Formation interuniversitaire de physique

Evolution and Measurement of Quantum States

Exercise class n°2: Quantum state estimation

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Qubits can be encoded in the polarization of single-mode photons. The $\hat{\sigma}_z$ vector basis is encoded as photons linearly polarized along the horizontal and vertical directions.

Preliminary question: What are the light polarizations corresponding to the $\hat{\sigma}_{x,y,z}$ basis states?

1 Measurements in the $\hat{\sigma}_{x,y,z}$ basis

1.1 Using a single copy

Let us consider a single qubit $|\psi\rangle$ represented by an unknow Bloch vector $\mathbf{r}$. One measures its polarization to be $\sigma = \pm 1$ along z.

1. What is the best guess $|\psi_e\rangle$ for the Bloch vector orientation?
2. What is the mean value of the fidelity $F = \overline{|\langle\psi|\psi_e\rangle|^2}$ for a fixed $|\psi\rangle$?
3. Show that the density matrix describing the outcome of this measurement is given by

$$\hat{\rho}_e = \frac{1}{2}\left[\mathbb{1} + r_z \hat{\sigma}_z\right],$$

   and that the fidelity F can be obtained as $F = \langle\psi|\,\hat{\rho}_e\,|\psi\rangle$.
4. Calculate the mean fidelity $\overline{F}$ when averaging over the orientation of the Bloch vector $\mathbf{r}$.

1.2 Using two copies: first strategy

A better estimation of the state $|\psi\rangle$ can be obtained from two independent polarization measurements performed on two identical copies of $|\psi\rangle$. Instead of measuring the qubit polarization twice along z, it is preferable to measure the spin projection along two orthogonal directions z and x.

1. Show that the density operator after the two measurement reads

$$\hat{\rho}' = \frac{1}{2}\left[\mathbb{1} + r_z \hat{\sigma}_z\right] \otimes \frac{1}{2}\left[\mathbb{1} + r_x \hat{\sigma}_x\right].$$

2. Suppose that one measures the outcomes $+1$ along z and x. What is the most probable quantum state?
3. As we know that the initial state is of the form $|\psi\psi\rangle$, we would prefer to write an estimate density matrix $\hat{\rho}_e$ acting on a single-spin Hilbert space. Using the previous question, show that it is given by the expression

$$\hat{\rho}_e = \frac{1}{2}\left[\mathbb{1} + \frac{1}{\sqrt{2}}\left(r_z \hat{\sigma}_z + r_x \hat{\sigma}_x\right)\right].$$

4. Calculate the fidelity of the measurement $F = \langle\psi|\,\hat{\rho}_e\,|\psi\rangle$.

5. Calculate the mean fidelity $\overline{F}$ averaged over the Bloch vector orientation.

1.3 Using three copies

The previous scheme can be generalized to the measurement on three copies of the same qubit. In that case the measurement is performed successively along x, y and z and thus appears to be more symmetric.

1. Show that the density operator describing the outcome of this measurement is given by

$$\hat{\rho}_e = \frac{1}{2}\left[\mathbb{1} + \frac{1}{\sqrt{3}}\left(r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z\right)\right].$$

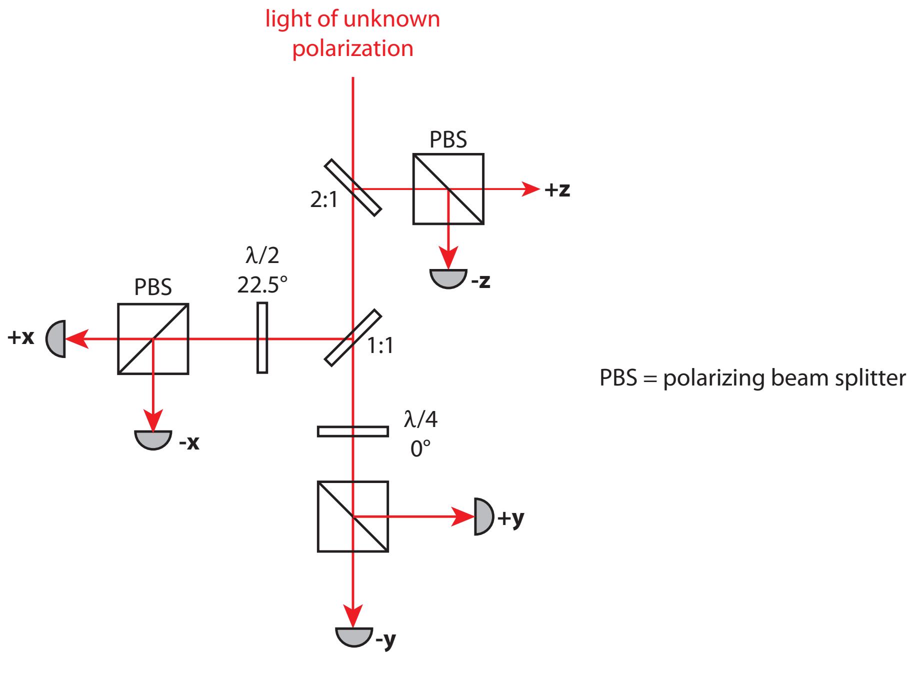
2. Calculate the fidelity of the measurement $F = \langle\psi|\,\hat{\rho}_e\,|\psi\rangle$.

3. Calculate the mean fidelity $\overline{F}$ averaged over the Bloch vector orientation.

1.4 Tomography with a large number of copies

We generalize to a large number of copies $N \gg 1$: we perform $N/3$ measurements along each direction $i = x, y, z$.

1. Show that the tomographic measurement can be performed using the scheme below.



2. One measures n_i occurrences of $+1$ among the measurement outcomes along i. Show that the best estimate of the state $|\psi\rangle$ is given by

$$|\psi_e\rangle\langle\psi_e| = \frac{1}{2}\left[\mathbb{1} + \frac{\mathbf{R}}{R}\cdot\hat{\sigma}\right],$$

where $R_i = 6n_i/N - 1$.

3. Show that the fidelity of the measurement is given by

$$F = \frac{1}{2}\left[1 + \frac{\mathbf{R}}{R}\cdot\mathbf{r}\right].$$

4. For large N values the vector $\mathbf{R}$ remains very close to the actual Bloch vector $\mathbf{r}$. Show that to minimal order the fidelity expands as

$$F = 1 - \frac{1}{4}\sum_i (1 - r_i^2)\delta r_i^2 + \frac{1}{2}\sum_{i<j} r_i r_j \delta r_i \delta r_j,$$

where one defines $\delta\mathbf{r} = \mathbf{R} - \mathbf{r}$.

5. Let us now calculate the mean value of the overlap when averaging over many estimates of the same vector $|\psi\rangle$. From the variance of n_i given by the binomial law show that $\overline{\delta r_i^2} = 3/N(1 - r_i^2)$. Deduce the value of the fidelity

$$F = 1 - \frac{3}{4N}\sum_i (1 - r_i^2)^2.$$

6. After averaging over the vectors $|\psi\rangle$ show that the mean fidelity of a tomographic measurement is given by

$$\overline{F} = 1 - \frac{6}{5N}.$$

2 Measurements along tetrahedron states

2.1 The tetrahedron POM

Let us consider the state estimation using a single qubit $|\psi\rangle$ of Bloch vector $\mathbf{r}$. We discuss in the following a measurement scheme based on a more complex probability operator measure than the $\hat{\sigma}_z$ measurement.

We define four vectors $\mathbf{a}_i$ $(1 \leq i \leq 4)$ as the vertices of a regular polyhedron inscribed in the Bloch sphere. For example:

$$|\mathbf{a}_1\rangle = -i\,|+\rangle\,, \quad |\mathbf{a}_2\rangle = -\frac{1}{\sqrt{3}}\,|+\rangle + \sqrt{\frac{2}{3}}\,|-\rangle\,, \quad |\mathbf{a}_3\rangle = -\frac{1}{\sqrt{3}}\,|+\rangle + \sqrt{\frac{2}{3}}j\,|-\rangle\,, \quad |\mathbf{a}_4\rangle = -\frac{1}{\sqrt{3}}\,|+\rangle + j^2\sqrt{\frac{2}{3}}\,|-\rangle\,,$$

where $j = e^{i2\pi/3}$.

1. Show that the operators $\hat{\pi}_i = \frac{1}{4}(\hat{\mathbb{1}} + \mathbf{a}_i \cdot \hat{\boldsymbol{\sigma}})$ constitute a probability operator measure. Are the $\hat{\pi}_i$ operators projectors?

2. What is the probability p_i to measure the result i? Show that the Bloch vector can be reconstructed from these probabilities as

$$\mathbf{r} = 3\sum_i p_i \mathbf{a}_i.$$

3. Show that, equivalently, the density matrix is given by

$$\hat{\rho} = \sum_i p_i(6\hat{\pi}_i - 1).$$

4. Give an expression of the Kraus operators $\hat{K}_i = \sqrt{\hat{\pi}_i}$ associated with the probability operators $\hat{\pi}_i$.

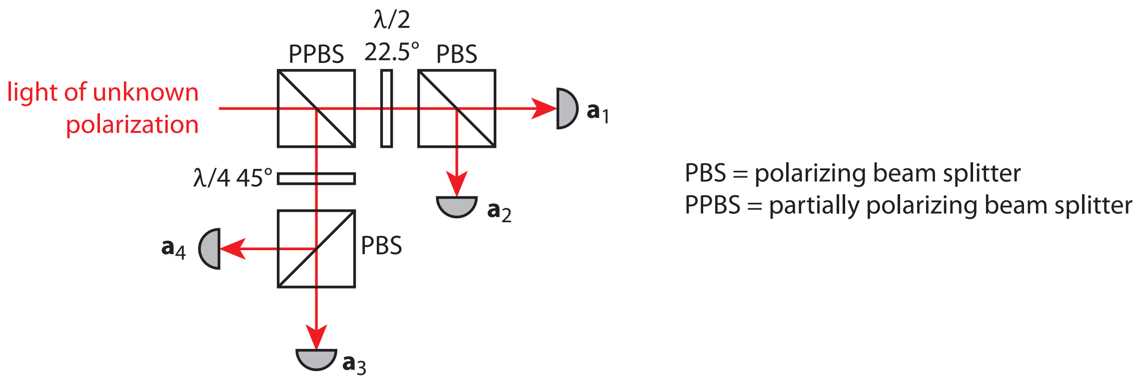
5. Let us consider the state estimation using a single qubit $|\psi\rangle$ of Bloch vector $\mathbf{r}$. Show that the density operator after measurement of the tetrahedron POM is given by

$$\hat{\rho}_e = \sum_i \hat{K}_i \hat{\rho} \hat{K}_i = \frac{1}{2}\left(\hat{\mathbb{1}} + \frac{1}{3}\mathbf{r}\cdot\hat{\boldsymbol{\sigma}}\right).$$

6. Show that the fidelity F is equal to $2/3$ for any qubit $|\psi\rangle$.

The isotropy of this POM makes it more suited that the $\sigma_{x,y,z}$ measure for more complex measurements (e.g. two qubit states).

7. Technical question for home: show that the scheme pictured below realizes the tetrahedron POM. How to choose the partially polarizing beam splitter?



2.2 Optimized collective measurement using two copies

The quantum state estimation discussed in the previous section consisted in interpreting the outcome of successive measurements performed on several copies of $|\psi\rangle$ one by one. We show here that a better strategy consists in performing a collective measurement on several copies simultaneously. We consider for simplicity the case of two copies. We thus aim at guessing the quantum state $|\psi\rangle$ by performing a measurement on the two-qubit state $|\psi\psi\rangle$.

8. Show that the states of the form $|\psi\psi\rangle$ generate the subspace of the Hilbert space corresponding to a total spin $S = 1$.

9. We postulate that an optimized POM is given by the measurement operators

$$\hat{\pi}_i = \frac{3}{4}|\mathbf{a}_i\mathbf{a}_i\rangle\langle\mathbf{a}_i\mathbf{a}_i|.$$

   Show that they indeed form a POM of the subspace $S = 1$. In particular one should show that $\sum_i \hat{\pi}_i = \hat{\mathbb{1}}_{S=1}$, and by symmetry it is sufficient to check the action on $|\mathbf{a}_1\mathbf{a}_1\rangle$ only.

10. Show that the probability to measure the outcome i is given by

$$p_i = \frac{3}{16}(1 + \mathbf{a}_i\cdot\mathbf{r})^2.$$

11. Deduce that the density operator describing the state estimation reads

$$\hat{\rho}_e = \frac{3}{32} \sum_i (1 + \mathbf{a}_i \cdot \mathbf{r})^2 \left(\hat{\mathbb{1}} + \mathbf{a}_i \cdot \hat{\boldsymbol{\sigma}}\right).$$

12. What is the fidelity F of the state estimation?

13. Average the fidelity over $|\psi\rangle$ to get the mean fidelity $\bar{F}$. Compare it with the strategy using successive measurements of $\hat{\sigma}_x$ and $\hat{\sigma}_z$.

Note: It can be shown that the optimal estimation of a quantum state using N copies has an average fidelity $\bar{F} = (N + 1)/(N + 2)$, which was reached for $N = 2$ with the collective measurement using the tetrahedron POM.