$

Integral sur ω: residues

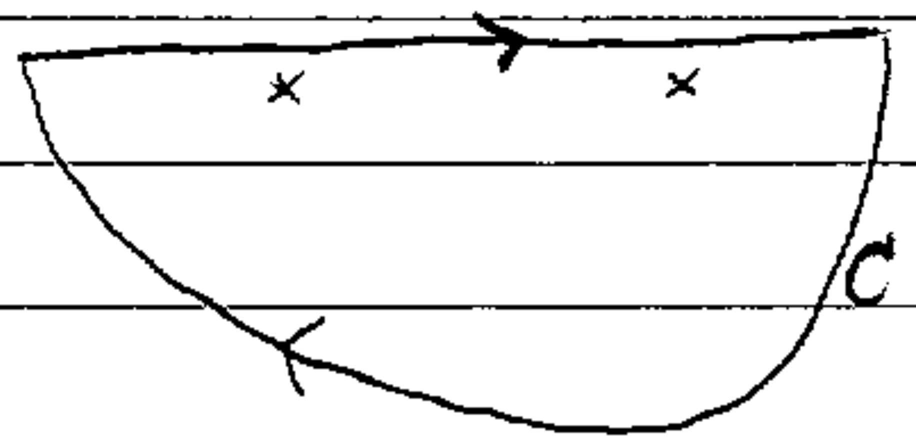
retardé: pôle à $\omega = -i\varepsilon \pm \sqrt{\vec{p}^{\,2}+m^2} = -i\varepsilon \pm E(\vec{p})$

$$\mathrm{Res}_+ = \text{residue à } \omega = -i\varepsilon + E(\vec{p}) = \left.\frac{-e^{-i\omega t}}{\omega+E}\right|_{\omega=E} = -\frac{e^{-iEt}}{2E};$$

$$\mathrm{Res}_- = -\left.\frac{e^{-i\omega t}}{\omega-E}\right|_{\omega=-E} = \frac{e^{iEt}}{2E};$$

pour $t<0$: $e^{-i\omega t} = e^{i\omega|t|}$, contour fermé à $\mathrm{Im}\,\omega > 0$:  $=0$

$t>0$:



$$\int_C \frac{d\omega}{2\pi}(\,) = -i\,(\mathrm{Res}_+ + \mathrm{Res}_-) = -i\left(\frac{e^{iE(p)t}}{2E(p)} - \frac{e^{-iE(p)t}}{2E(p)}\right)$$

orientation du contour!

$$\Delta_{ret}(x,t) = i\,\theta(t)\int\frac{d^3p}{(2\pi)^3}\,\frac{e^{i\vec{p}\cdot\vec{x}}}{2E(\vec{p})}\left(e^{-iE(p)t} - e^{iE(p)t}\right) =$$

$$= i\theta(t)\int\frac{d^3p}{(2\pi)^3 2E(p)}\left(e^{i(\vec{p}\cdot\vec{x}-p^0t)} - e^{i(\vec{p}\cdot\vec{x}+p^0t)}\right)$$

$$= i\,\theta(t)\int\frac{d^3p}{(2\pi)^3 2E(\vec{p})}\left(e^{ip_\mu x^\mu} - e^{-ip_\mu x^\mu}\right)$$

réelle

non-invariante de Lorentz

avancé : même pôles, pour $t<0$:  orientation différente

$$\Delta_{adv} = -i\,\theta(-t)\int\frac{d^3p}{(2\pi)^3 E(\vec{p})}\left(e^{ipx}-e^{-ipx}\right)$$

En term de $\Delta_+(x,t) = \int\frac{d^3p}{(2\pi)^3 2E(p)}\, e^{ipx}$:

$$\Delta_{\substack{ret\\adv}} = \pm i\,\theta(\pm t)\left(\Delta_+(x,t) - \Delta_+(-x,-t)\right)$$

Δ_+ est manifestement invariante de Lorentz (orthochrone):

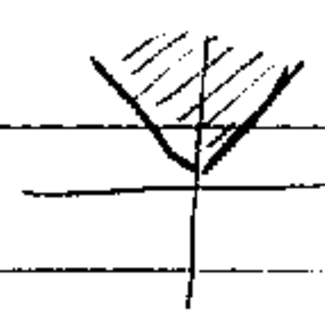
$$\int\frac{d^3p}{2E(p)} = \int\frac{d^3p}{2p^0} = \int\delta(p^2+m^2)\,d^4p \quad \text{si } p^0>0$$

plus exactement: $\delta(p^2+m^2) = \frac{1}{|E(p)|}\delta(p^0+E(p)) + \frac{1}{|-E(p)|}\delta(p^0-E(p))$

$$\left[\delta(f(x)) = \sum_{f(x_*)=0}\frac{1}{|f'(x_*)|}\,\delta(x-x_*)\right]$$

pour $x^2>0$ (genre espace), x et $-x$ connecté par une transf. de Lorentz. boost à $(0,\vec{x})$, rotation à $(0,-\vec{x})$

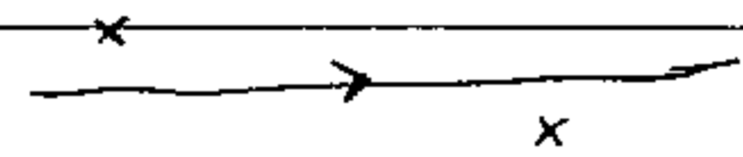
$\Delta(x,t) = \Delta(-x,-t)$

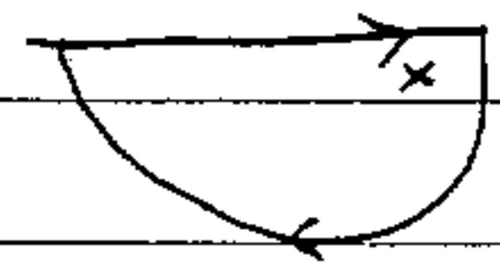
$\Rightarrow$ $\Delta_{\substack{ret\\adv}} = 0$ pour $x^2>0$; Δ_{ret} a support dans le cône lumière futur: 

Propagateur de Feynman: $\Delta_F = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \frac{1}{p^2+m^2-i\varepsilon}$ (Lorentz-invariant)

pôle : $-\omega^2+\vec{p}^2+m^2-i\varepsilon=0$:

$$\omega = \pm\sqrt{\vec{p}^2+m^2-i\varepsilon} = \pm\left(E(\vec{p})-i\varepsilon\right)$$



$t>0$:  $-i\, Res_+ = i\int \frac{d^3p}{(2\pi)^3 2E(p)} e^{-iE(p)t+i\vec{p}\cdot\vec{x}} = i\Delta_+(x,t)$

$t<0$:  $i\, Res_- = i\int \frac{d^3p}{(2\pi)^3 2E(p)} e^{iE(p)t+i\vec{p}\cdot\vec{x}} =$

$$= i\int \frac{d^3p}{(2\pi)^3 2E(p)} e^{ip(-x)} = i\Delta_+(-x,-t)$$

$$\Delta_F = i\left(\theta(t)\Delta_+(x,t) + \theta(-t)\Delta_+(-x,-t)\right)$$

$x^2>0$: $\Delta_F = i\Delta_+$

$x^2<0$, $x^0>0$: $\Delta_F = i\Delta_+$

$x^0<0$: $\Delta_F = i\Delta_+(-x,-t) = i\Delta_+^*(x,t)$

b)

$$\frac{\partial^2}{\partial t^2}\,\theta(t)\left(e^{-iEt}-e^{iEt}\right) = \delta'(t)\left(e^{-iEt}-e^{iEt}\right)+2\,\delta(t)(-i)E\left(e^{-iEt}+e^{iEt}\right)$$

$$+\theta(t)\left(-E^2\right)\left(e^{-iEt}-e^{iEt}\right)$$

$$\left(\frac{\partial^2}{\partial t^2}+E(p)^2\right)\Delta_{ret} = i\int\frac{d^3p}{(2\pi)^3 2E(p)}\,e^{i\vec p\cdot\vec x}\left(\underbrace{\delta'(t)\left(e^{-iEt}-e^{iEt}\right)}_{\sim-\delta(t)(-iE)\left(e^{-iEt}+e^{iEt}\right)}+2(-i)E(p)\,\delta(t)\left(e^{-iEt}+e^{iEt}\right)\right)$$

$$\Rightarrow \delta(t)\int\frac{d^3p}{(2\pi)^3}\,e^{i\vec p\cdot\vec x} = \delta(t)\,\delta^{(3)}(\vec x)$$

c)

$$m=0:\quad \frac{i\,\theta(t)}{(2\pi)^3}\int\frac{p^2\,dp\,d\cos\theta\,d\varphi}{2\,p}\left(e^{-itp}-e^{+itp}\right)e^{ipr\cos\theta}$$

$$=\frac{i\,\theta(t)}{(2\pi)^2}\int\frac{p\,dp}{2}\left(e^{-itp}-e^{itp}\right)\frac{1}{ipr}\left(e^{ipr}-e^{-ipr}\right)=$$

$$=\frac{\theta(t)}{2(2\pi)^2}\,\frac{1}{r}\int_0^\infty dp\left(e^{-itp}-e^{itp}\right)\left(e^{ipr}-e^{-ipr}\right)$$

$$=\frac{\theta(t)}{4(2\pi)^2}\,\frac{1}{r}\int_{-\infty}^{\infty}dp\;2\left(e^{-ip(t-r)}-e^{ip(t+r)}\right)=$$

$$=\frac{\theta(t)}{4\pi}\,\frac{1}{r}\left(\delta(t-r)-\underset{\uparrow\ t>0:\ \text{vanishes}}{\delta(t+r)}\right)=\frac{1}{4\pi r}\,\delta(t-r)$$

Solution de $\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) A_\mu(x,t) = j_\mu(x,t)$

$$A_\mu(x,t) = \int d^3x' dt' \; \Delta_{ret}(x-x', t-t') \; j_\mu(x',t')$$

$$= \int d^3x' dt' \frac{1}{4\pi |x-x'|} \delta\left(t-t' - |x-x'|\right) j_\mu(x',t') =$$

$$= \int d^3x' \frac{1}{4\pi |x-x'|} j_\mu\left(x', t-|x-x'|\right)$$

si on suppose $j_\mu \to 0$ pour $t \to -\infty$, cette solution a la même propriété.

On peut utiliser Δ_{ret} aussi pour le problème de Cauchy.

conditions initiales $\phi(x)$, $\partial_t \phi(x)$ à $t=0$

prenons ϕ qui satisfait $\left(\partial_t^2 - \vec{\nabla}^2\right)\phi = 0$

théorème de Green: $\displaystyle\int_V \left(\phi \vec{\nabla}^2 \psi - \psi \vec{\nabla}^2 \phi\right) = \int_{\partial V} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right)$

n normale au bord orientée à l'intérieur

si ϕ et $\partial_t \phi$ ont support dans V, le membre droit $= 0$.

$\psi = \Delta_{ret}(y-x, \tau - t)$;

$$0 = \int_{t_0}^{t_1} dt \int_V \left(\phi \nabla_x^2 \psi - \psi \nabla_x^2 \phi\right) = \int_{t_0}^{t_1}\int_V \phi(x,t)\left(\partial_t^2 \psi - \delta(\tau-t)\,\delta^3(y-x)\right) - \psi \partial_t^2 \phi =$$

$$= -\phi(y,\tau) + \int_{t_0}^{t_1}\int_V \left(\phi \partial_t^2 \psi - \psi \partial_t^2 \phi\right) = -\phi(y,\tau) + \int_V \left(\phi \partial_t \psi - \psi \partial_t \phi\right)\Big|_{t_0}^{t_1}$$

$\tau < t_1$: $\psi = 0 \Rightarrow \phi(y,\tau) = -\int_V \phi(x,t_0)\, \partial_t \Delta(y-x, \tau-t)\Big|_{t_0} - \partial_t\phi(x,t_0)\, \Delta(y-x, \tau-t_0)$

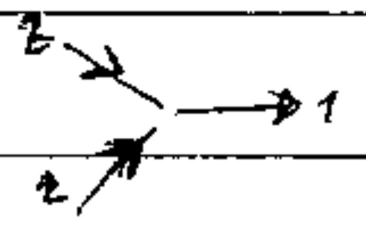
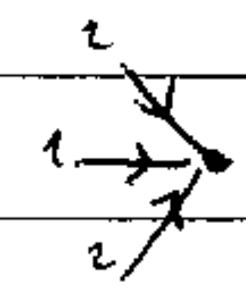
$$= \int_V \phi(x,t_0)\, \partial_\tau \Delta(y-x, \tau-t_0) + \partial_t \phi(x,t_0)\, \Delta(y-x, \tau-t_0)$$

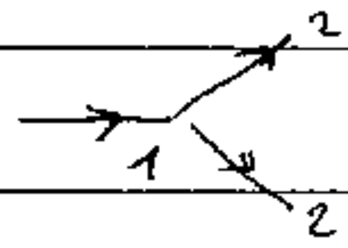
5.7.2 Interaction entre deux champs scalaires

a) $\mathcal{H}_{int} = \frac{g}{2} :\phi_1(x)\,\phi_2^2(x):$ $\qquad\qquad \langle 0 | :\mathcal{H}_{int}: | 0 \rangle = 0$

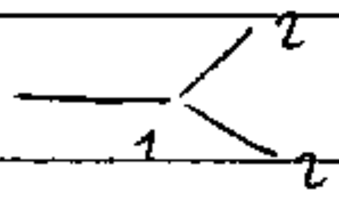
$$\phi(x) = \int \frac{d^3p}{\left((2\pi)^3 2p^0\right)^{1/2}} \left(a(p)\, e^{ipx} + a^+(p)\, e^{-ipx}\right)$$

$$:\phi_1\,\phi_2^2: \;\sim\; :(a_1 + a_1^+)(a_2 + a_2^+)(a_2 + a_2^+):$$

$a_1 a_2^2$:  mais aussi 

$a_1^+ (a_2^+)^2$: 

⋮

on obtient tous les processus  avec n'importe quelle orientation

$$\langle 0 | a(\vec{p}_2, 2)\, a(\vec{p}_1, 2) \int d^4x\, \mathcal{H}_{int}(x)\, a^+(\vec{p}, 1) | 0 \rangle =$$

$$= \frac{g}{2} \int d^4x \int \frac{d^3q_1\, d^3q_2\, d^3q_3\, e^{i(q_1 - q_2 - q_3)x}}{\left((2\pi)^3 2q_1^0\right)^{\frac{1}{2}} \cdots \left((2\pi)^3 2q_3^0\right)^{\frac{1}{2}}} \langle 0 | a_2(p_2) a_2(p_1)\, a_2^+(q_2)\, a_2^+(q_3)\, a_1(q_1)\, a_1^+(\vec{p}) | 0 \rangle =$$

$$= \frac{g}{2} \int \frac{(2\pi)^4}{(2\pi)^{9/2}} \frac{\delta^{(4)}(q_1 - q_2 - q_3)}{(8 q_1^0 q_2^0 q_3^0)^{1/2}} dq_1 dq_2 dq_3 \; \delta(q_1 - p)\; \delta(p_1 - q_2)\, \delta(p_2 - q_3) \times 2 =$$

$$= \frac{g}{(2\pi)^{1/2}} \frac{1}{(8 p^0 p_1^0 p_2^0)^{1/2}} \delta^{(4)}(p - p_1 - p_2)$$

$$S = T\exp\left(-i\int \mathcal{H}_{int}\right)$$

au premiere ordre, $S = 1 - i\int dx\, \mathcal{H}_{int}(x)$

$$S_{p_1 p_2, P} = -2\pi i \frac{g}{\sqrt{(2\pi)^3\, 8\, P^0 p_1^0 p_2^0}}\, \delta^{(4)}(P - p_1 - p_2)$$

$$M_{\alpha\beta} = \frac{g}{\sqrt{(2\pi)^3\, 8\, p^0 p_1^0 p_2^0}}$$

$$d\Gamma = 2\pi\, |M_{\alpha\beta}|^2\, \delta^{(4)}(P - p_1 - p_2)\, dp_1\, dp_2$$

référentiel du c.m. : $P = (m_1, 0)$;

$$p_1^0 + p_2^0 = 2\sqrt{|p_1|^2 + m_2^2} = m_1; \quad |\vec{p}_1|^2 = \left(\frac{m_1}{2}\right)^2 - m_2^2$$

la distribution est isotrope (comme il doit etre pour la desintegration d'une particule scalaire)

$$\Gamma = 2\pi\int d^3p_1\, d^3p_2\, \delta(E - p_1^0 - p_2^0)\, \delta^{(3)}(\vec{p}_1 + \vec{p}_2)\, \frac{g^2}{(2\pi)^3\, 8\, P^0 p_1^0 p_2^0} =$$

$$= \frac{g^2}{(2\pi)^2}\int \frac{d^3p_1\, \delta(m_1 - 2p_1^0)}{8\, m_1\, (p_1^0)^2} = \frac{g^2}{(2\pi)^2}\frac{4\pi}{8 m_1}\int \frac{|p_1|^2\, d|p_1|}{(p_1^0)^2}\, \delta(m_1 - 2p_1^0)$$

$$= \frac{g^2}{8\pi m_1}\frac{1}{\left(\frac{m_1}{2}\right)^2}\int d|p_1|\, |p_1|^2\, \delta\big(m_1 - 2E(|p_1|)\big) = \frac{g^2}{2\pi m_1^3}\int d|p_1|\, |p_1|^2 \frac{1}{|2E'(p_1)|}\delta(|p_1| - x),$$

$$E'(|p|) = \frac{1}{2}\frac{2|p|}{\sqrt{|p|^2 + m_2^2}} = \frac{|p|}{p^0} = \frac{|p|}{(m_1/2)}; \quad \Gamma = \frac{g^2}{2\pi m_1^3}\frac{m_1}{4}|\vec{p}_1| = \frac{g^2}{8\pi m_1^2}\sqrt{\frac{m_1^2 - 4m_2^2}{4}}$$

$$= \frac{g^2}{16\pi}\frac{\sqrt{m_1^2 - 4m_2^2}}{m_1^2} = \frac{g^2}{16\pi m_1}\sqrt{1 - 4\left(\frac{m_2}{m_1}\right)^2};$$

$\frac{m_2}{m_1} = \frac{135}{780} \sim 0{,}1$; $\Gamma \simeq \frac{g^2}{16\pi m_1} \sim (70 \pm 30)$ MeV; $g \sim 1856 \pm 400$ MeV

d) Symétries discrètes:

ϕ_1, ϕ_2 champs scalaires: $\quad P\,\phi(x)\,P^{-1} = \phi(Px)$

$$P\,\mathcal{H}_{int}(x)\,P^{-1} = \mathcal{H}_{int}(Px)$$

$$P\,H_{int}\,P^{-1} = \int d^4x\, \mathcal{H}_{int}(Px) = \int d^4x'\, \mathcal{H}_{int}(x') = H_{int}$$

La parité est conservée alors si ϕ_2 est pseudoscalaire, mais ϕ_1 doit être scalaire.

pour la matrice S, $\Rightarrow$ $S_{p_1,p_2;P} = S_{Pp_1,Pp_2;PP}$, $P(p^0,\vec{p}) = (p^0,-\vec{p})$

dans ce cas P ne donne aucune information supplémentaire, l'invariance pour rotations donnait le même résultat.

Inversion temporelle: $\quad T\phi(x)T^{-1} = \phi(-Px) \quad \Rightarrow \quad T H_{int} T^{-1} = H_{int}$

$$S_{p_1,p_2;P} = S_{PP;Pp_1,Pp_2}$$