4.4.3 Désintégration du muon

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$$

$$p \qquad p_1 \qquad p_2 \qquad p_3$$

$m_\mu \sim 100$ MeV , $m_e \sim 0{,}5$ MeV , $m_\nu \approx 0$

$$\overline{|M_{\beta d}|^2} = \frac{G^2}{128\pi^6} \frac{p_{1\mu} p_3^\mu \, p_\nu p_2^\nu}{p^0 p_1^0 \, p_2^0 p_3^0}$$ (sommé sur polar. finales, moyenné sur les initiales)

$$d\Gamma(\alpha \to \beta) = 2\pi \overline{|M|^2} \, \delta^{(4)}(p_1 + p_2 + p_3 - p) \, d^3p_1 \, d^3p_2 \, d^3p_3$$

$$= \frac{d^3p_1 \, p_{1\mu} p_\nu}{p_0 \, p_1^0} \, I^{\mu\nu}(p - p_1)$$

$$I^{\mu\nu} = 2\pi \frac{G^2}{128\pi^6} \int d^3p_2 \, d^3p_3 \, \delta(E - p_1^0 - p_2^0 - p_3^0) \, \delta^{(3)}(p_2 + p_3 + p_1 - p) \frac{p_2^\nu p_3^\mu}{p_2^0 p_3^0}$$

$I^{\mu\nu}(q)$ tenseur de Lorentz : il peut être $A(q^2) \, q^\mu q^\nu + B(q) \, q^2 g^{\mu\nu}$

A, B scalaires : on les calcules dans le système où $\vec{q} = 0$; $(q = p - p_1)$

$$I^{\mu\nu}(q) = \frac{G^2}{64\pi^5} \int d^3p_2 \, \delta(2p_2^0 - q^0) \frac{+p_2^\mu p_3^\nu}{p_2^0 p_3^0} =$$

$$= + \frac{G^2}{64\pi^5} \int |p_2|^2 d|p_2| \, d\Omega \, \delta(2|p_2| - q^0) \frac{p_2^\mu p_3^\nu}{p_2^0 p_3^0}$$

$$I^{00} = \frac{G^2}{64\pi^5} \int |p_2|^2 \, d|p_2| \, d\Omega \, \delta(2|p_2| - q^0) =$$

$$= \frac{G^2}{64\pi^5} \times \frac{1}{2} \left(\frac{q^0}{2}\right)^2 4\pi = \frac{G^2}{128\pi^4} (q^0)^2$$

$$I^{11} = \frac{G^2}{64\pi^5} \int d|p_2| \cancel{|p_2|^2} \, d\Omega \, \delta(2|p_2| - q^0) \frac{-(p_2^1)^2}{\cancel{|p_2|^2}} =$$

$$= - \frac{G^2}{64\pi^5} \frac{1}{2} 2\pi \int d(\cos\theta) \left(\frac{q^0}{2}\right)^2 \cos^2\theta =$$

$$= - \frac{G^2}{64\pi^4} (q^0)^2 \frac{1}{4} \cdot \frac{2}{3} = \frac{G^2}{128\pi^4} (q^0)^2 \left(-\frac{1}{3}\right);$$

$$I^{00} = A (q^0)^2 + B q^2 \eta^{00} = (q^0)^2 (A + B)$$

$$I^{11} = B q^2 = (q^0)^2 (-B)$$

$$B = \frac{G^2}{128\pi^4} \frac{1}{3}; \qquad A + B = \frac{G^2}{128\pi^4} 1; \qquad A = \frac{G^2}{128\pi^4} \times \frac{2}{3}$$

$$I^{\mu\nu} = \frac{2}{3} \frac{G^2}{128\pi^4} \left(q^\mu q^\nu + \frac{1}{2} q^2 \eta^{\mu\nu}\right)$$

$$p_\mu q^\mu = p \cdot (p - p_1) = p^2 - p \cdot p_1 = - m_\mu^2 + E_1 m_\mu \quad \leadsto \quad p^\mu = (m_\mu, \vec{0})$$

$$p_1 \cdot q = p \cdot p_1 - p_1^2 = - m_\mu E_1 + m_e^2$$

$$q^2 = (p - p_1) \cdot q = 2 m_\mu E_1 - m_\mu^2 - m_e^2;$$

$$p_1^\mu p^\nu \left(q_\mu q_\nu + \frac{1}{2} q^2 \eta_{\mu\nu}\right) = (p_1 \cdot q)(p \cdot q) + \frac{1}{2} q^2 p_1 \cdot p =$$

$$= \left(m_e^2 - m_\mu E_1\right)\left(-m_\mu^2 + m_\mu E_1\right) + \frac{1}{2}\left(2 m_\mu E_1 - m_\mu^2 - m_e^2\right)\left(-m_\mu E_1\right) =$$

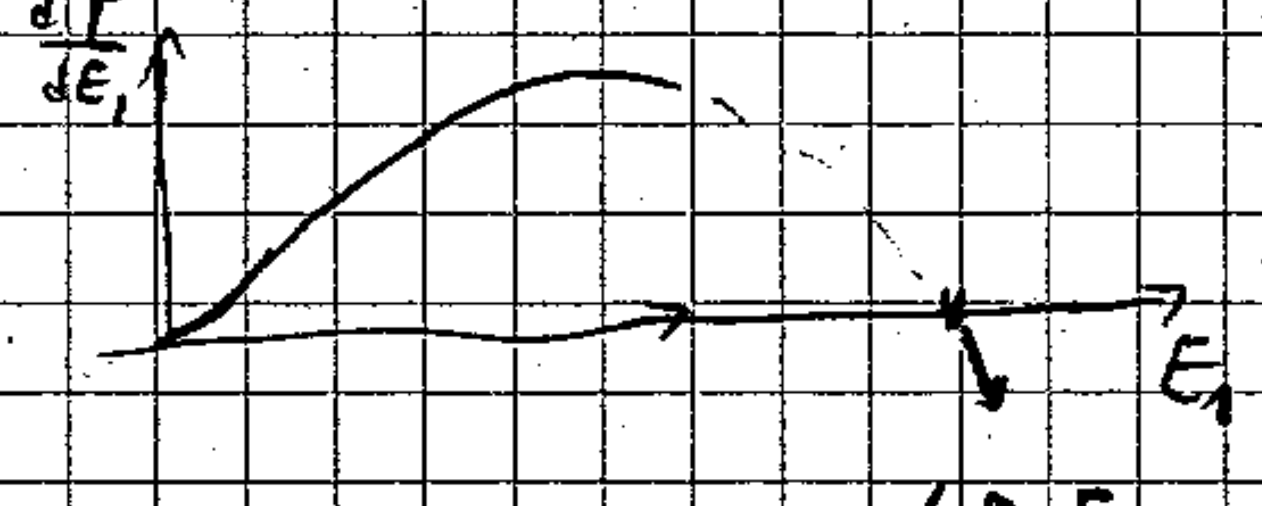
$$= - m_e^2 m_\mu^2 + m_\mu m_e^2 E_1 + m_\mu^3 E_1 - m_\mu^2 E_1^2 - m_\mu^2 E_1^2 + \frac{1}{2} m_\mu^3 E_1 + \frac{1}{2} m_e^2 m_\mu E_1$$

$$= -2 m_\mu^2 E_1^2 + \frac{3}{2} m_\mu^3 E_1 + \frac{3}{2} m_e^2 m_\mu E_1 - m_e^2 m_\mu^2$$

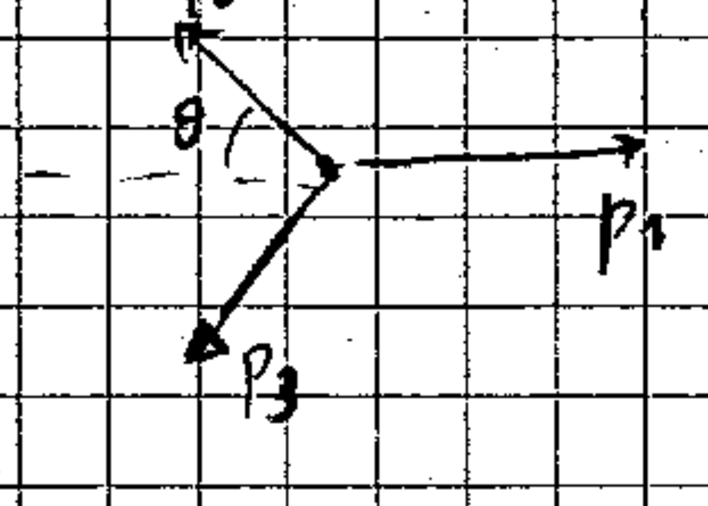
$$\Gamma = \frac{2}{3} \frac{G^2}{128\pi^4} \int \frac{d^3 p_1}{p_1^0} \left(-2 m_\mu E_1^2 + \frac{3}{2} m_\mu^2 E_1 + \frac{3}{2} m_e^2 E_1 - m_e^2 m_\mu\right)$$

$$(|p_1| \sim E_1) = \frac{1}{3} \frac{4\pi G^2}{128\pi^4} \int dE_1 \left(-4 m_\mu E_1^3 + 3 m_\mu^2 E_1^2 + 3 m_e^2 E_1^2 - m_e^2 m_\mu E_1\right)$$

$$\sim \frac{G^2}{96\pi^3} \int_0^{E_{max}} dE_1 \left(-4 m_\mu E_1^3 + 3 m_\mu^2 E_1^2\right)$$



$$4 E_1 - 3 m_\mu = 0 ; \quad E_1 = \frac{3}{4} m_\mu$$



$$\vec{p} = 0 : \quad \vec{p}_1 = -(\vec{p}_2 + \vec{p}_3)$$

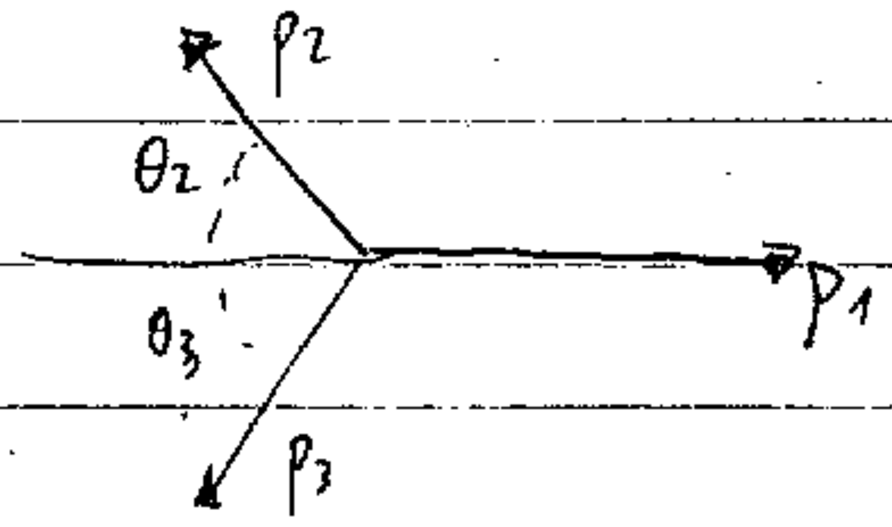
$$|\vec{p}_1| = 2 |p_2| \cos\theta ;$$

$$m_\mu = E_M = |p_1| + 2|p_2| = 2|p_2|(1 + \cos\theta)$$

$$E_1 = m_\mu \frac{\cos\theta}{1 + \cos\theta} = \frac{x}{1+x}$$

$$E_{max} = E(\cos\theta = 1) = \frac{1}{2} m_\mu$$

Mieux:



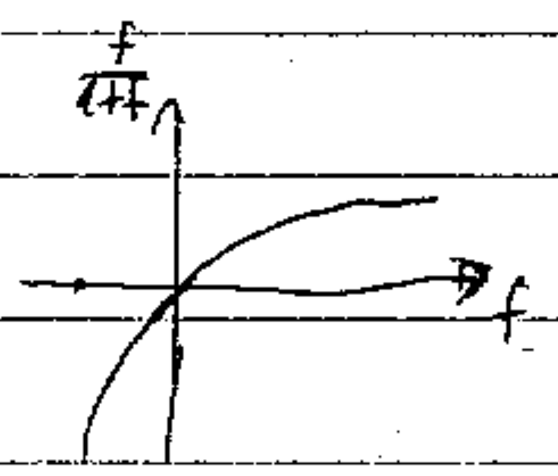
$$|p_2| \,\text{sen}\,\theta_2 = |p_3|\,\text{sen}\,\theta_3 \; ; \qquad |p_1| + |p_2| + |p_3| = m_\mu$$

$$\Rightarrow |p_2| = \frac{\text{sen}\,\theta_3}{\text{sen}\,\theta_2 + \text{sen}\,\theta_3}\left(m_\mu - |p_1|\right), \qquad |p_3| = \frac{\text{sen}\,\theta_2}{\text{sen}\,\theta_2 + \text{sen}\,\theta_3}\left(m_\mu - |p_1|\right)$$

$$|p_1| = |p_2|\cos\theta_2 + |p_3|\cos\theta_3 = \frac{\cos\theta_2\,\text{sen}\,\theta_3 + \text{sen}\,\theta_2\cos\theta_3}{\text{sen}\,\theta_2 + \text{sen}\,\theta_3}\left(m_\mu - |p_1|\right)$$

$$\equiv f(\theta_2, \theta_3)\left(m_\mu - |p_1|\right)$$

$$|p_1| = \frac{f}{1+f}\, m_\mu \; ; \qquad f = \frac{\text{sen}\,(\theta_2 + \theta_3)}{\text{sen}\,\theta_2 + \text{sen}\,\theta_3}$$



maximiser $|p_1|$ = maximiser $f(\theta_2, \theta_3)$

$$\frac{\partial f}{\partial \theta_2} = \frac{\cos(\theta_2+\theta_3)(\text{sen}\,\theta_2 + \text{sen}\,\theta_3) - \text{sen}\,(\theta_2+\theta_3)\cos\theta_2}{(\text{sen}\,\theta_2 + \text{sen}\,\theta_3)^2} =$$

$$= \frac{(\cos 2 \cos 3 - \text{sen}\, 2\, \text{sen}\, 3)(\text{sen}\, 2 + \text{sen}\, 3) - (\text{sen}\, 2 \cos 3 + \cos 2\, \text{sen}\, 3)\cos 2}{''} =$$

$$= \frac{\text{sen}\,\theta_3\left(\cos\theta_2\cos\theta_3 - \text{sen}\,\theta_2\,\text{sen}\,\theta_3 - 1\right)}{(\;)^2} = 0. \qquad \frac{\partial f}{\partial \theta_3} = \frac{\text{sen}\,\theta_2\left(\cos(\theta_2+\theta_3) - 1\right)}{(\;)^2} = 0$$

$\Rightarrow \theta_2 = \theta_3 = 0$, le denominateur est zero aussi, mais pour $\theta_2, \theta_3 \sim 0$ on a

$$f = \frac{\theta_2 + \theta_3 - \frac{1}{6}(\theta_2+\theta_3)^3}{\theta_2 - \frac{\theta_2^3}{6} + \theta_3 - \frac{\theta_3^3}{6}} = \frac{1 - \frac{(\theta_2+\theta_3)^2}{6}}{1 - \frac{\theta_2^3 + \theta_3^3}{6(\theta_2+\theta_3)}} \sim 1 - \frac{1}{6}\,\frac{(\theta_2+\theta_3)^3 - \theta_2^3 - \theta_3^3}{(\theta_2+\theta_3)} = 1 - \frac{1}{2}(\theta_2 + \theta_3)$$

$$\Rightarrow f_{max} = 1 \; , \quad E_{max} = \frac{1}{2} m_\mu$$

4.4.4

$$F[j,h] = \sum_{n,m} \frac{1}{n!\,m!} \int dq_1' \dots dq_n' \, dq_1 \dots dq_m \; j(q_1') \dots j(q_n') \, S_{q_1' \dots q_n';\, q_1 \dots q_m} \, h(q_1) \dots h(q_m)$$

a)

$$\langle 0 | e^{\int dq' j(q') a(q')} \, S \, e^{\int dq\, h(q) a^\dagger(q)} | 0 \rangle =$$

$$= \sum_{n,m} \frac{1}{n!\,m!} \int dq_1' \dots dq_n \; j(q_1') \dots j(q_n') \langle 0 | a(q_1') \dots a(q_n') \, S \int dq\, h(q)\, a^\dagger(q) \dots a^\dagger(q_m) | 0 \rangle =$$

$$= \sum_{n,m} \frac{1}{n!\,m!} \int dq'\, dq \quad \langle \Phi_{q_1' \dots q_n'} | S | \Phi_{q_1 \dots q_m} \rangle \equiv F[j,h]$$

b) $F^c = j(q') \, S^c_{q';q} \, h(q) + \frac{1}{2^2} \, j(q_1') j(q_2') \, S^c_{q_1'q_2';\, q_1 q_2} \, h(q_1)\, h(q_2)$

$$e^{F^c} = 1 + F + \frac{1}{2} F^2 \rightarrow : \frac{1}{2^2} \, j(q_1') j(q_2') \, S^c_{q_1'q_2';\, q_1 q_2} \, h(q_1)\, h(q_2) +$$

$$+ \frac{1}{2} j(q_1') \, S^c_{q_1';q_1} \, h(q_1) \; j(q_2') \, S^c_{q_2';q_2} \, h(q_2) =$$

$$= \frac{1}{2^2} \, j(q_1') j(q_2') \; h(q_1) h(q_2) \underbrace{\left(S^c_{q_1'q_2';\, q_1 q_2} + S^c_{q_1';q_1} S^c_{q_2';q_2} + S^c_{q_2';q_1} S^c_{q_1';q_2} \right)}_{S_{q_1'q_2';\, q_1 q_2}}$$

en symetrisant sur (q_1', q_2') le deuxième term.

c) $F^c = \sum_{n \geq 1} F_n^c$

$$e^{F^c} = \prod_{n \geq 1} e^{F_n^c} = \prod_i \left(\sum_{n_i} \frac{1}{n_i!} (F_i^c)^{n_i} \right) = \sum_{n_i} \prod_i \frac{1}{n_i!} (F_i^c)^{n_i} =$$

$$= \sum_n \sum_{\{\sum i n_i = n\}} \prod_i \frac{1}{n_i!} (F_i^c)^{n_i} \qquad \left(\sim \sum_n j^n h^n \right)$$

$$n := \sum_{\{\sum i n_i = n\}} \prod_i \frac{1}{n_i!} (F_i^c)^{n_i}.$$

$$\frac{\delta^n}{\delta j(p_1') \ldots \delta j(p_k')}\Bigg|_{j=0} \frac{\delta^n}{\delta h(p_1) \ldots \delta h(p_k)}\Bigg|_{h=0} (F_k^c) = \frac{1}{k!^2} \int dq_1' \ldots dq_k' \, dq_1 \ldots dq_k \left(\sum_\sigma \delta(q_1' - p_{\sigma(1)}') \ldots \delta(q_k' - p_{\sigma(k)}') \right)$$
$$\left(\sum_{\tilde\sigma} \delta(q_1 - p_{\tilde\sigma(1)}) \ldots \delta(q_k - p_{\tilde\sigma(k)}) \right)$$
$$\times \, S^{c}_{q_1' \ldots q_k'; q_1 \ldots q_k} =$$

$$= S^{c}_{p_1' \ldots p_k'; p_1 \ldots p_k}$$

$$\frac{\delta^n}{\delta j^n} \frac{\delta^n}{\delta h^n} \left(\prod_i \frac{1}{n_i!} (F_i^c)^{n_i} \right) :$$

$$\left. \begin{matrix} \frac{\delta}{\delta j} - F_1 - \frac{\delta}{\delta h} \\ - F_1 - \end{matrix} \right\} n_1$$

$$\left. \begin{matrix} = F_2 = \\ = F_2 = \end{matrix} \right\} n_2$$

$$\left. \begin{matrix} \equiv F_k \equiv \\ \equiv F_k \equiv \end{matrix} \right\} n_k$$

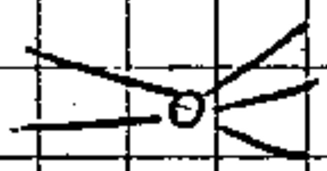
$$= \sum_{\text{all those } \ldots} \left(\prod_i \frac{1}{n_i!} \right) S^{c}_{p_1', p_1} \ldots S^{c}_{p_{n_1}', p_{n_1}} \; S^{c}_{p_1' p_2'; p_1 p_2} \ldots S^{c}_{p_{k-1}' p_k'; p_{k-1} p_k} \ldots$$

$$= \sum_{\substack{\text{partitions}\\ \alpha=\alpha_1\cup\dots\cup\alpha_k\\ \beta=\beta_1\cup\dots\cup\beta_k}} S^c_{\alpha_1\beta_1}\dots S^c_{\alpha_k\beta_k}$$

$n_i!$ compte les permutations dans chaque groupe: $S_{p_1' p_1} S_{p_2' p_2} = S_{p_2' p_2} S_{p_1' p_1}$

$\neq S_{p_1' p_2} S_{p_2' p_1}$ (not $n_i!^2$)

$$\Rightarrow e^{F^c} = F \quad ;$$

d) si $n \neq m$, il faut aussi considérer 

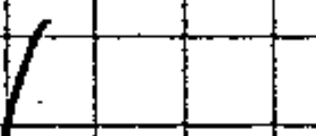
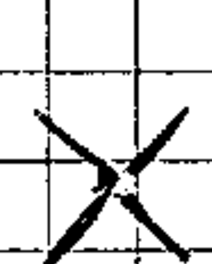
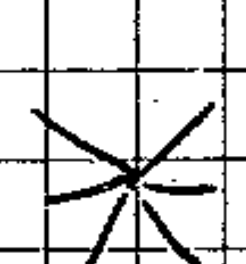
$$F^c = \sum F^c_{n,m} \quad , \quad \frac{\delta}{\delta j^n}\frac{\delta}{\delta h^m} F^c_{n,m} = S^c_{p_1'\dots p_n';\,p_1\dots p_m}$$

considère

$$e^{F^c} = \prod_{n,m} e^{F^c_{ij}} = \sum_{n_{ij}} \prod_{i,j} \frac{1}{n_{ij}!}\left(F^c_{ij}\right)^{n_{ij}} =$$

$$= \sum_{n,m} \sum_{\left\{\substack{\sum i\, n_{ij}=n\\ \sum j\, n_{ij}=m}\right\}} \frac{1}{(n_{ij})!}\left(F^c_{ij}\right)^{n_{ij}}$$

toujours la dégénérescence est liée aux diagrammes du même type.

exp ( +  +  + …) $= 1 + \sum$ (toutes diagrammes déconnectés) expansion en cumulants:

$$Z(t) = \langle e^{tX} \rangle$$

$$\log Z(t) = \sum_n \frac{t^n}{n!} M_n(X),$$

$$M_1(X) = \langle X \rangle$$

$$M_2(X) = \langle X^2 \rangle - \langle X \rangle^2$$

$$M_3 = \langle X^3 \rangle - 3\langle X \rangle \langle X^2 \rangle + 2\langle X \rangle^3$$

X, Y indépendants: $M_n(X+Y) = M_n(X) + M_n(Y)$

$Z(0) = \langle \ldots \rangle = C$; $\quad \langle\langle A \rangle\rangle = \frac{1}{C}\langle A \rangle$,

$$M_1(X) = \langle\langle X \rangle\rangle$$

$$M_2 = \langle\langle X^2 \rangle\rangle - \langle\langle X \rangle\rangle^2 = \frac{1}{C}\langle X^2 \rangle - \frac{1}{C^2}\langle X \rangle^2$$

$$F(t) = \log \frac{Z(t)}{Z(0)};$$

Diagrams: $S_{\circ\circ} = e^{\bigcirc}$

$$\exp\left(\bigcirc + \text{—} + \times + \ldots\right) = \sum \text{toutes diagrammes} =$$

$$= \sum (\text{diagrammes connexes}) \times e^{\bigcirc}$$

F