M1 ICFP

Evolution and Measurement of Quantum States

Examination of November 15 2013

3 hours - lecture notes authorized

Answers to questions can be written either in French or in English

I. EXERCISE 1 : TRIPARTITE ENTANGLEMENT

One considers a source generating three spin 1/2 particles in a correlated way. These three particles are then sent to three remote observers Alice, Bob and Claude that can measure by a Stern and Gerlach apparatus the Pauli matrices σ_x or σ_y on their respective particles. One names respectively A_x, A_y, B_x, B_y and $C_x, C_y$the values they obtain in their measurement. One recalls :

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad ; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

1) The produced state is the entangled state of the three particles (ordered as $|A, B, C\rangle$), called W state :

$$|\Psi_1\rangle = \frac{1}{\sqrt{3}}(|-,+,+\rangle + |+,-,+\rangle + |+,+,-\rangle)$$

a) Claude does not make any measurement. Give the expression of the mathematical object which describes the bipartite quantum state shared by Alice and Bob.

b) Determine its purity. What is the minimum possible value of this quantity ?

c) Use the Positive Partial Transpose criterion to show that this state is indeed entangled.

2) The produced state is now the entangled state, called GHZ state :

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|+,+,+\rangle - |-,-,-\rangle)$$

a) Same question as 1a. Is the new bipartite state separable or entangled ? What do you conclude concerning the respective robustness of entanglement in states $|\Psi_1\rangle$ and $|\Psi_2\rangle$?

b) Show that

$$\sigma_x^A \sigma_y^B \sigma_y^C |\Psi_2\rangle = \sigma_x^A \sigma_y^B \sigma_y^C |\Psi_2\rangle = \sigma_x^A \sigma_y^B \sigma_y^C |\Psi_2\rangle = |\Psi_2\rangle$$

and

$$\sigma_x^A \sigma_x^B \sigma_x^C |\Psi_2\rangle = -|\Psi_2\rangle$$

c) Give the probability of that Alice gets the values +1 or −1 for A_x in her measurement of σ_x^A. What about the probabilities for the different possible values of A_y, B_x, B_y and C_x, C_y ?

d) The three observers now compare the results of their measurements. Show that they always find that :

$$A_x B_y C_y = A_y B_x C_y = A_y B_y C_x = 1 \quad ; \quad A_x B_x C_x = -1 \tag{1}$$

e) Let us assume in this question that the precise values of the quantities $A_x, A_y, B_x, B_y, C_x, C_y$ are determined by an unknown random process occuring when the three particles are generated and cannot change when the particles propagate in the directions of Alice, Bob and Claude. Show that this assumption is incompatible with the properties given in relations (**??**). Conclusion ?

II. EXERCISE 2 : DISCRIMINATING BETWEEN QUANTUM STATES

Suppose that one prepares an individual qubit in one of the two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$, the state $|\psi_i\rangle$ being chosen with probability p_i ($i = 1$ or 2).

We are interested in the optimal strategy for discriminating between these two possible quantum states, i.e. for determining as certainly as possible the value of i from a quantum measurement. More precisely, we want to know whether a single measurement can allow one to determine the value of i with certainty. This means that there should exist a Probability Operator Measure $\{\hat{\pi}_1, \hat{\pi}_2\}$ such that

$$\langle\psi_1|\,\hat{\pi}_1\,|\psi_1\rangle = 1,\ \langle\psi_1|\,\hat{\pi}_2\,|\psi_1\rangle = 0,$$
$$\langle\psi_2|\,\hat{\pi}_1\,|\psi_2\rangle = 0,\ \langle\psi_2|\,\hat{\pi}_2\,|\psi_2\rangle = 1.$$

1) Let us first assume that the $|\psi_i\rangle$ states are orthogonal. Is it possible to discriminate between them with certainty ?

2) From now on, we will consider the case of non-orthogonal quantum states : $\langle\psi_1\,|\psi_2\rangle \neq 0$. We wonder whether one can discriminate between them with certainty. The probability operator $\hat{\pi}_1$ can be written as

$$\hat{\pi}_1 = |\psi_1\rangle\,\langle\psi_1| + \hat{A},$$

a) Show that $\hat{A}$ is hermitian and positive.

b) Show that this implies that

$$\langle\psi_2|\,\hat{\pi}_1\,|\psi_2\rangle \geq |\langle\psi_1\,|\psi_2\rangle\,|^2,$$

Conclusion ?

3) We want now to construct a measurement scheme which would lead to a minimal error probability in the state discrimination.We are interested in finding the POM $\{\hat{\pi}_1, \hat{\pi}_2\}$ that minimizes the error probability.

a) Explain why the error probability reads

$$P_{\rm err} = p_1\,\langle\psi_1|\,\hat{\pi}_2\,|\psi_1\rangle + p_2\,\langle\psi_2|\,\hat{\pi}_1\,|\psi_2\rangle\,.$$

b) Show that the probability operator $\hat{\pi}_2$ should be chosen as the projector onto the eigenstate of $p_1\,|\psi_1\rangle\,\langle\psi_1| - p_2\,|\psi_2\rangle\,\langle\psi_2|$ of smallest eigenvalue.

(rewrite the error probability as $P_{\rm err} = p_2 + {\rm Tr}[(p_1\,|\psi_1\rangle\,\langle\psi_1| - p_2\,|\psi_2\rangle\,\langle\psi_2|)\hat{\pi}_2)$

c) What should then be the operator $\hat{\pi}_1$?

4) As an example, we consider a qubit prepared with equal probability in one of the two states

$$|\psi_1\rangle = \cos\theta\,|-\rangle + \sin\theta\,|+\rangle\,,\quad |\psi_2\rangle = \cos\theta\,|-\rangle - \sin\theta\,|+\rangle\,,$$

where $0 \leq \theta \leq \pi/4$.

a) What is the value of $\langle\psi_2|\psi_1\rangle$? What are the expressions of the projectors $\hat{\pi}_1$ and $\hat{\pi}_2$?

b) Check that the Bloch vectors associated with the states $|\psi_1\rangle$, $|\psi_2\rangle$ and with the projectors $\hat{\pi}_1$, $\hat{\pi}_2$ belong to the xz plane and plot them for $\theta = \pi/6$.

c) Show that the minimal error probability is given by

$$P_{\rm err} = \frac{1}{2}[1 - \sin(2\theta)].$$