2.6.1. a)

$$[J^{\mu\nu}, J^{\rho\sigma}] = -i\left(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\right)$$

$$[J_1, J_2] = \cancel{4}\,[J^{23}, J^{31}] = -\frac{i}{\cancel{4}}\left(J^{21}\right) = i J_3, \Rightarrow [J_i, J_j] = i\,\varepsilon_{ijk} J_k;$$

$$[J^{ij}, J^{0l}] = +i\left(\delta^{jl} J^{i0} - \delta^{il} J^{j0}\right)$$

$$[J_k, N_l] = -\frac{i}{2}\,\varepsilon_{kij}\left(\delta^{jl} N^i - \delta^{il} N^j\right) = -i\,\varepsilon_{kil} N_i = i\,\varepsilon_{klm} N_m,$$

$$[J^{0k}, J^{0l}] = -i\left(J^{kl}\right); \quad [J^{01}, J^{02}] = -i J^{12} = -i J_3; \quad [N^k, N^l] = -i\,\varepsilon_{klm} J_m;$$

b) $J^L = \frac{1}{2}(J + iN)$

$J^R = \frac{1}{2}(J - iN)$ non-hermitiques!

$$[J_i^L, J_j^{L/R}] = \frac{1}{4}[J_i + iN_i, J_j \pm i N_j] =$$

$$= \frac{1}{4}\left[i\varepsilon_{ijk} J_k \pm i\,i\,\varepsilon_{ijk} N_k - i\,i\,\varepsilon_{jik} N_k \mp (-i)\,\varepsilon_{ijk} J_k\right] =$$

$$= \frac{1}{4}\left[(i \pm i)\,\varepsilon_{ijk} J_k + (-1 \mp 1)\,\varepsilon_{ijk} N_k\right]; \quad [J^L, J^R] = 0$$

$$= \frac{1}{2}\varepsilon_{ijk}\left[i J_k - N_k\right] = +\frac{i}{2}\varepsilon_{ijk}\left(J_k + i N_k\right) = i\,\varepsilon_{ijk} J^L_k$$

c) $P\vec{J}P^{-1} = \vec{J}$, $P\vec{K}P^{-1} = -\vec{K}$

d) $e^{i\theta^a J_a + i v^a N_a} = U(\Lambda)$, infinitesimally $\Lambda = \delta_\mu^{\ \nu} + \omega_\mu^{\ \nu}$

$$\omega_{\mu\nu} J^{\mu\nu} = \theta^a J_a + v^a N_a$$

$$e^{i v^a N_a} = U(\Lambda(v)); \quad \frac{d}{dv}\Big|_{v=0}: \frac{d\Lambda()}{dv} = \omega^{(01)}_{\mu\nu} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \cdot & \\ & & & \cdot \end{pmatrix}; \quad \Lambda(v) = e^{v\omega^{(01)}} = \begin{pmatrix} \cosh v & \sinh v & & \\ \sinh v & \cosh v & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$(\omega^{(01)})^2 = -1; \quad (\omega^{(01)})_\mu^{\ \nu} = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix};$$

$$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & & \\ +\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} ch\,\nu & sh\,\nu \\ sh\,\nu & ch\,\nu \end{pmatrix}$$

$$\frac{v}{c} = \beta = \frac{\beta\gamma}{\gamma} = \frac{sh\,\nu}{ch\,\nu} = tgh\,\nu\,,$$

e) Irreps of $SU(2)$: $\quad j=0: \quad D(J_a) = 1$

$$j = \frac{1}{2}: \quad D_{1/2}(J_a) = \sigma_a$$

$$j = 1: \quad D(J_3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$D(J_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D(J_2) = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$J_1^2 + J_2^2 + J_3^2 = j(j+1) = 2$$

f) $$D_{(j_1,j_2)}\left(e^{i\theta J + i\nu N}\right) = \exp\left(i\,\xi^a D_{j_1}(J_a^L)\right)\, \exp\left(i\,\xi^{a*}\, D_{j_2}(J_a^R)\right)$$

ξ, ξ^* réelles, indépendantes $\Rightarrow$ irrep of $SU(2)_L \times SU(2)_R$, unitaire

ξ^* c.c. of ξ, $\Rightarrow$ rep of $SO(3,1)$, non unitaire

boost N_3: $\xi^3 = -i\nu^3$,

$$i D_{(\frac{1}{2},0)}(\nu^3) \begin{pmatrix} a \\ b \end{pmatrix} = \nu^3 D_{\frac{1}{2}}(J_3) \begin{pmatrix} a \\ b \end{pmatrix} = \nu^3 \frac{1}{2} \begin{pmatrix} a \\ -b \end{pmatrix}$$

$$i D_{(0,\frac{1}{2})}(\nu^3) \begin{pmatrix} a \\ b \end{pmatrix} = \qquad -\nu^3 \frac{1}{2} \begin{pmatrix} a \\ -b \end{pmatrix}$$

$$i D_{(\frac{1}{2},\frac{1}{2})}(\nu) = i D_{(\frac{1}{2},0)}(\nu) \otimes 1 + 1 \otimes i D_{(0,\frac{1}{2})}(\nu) \qquad \dim 2 \times 2 = 4$$

$$D_{(\frac{1}{2},\frac{1}{2})}\left(e^{i\theta J + i\nu N}\right) = \underbrace{\exp\left(\frac{\nu}{2}\sigma_3\right)}_{\lambda} \otimes \underbrace{\exp\left(-\frac{\nu}{2}\sigma_3\right)}$$

$$V_L(\zeta) = e^{i\zeta D(\zeta)} \qquad \psi \mapsto V(\zeta)\psi$$

$$V_R = e^{i\zeta^* D(\zeta)} \qquad \psi^+ \to \psi^+ V^+ = \psi^+ \underbrace{e^{-i\zeta^* D(\zeta)}}_{V_R^{-1} = V_L^+}$$

$$V_L \underbrace{\psi_L \psi_R}_{X} V_R^{-1} = V X V^+,$$

$$X = \begin{pmatrix} -x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 + x^3 \end{pmatrix} \in \text{matrices hermitiques}$$

$$\det X = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

$$\det(V X V^+) = \det X \, |\det V|^2 \Rightarrow \text{Lorentz transf. if } |\det V| = 1$$

on peut toujours choisir $\det V = 1$

Lorentz group $SO(3,1) \simeq SL(2, \mathbb{C})/\mathbb{Z}_2$

($V \sim -V$ donnent la même transformation)

Boost : $V = e^{-\frac{\nu}{2}\sigma_3} = V^+$,

$$X \mapsto \begin{pmatrix} e^{-\frac{\nu}{2}} & \\ & e^{\frac{\nu}{2}} \end{pmatrix} \begin{pmatrix} x_+ & y \\ y^* & x_- \end{pmatrix} \begin{pmatrix} e^{-\frac{\nu}{2}} & \\ & e^{\frac{\nu}{2}} \end{pmatrix} = \begin{pmatrix} x_+ e^{-\frac{\nu}{2}} e^{-\frac{\nu}{2}} & y e^{-\frac{\nu}{2}} e^{\frac{\nu}{2}} \\ y^* e^{\frac{\nu}{2}} e^{-\frac{\nu}{2}} & x_- e^{\frac{\nu}{2}} e^{\frac{\nu}{2}} \end{pmatrix} = \begin{pmatrix} x_+ e^{-\nu} & y \\ y & x_- e \end{pmatrix}$$

$$\mapsto \begin{cases} x_0 \mapsto x_0 \cosh \nu \\ \cancel{x}_3 \mapsto x_3 \, \mathrm{sh}\, \nu \end{cases}$$

h) $\left(\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right)\otimes\left(\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right)=$

$$= (0,0)_S \oplus (1,0)_A \oplus (0,0) \oplus (0,1) \oplus 2\left(\frac{1}{2},\frac{1}{2}\right)$$

symétrie: $(0,0)$

2.6.2. a) Considérons $U(\Lambda,0)$: $J^{\mu\nu}$, P^μ sont tenseurs

$$T_{\mu_1\ldots\mu_n}{}^{\mu_1\ldots\mu_l} \to T_{\nu_1\ldots\nu}{}^{\rho\ldots}\Lambda_{\mu_1}{}^{\nu_1}\ldots\Lambda^{\mu_\rho}{}_{\nu_\rho}\ldots$$

P tenseur pour $U(\Lambda,a)$

$$U(\Lambda,a)\,J^{\rho\sigma}\,U^{-1} = \Lambda_\mu{}^\rho\left(J^{\mu\nu}-a^\mu P^\nu + a^\nu P^\mu\right)\Lambda_\nu{}^\sigma$$

$$U\,J^{\rho\sigma}_{\mu}J_{\rho\sigma}\,U^{-1} = \left(J^{\mu\nu}-a^\mu P^\nu+a^\nu P^\mu\right)\left(J_{\alpha\beta}-a_\alpha P_\beta + a_\beta P_\alpha\right)\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\Lambda^\alpha{}_\rho\Lambda^\beta{}_\sigma$$

$$= \left(J^{\mu\nu}-a^\mu P^\nu+a^\nu P^\mu\right)\left(J_{\mu\nu}\underbrace{-a_\mu P_\nu + a_\nu P_\mu}_{-2a_\mu P_\nu}\right)$$

$$= J^{\mu\nu}J_{\mu\nu} - 2a\cdot J\cdot P + 2P\cdot J\cdot a + 2a^2P^2 - 2(a\cdot P)^2$$

$\Lambda'_{\rho'}$ est antisymétrique dans toutes les indices, donc

$$\Lambda = 4!\ \det \Lambda = C\, \varepsilon^{\mu\nu\lambda\rho}\varepsilon_{\mu\nu\lambda\rho} = \varepsilon \cdot 4!\ ,\quad c = 1.$$

$J_{\rho\sigma}$; 4-vecteur de Lorentz

$$\varepsilon^{\mu\nu\rho\sigma} P_\nu \left(J_{\rho\sigma} - a_\rho \cancel{P_\sigma} + a_\sigma \cancel{P_\rho}\right) = W^\mu ;$$

$$P^\mu = (m, 0, 0, 0) ;$$

$$W^\mu = \left(0, -P_0 \tfrac{1}{2}\varepsilon^{ijk} P_{jk}\right) = (0, m\vec{J})$$

$$W^2 = m^2 J^2 = m^2 s(s+1) ;$$

$\vdots K^\mu |K\rangle \quad |K$

$= \cancel{K J^3}$

3.4.1

$$H_0 = \omega\, a^+ a$$

$$V = \frac{\omega}{1-\gamma^2}\left[2\gamma^2 a^+ a + \gamma^2 + \gamma a^2 + \gamma (a^+)^2\right]$$

Spectrum: $|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle$; $\quad H_0|n\rangle = \omega n|n\rangle$, $\quad \begin{cases} a|n\rangle = \sqrt{n}\,|n-1\rangle \\ a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases}$;

$\hookrightarrow$ normalization!

$$b = \frac{1}{\sqrt{1-\gamma^2}}(a+\gamma a^+); \qquad [b,b^+] = \frac{1}{1-\gamma^2}\left([a,a^+] + \gamma^2[a^+,a]\right) = 1.$$

$$b^+ = \frac{1}{\sqrt{1-\gamma^2}}(a^+ + \gamma a),$$

$$b^+ b = \frac{1}{1-\gamma^2}(a^+ + \gamma a)(a+\gamma a^+) = \frac{1}{1-\gamma^2}\left(a^+ a + \gamma^2 a a^+ + \gamma a^2 + \gamma (a^+)^2\right) =$$

$$= \frac{1}{1-\gamma^2}\left((1+\gamma^2)a^+ a + \gamma^2 + \gamma a^2 + \gamma (a^+)^2\right)$$

$$= a^+ a + \frac{1}{1-\gamma^2}\left(2\gamma^2 a^+ a + \gamma^2 + \gamma a^2 + \gamma (a^+)^2\right),$$

$$0 = b|\tilde{0}\rangle = (a + \gamma a^+)|\tilde{0}\rangle = \sum_{n\geq 0} b_n\left(\sqrt{n}\,|n-1\rangle + \gamma\sqrt{n+1}\,|n+1\rangle\right) =$$

$$= \sum_{n\geq 0} b_{n+1}\sqrt{n+1}\,|n\rangle + \gamma\sum_{n\geq 1} b_{n-1}\sqrt{n}\,|n\rangle;$$

$\langle 0|$: $b_1 = 0$; $\quad b_3 = 0$ ...

$\langle 2k+1|$ $\quad b_{2k+2}\sqrt{2k+2} + \gamma\, b_{2k}\sqrt{2k+1} = 0$; $\quad b_{2k+2} = -\sqrt{\frac{2k+1}{2k+2}}\,\gamma\, b_{2k} = (-\gamma)^{k+1}\sqrt{\frac{(2k+1)!!}{(2k+2)!!}}\, b_0$

$$= (-\gamma)^{k+1}\sqrt{\frac{(2k+1)!}{(2k)!!\,(2k+2)!!}}\, b_0 =$$

$$= (-\gamma)^{k+1}\sqrt{\frac{(2k+1)!}{2^{2k+1}\,k!\,(k+1)!}}\, b_0; \qquad b_{2n} = (-\gamma)^n\sqrt{\frac{(2n-1)!}{2^{2n-1}(n-1)!\,n!}}\, b_0;$$

$$|\tilde{0}\rangle = b_0 \sum_{n\geq 0} (-\gamma)^n \sqrt{\frac{(2n-1)!}{2^{2n-1}(n-1)!\,n!}}\, \frac{1}{\sqrt{(2n)!}}\, (a^+)^{2n}\, |0\rangle =$$

$$= b_0 \sum_{n} (-\gamma)^n \sqrt{\frac{1}{2^{2n}(n!)^2}}\, (a^+)^{2n} = b_0 \sum \left(-\frac{\gamma}{2}\right)^n (a^+)^{2n} \frac{1}{n!} =$$

$$= b_0\, e^{-\frac{\gamma}{2}(a^+)^2} |0\rangle,$$

$$\langle\tilde{0}|\tilde{0}\rangle = (b_0)^2 \sum_n (-\gamma)^{2n} \frac{(2n-1)!}{2^{2n-1}(n-1)!\,n!} = (b_0)^2 \sum_n \left(\frac{-\gamma}{2}\right)^{2n} \frac{(2n)!}{n!^2} =$$

$$= (b_0)^2 \frac{1}{\sqrt{1-\gamma^2}}\,, \quad b_0 = (1-\gamma^2)^{\frac{1}{4}} \equiv N;$$

$$[H_0, a] = \omega[a^+a, a] = -\omega a, \quad H_0 a = a(H_0 - \omega),$$

$$e^{iH_0t}\, a\, e^{-iH_0t} = a\, e^{i(H_0-\omega)t} e^{-iH_0t} = a\, e^{-i\omega t},$$

$$e^{iH_0t}\, a^+ e^{-iH_0t} = \left(e^{iH_0t}\, a\, e^{-iH_0t}\right)^+ = a^+ e^{i\omega t},$$

$$V_I(t) = e^{iH_0t}\, V\, e^{-iH_0t} = V(\gamma \to \gamma e^{-2i\omega t}) = \frac{\omega}{1-\gamma^2}\left[2\gamma^2 a^+a + \gamma^2 + \gamma e^{-2i\omega t} a^2 + \gamma e^{2i\omega t}(a^+)^2\right]$$

$$H_I(t) = e^{iH_0t}(H_0 + V)e^{-iH_0t} = H_0 + V_I(t),$$

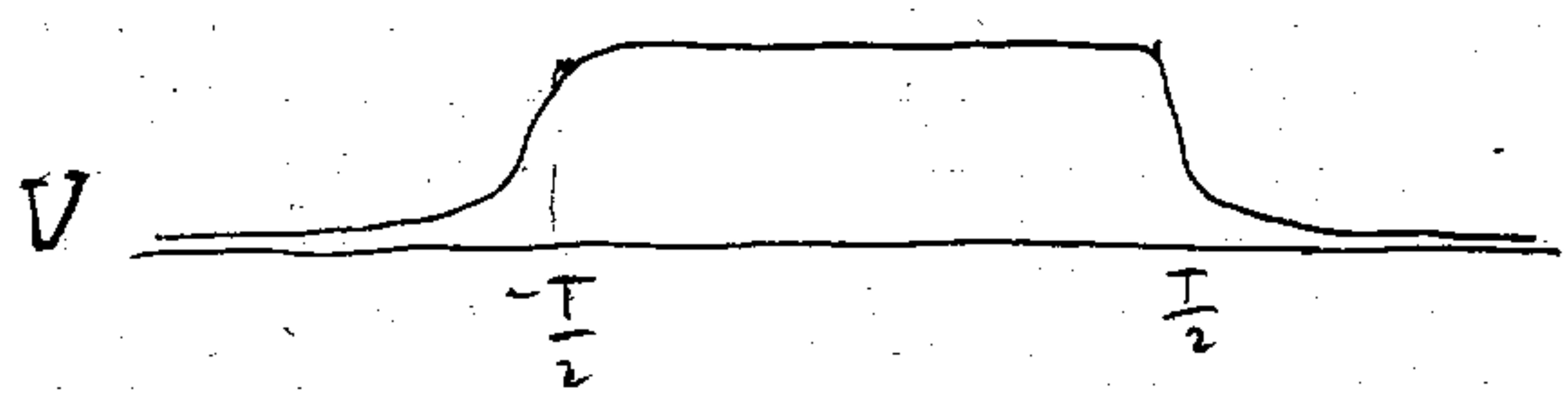
$$U_I(t) \equiv e^{iH_0t}\, e^{iHt}\, e^{-iH_0t} = e^{iH_I(t)t}\,;$$ H_I est le générateur de l'évolution temporelle dans le rep. d'interaction.

$$\Omega(t) = e^{iHt}\, e^{-iH_0t} = e^{-iH_0t}\, e^{iH_I(t)t};$$

$$\Psi_H = \Omega(t)\, \Psi_I(t)$$

L'interaction n'est jamais éteinte, à tout temps les "particules" interagissent.

On pourrait considérer un "switch-on":



$$S_{00} = \langle 0 | e^{-iHT} | 0 \rangle$$

$$a = \frac{1}{\sqrt{1-\gamma^2}} (b - \gamma b^+) \Rightarrow |0\rangle = N e^{\frac{\gamma}{2}(b^+)^2} |\tilde{0}\rangle$$

$$S_{00} = N^2 \langle \tilde{0} | e^{\frac{\gamma}{2} b^2} e^{-i\omega T b^+ b} e^{\frac{\gamma}{2}(b^+)^2} |\tilde{0}\rangle =$$

$$= N^2 \langle \tilde{0} | e^{\frac{\gamma}{2} b^2} e^{\frac{\gamma}{2} e^{2i\omega T}(b^+)^2} |\tilde{0}\rangle$$

même calcul, mais avec $\gamma^2 \to \gamma^2 e^{2i\omega T}$: $\left(\frac{1-\gamma^2}{1-\gamma^2 e^{2i\omega T}} \right)^{\frac{1}{4}}$

Probability $P_{00} = |S_{00}|^2 = \dfrac{\sqrt{1-\gamma^2}}{\left(1+\gamma^4 - 2\gamma^2 \cos(2\omega T)\right)^{\frac{1}{4}}}$ oscille entre 1 et $\sqrt{\dfrac{1-\gamma^2}{1+\gamma^2}}$