Master ICFP

Ultracold Atoms

Exercise class n°2: Sagnac interferometer

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We consider an atom interferometer, linked to a reference frame $\mathcal{R}$ in uniform rotation with respect to a Galilean reference frame $\mathcal{R}'$. The rotation vector $\mathbf{\Omega}$ is assumed along the Oz axis of a reference frame linked to the reference frame $\mathcal{R}$, the atoms evolving in the plane (xOy). We will evaluate the phase shift of the wave function of a particle between the two arms of the interferometer due to the rotation of $\mathcal{R}$ with respect to $\mathcal{R}'$. For this purpose, we consider a particle of mass m, free in $\mathcal{R}'$. We note $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{p}$ (respectively $\mathbf{r}'$, $\mathbf{v}'$ and $\mathbf{p}'$) its position, velocity and momentum in $\mathcal{R}$ (respectively $\mathcal{R}'$). We recall:

$$x = x' \cos(\Omega t) + y' \sin(\Omega t) \qquad y = -x' \sin(\Omega t) + y' \cos(\Omega t)$$

1 Physics in the rotating frame

1. Give the expression linking $\mathbf{v}'$, $\mathbf{v}$ and $\mathbf{r}$, then recall the Lagrangian $L'(\mathbf{r}', \mathbf{v}')$ of the particle in $\mathcal{R}'$ (gravity will be neglected).
2. We introduce the Lagrangian $L(\mathbf{r}, \mathbf{v}) = L'(\mathbf{r}', \mathbf{v}')$. Write explicitly $L(\mathbf{r}, \mathbf{v})$ in terms of $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{\Omega}$.
3. It should be recalled that in Lagrangian formalism, the momentum is defined by $p_i = \partial L/\partial \dot{x}_i$.
   (a) Determine the momentum $\mathbf{p}$ as a function of $\mathbf{r}$ and $\mathbf{v}$. Does this result evoke an analogy with another physical situation?
   (b) Compare $\mathbf{p}$ and $\mathbf{p}'$, and deduce that the Lagrangian L is a constant of the motion.
4. Show that the hamiltonian $H(\mathbf{r}, \mathbf{p})$ of the particule (defined by $H = \mathbf{p} \cdot \mathbf{v} - L$ is written:
$$H = \frac{\mathbf{p}^2}{2m} - \mathbf{\Omega} \cdot \mathbf{L} \tag{1}$$
   where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum of the particle.

2 Wavefunction propagation

We note $\Psi(\mathbf{r}, t)$ the wave function associated with the particle. It is assumed that at $t = 0$, this wave function is that of a plane wave of momentum $\mathbf{p}_0$: $\Psi(\mathbf{r}, 0) = \exp(i\mathbf{r} \cdot \mathbf{p}_0)$. If the Lagrangian is quadratic in velocity and position, we admit that the wavefunction at a point B at time T is given by:

$$\Psi(\mathbf{r}_B, T) = G(T)\Psi(\mathbf{r}_A, 0) \exp\left(\frac{i}{\hbar} S(\mathbf{r}_B, T; \mathbf{r}_A, 0\right) \tag{2}$$

where the point $\mathbf{r}_A$ is such that the trajectory passing in $\mathbf{r}_A$ at the time $t = 0$ with the momentum $\mathbf{p}_0$ passes in $\mathbf{r}_B$ at the time T. The function $G(T)$ depends only on T (not on the path traveled), and S is the classical action calculated on the path.

1. Calculate $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$. We recall the Hamilton-Jacobi equations :

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}}.$$

2. Integrate the equation over $\mathbf{p}$, and then the one over $\mathbf{r}$, with initial conditions the particle in $\mathbf{r}_A$ with momentum $\mathbf{p}_0$.

3. Show that the classical action S, in first order in Ω, is given by:

$$S = S_0 + m\Omega(x_A y_B - y_A x_B) \quad \text{with} \quad S_0 = \frac{m}{2T}\left((x_B - x_A)^2 + (y_B - y_A)^2\right). \tag{3}$$

4. Show that at the zeroth order, we have:

$$x_A y_B - y_A x_B = \mathbf{u}_z \cdot \int_0^T \mathbf{r}(t) \times \mathbf{v}(t) dt$$

   where the integral is taken along the AB trajectory.

3 Phase shift

We can then show that the wavefunction in $\mathbf{r}_B$ at T is written:

$$\Psi(\mathbf{r}_B, T) = \Psi^{(0)}(\mathbf{r}_B, T) \exp\left(\frac{i}{\hbar} m\mathbf{\Omega} \cdot \int_0^T \mathbf{r}(t) \times \mathbf{v}(t) dt\right) \tag{4}$$

where $\Psi^{(0)}(\mathbf{r}_B, T)$ represents the wavefunction for $\Omega = 0$.
This point is more difficult to show than it seems, and requires careful inspection of the trajectories.

1. The interferometer is assumed to be in the shape of an $ACBD$ quadrilateral. A particle from point A can therefore take two paths (ACB or ADB). Deduce from the previous questions the phase shift of the particle wavefunction between the two arms of the interferometer according to Ω and the area of the interferometer.

2. In a 1996 experiment in California, Mark Kasevich and his group demonstrated this effect on cesium atoms (mass $m = 2.2 \times 10^{-25}$ kg). The velocity of the atomic beam used is 300 m/s. At point A, the beam is divided into two components. One (AD) is transmitted without modification; the other (AC) is deflected by interaction with a laser beam. More precisely, the transverse velocity of the deflected atoms is modified by two recoil velocities, using a stimulated Raman pulse. The same phenomenon is used at points C and D to deflect the atoms again, and at B to close the interferometer.
   (a) The lasers wavelength is $\lambda = 852$ nm. Calculate the recoil velocity.
   (b) We give $AD = CB = 1$ m. Calculate the area of the interferometer.
   (c) Calculate the phase shift related to the rotation of the earth and compare it to the result found by Mr. Kasevich: 4.2 radian.