Heuristic analysis of the Pure Random Walk search procedure for random SAT problems

(on the importance of large deviations for the…)

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Pure Random Walk procedure for 3-SAT

1. Pick up an initial (random) configuration of variables;
2. If all clauses are satisfied, STOP.
3. Else, choose randomly one unsat clause and flip one of its attached variables; Goto 2.

Example:

set of clauses

(NOT x_1 or x_2 or x_3) → F
(x_1 or x_2 or NOT x_4) → T
(NOT x_1 or x_2 or x_4) → T

1. random configuration x_1=T, x_2=F, x_3=F, x_4=T
2. pick up first clause and flip one variable it contains

   e.g. x_2=F ö x_2=T then all clauses are satisfied

   e.g. x_1=T ö x_1=F then clause 1 is satisfied, but clause 2 is not any longer.

Number of unsat clauses (=energy) is not guaranteed to decrease!

Rigorous results for 2-Sat and 3-Sat

- Papadimitriou (1992)

  2-SAT: if there is a solution, it is found w.h.p. in a time growing as $O(N^2)$.

- Schöning (2000):

  3-SAT: the probability that a satisfiable instance is not solved after T runs (of 3N steps each) is bounded from above by $\exp(-T\mu(3/4)^N)$
  recently: 1.329… instead of 1.333…

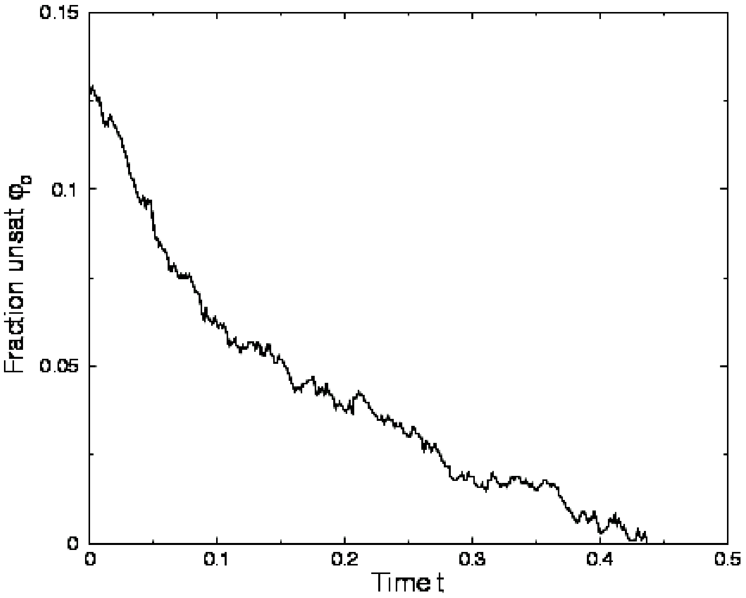
- Alekhnovich and Ben-Sasson (2002):

  random 3-SAT: a solution is almost surely reached in polynomial time if ratio α of clauses/variable is smaller than 1.63

Phenomenology

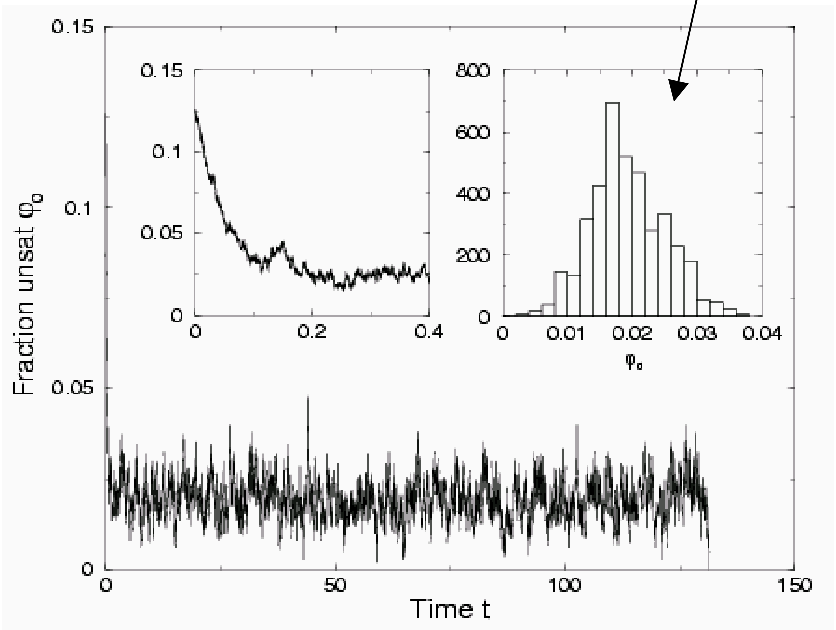
Single runs on instances with N=500 variables, M= a N clauses

a = 2



a = 3

$P_N(j_0)$
o
$\exp(-M\ p(j_0))$



(Approximate) Analysis

- Polynomial regime: what is the average solving time?

$$T_{res}(N,a) = N\ t_{res}(a)$$

- Characterization of the polynomial/exponential transition.

  (average behaviour: plateau properties)

- Exponential regime: what is the average solving time?

$$T_{res}(N,a) = \exp(N\ z(a))$$
$$= 1 / \mathrm{Proba}(\ j_0 = 0\ ) = \exp(M\ p(0))$$

(large deviations: escape from plateau)

Average running time in the polynomial regime

- if a is small enough, the graph of interactions is essentially made of small clusters C
- $t_{res} = \underset{C}{S}\ p(C)\ \ t_{res}(C)$
- results for K-SAT *(exact provided t_{res} can be expanded):*

$$t_{res}(\alpha, K) = \frac{1}{2^K} + \frac{K(K+1)}{K-1}\,\frac{1}{2^{2K+1}}\,\alpha + \frac{4K^6 + K^5 + 6K^3 - 10K^2 + 2K}{3(K-1)(2K-1)(K^2-2)}\,\frac{1}{2^{3K+1}}\,\alpha^2 + O(\alpha^3)$$

But cannot capture the plateau regime …

Average behaviour (I)

Parameters:

N = number of variables
M = number of clauses = a N
K = number of variables in each clause (K-SAT)

- Define $M_j(T)$ the number of clauses satisfied by j literals after T clause flips with j = 0, 1,…, K.
  ($M_0(T)$ is the number of unsatisfied clauses)

- Normalization condition:

$$\sum_{j=0}^{K} M_j(T) = M$$

- Define fractions of j-clauses:

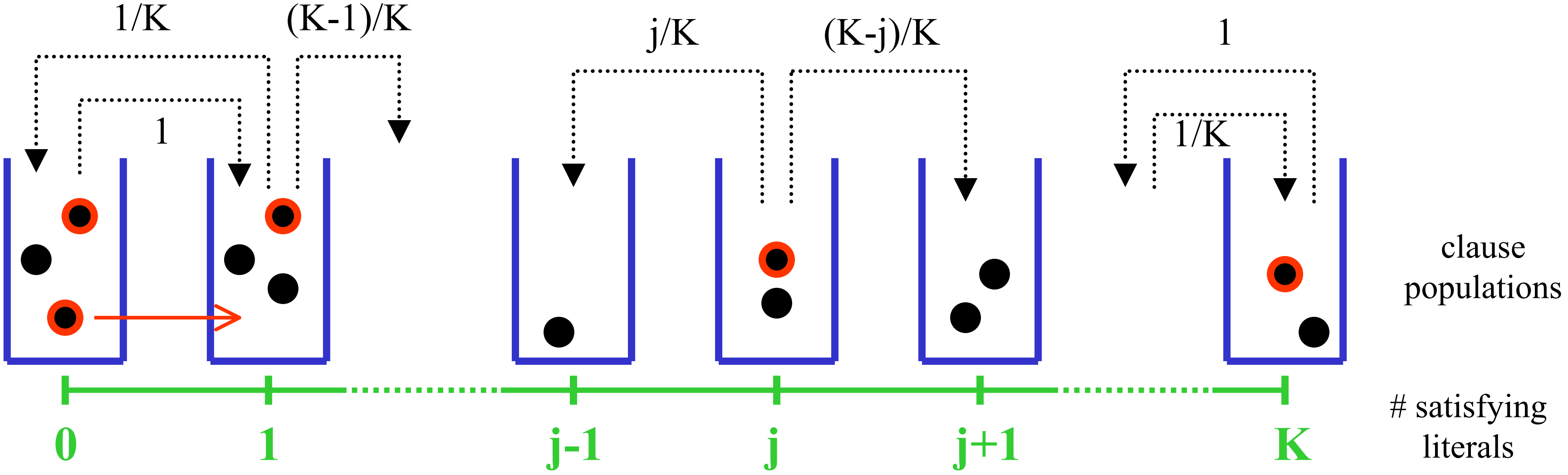
$$\phi_j(t) = M_j(T)/M$$

Question: evolution equations for the fractions when T Ø T+1 ?

Average behaviour (II)

Initial conditions for the fractions of j-clauses: $$\phi_j(0) = \frac{1}{2^K}\binom{K}{j}$$

After a new clause flip, i.e. from T Ø T+1:



$$M_j(T+1) - M_j(T) = \frac{K}{N}\Big((K-j+1)/K\;\; M_{j-1}(T) + (j+1)/K\;\; M_{j+1}(T)\Big) - d_j + d_{j-1}$$

Average behaviour (III)

DT = 1 leads to DM$_j$ = q(1). Thus, fractions F_j vary by q(1) on time scale DT = q(M). Define reduced time: t = T/M.

Evolution equation:

$$\frac{d\vec{\phi}}{dt} = \vec{v} + \alpha\, K\, W_r . \vec{\phi}$$

with

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ .. \\ 0 \end{pmatrix}, \qquad W_r = \begin{pmatrix} -1 & \frac{1}{K} & 0 & 0 & 0 & .. & 0 \\ 1 & -1 & \frac{2}{K} & 0 & 0 & .. & 0 \\ 0 & \frac{K-1}{K} & -1 & \frac{3}{K} & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & ... \\ 0 & 0 & 0 & 0 & .. & & -1 \end{pmatrix}$$

NB: normalization of fractions comes from nullity of matrix elements summed over columns.

Average behaviour (IV)

Existence of a critical value of
the ratio clauses/variable

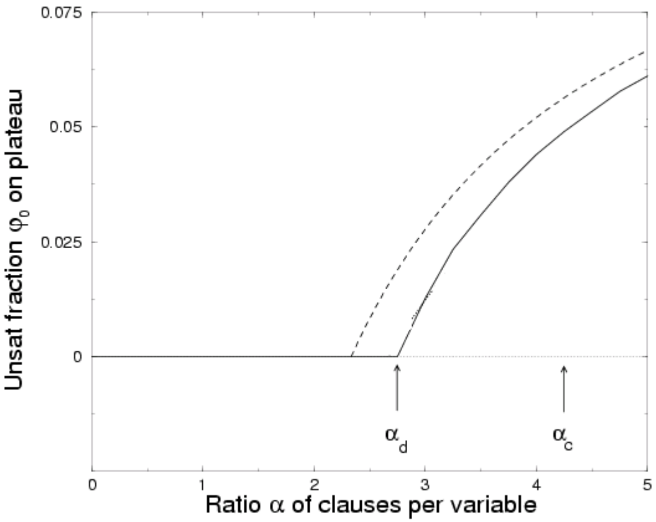
$$\boxed{\alpha_d(K) = \frac{2^K - 1}{K}}$$

- If $a < a_d$, then the procedure stops after a finite time with no unsat clause left.

$$t_{res}(a,K) = \frac{1}{a}\ \mathrm{Tanh}^{-1}\left[\ \left(1 - a/a_d(K)\right)^{-1/K} - 1\ \right]$$

- If $a > a_d$, then the procedure never stops, and a finite fraction of clauses remain unsatisfied.

$$\phi_0^* = \frac{1}{2^K} - \frac{1}{\alpha K}\left(1 - \frac{1}{2^K}\right) = \frac{1}{2^K}\left(1 - \frac{\alpha_d(K)}{\alpha}\right)$$



Large deviations and escape from plateau (I)

Evolution equation for the vector of clause types populations:

$$\mathrm{Prob}[\mathbf{M}', T+1] = \sum_{\mathbf{M}} [A_{\mathbf{M}'\mathbf{M}}] \, \mathrm{Prob}[\mathbf{M}, T] \quad ,$$

where

$$[A_{\mathbf{M}'\mathbf{M}}] = \sum_{\mathbf{Z}} \frac{N Z_0}{K M_0} \, \delta(\mathbf{M}' - \mathbf{M} - \Delta \cdot \mathbf{Z}) \; P(\mathbf{Z}|\mathbf{M})$$

Z_i = number of clauses of type i in which the flipped variable appears,
Z_i^s = number of clauses of type i in which the flipped variable was one of the i satisfying literal.

$$P(\mathbf{Z}|\mathbf{M}) = \prod_{i=0}^{K} \binom{M_i}{Z_i} \left(\frac{K}{N}\right)^{Z_i} \left(1 - \frac{K}{N}\right)^{M_i - Z_i} \prod_{i=1}^{K-1} \binom{Z_i}{Z_i^s} \left(\frac{i}{K}\right)^{Z_i^s} \left(1 - \frac{i}{K}\right)^{Z_i - Z_i^s}$$

Large deviations and escape from plateau (II)

Scaling hypothesis for clause type populations:

$$\mathrm{Prob}[\mathbf{M},T] \;^{o}\; \exp(-M\ p(\mathbf{j}, t)) \qquad \text{where} \qquad \mathbf{j}_j = \mathbf{M}_j/M, \quad t = T/M$$

Resulting partial differential equation for p :

$$p_t = -p_0 + p_1 + a \overset{K}{\underset{j=0}{S}} \mathbf{j}_j \left((K-j)\, e^{\,p_{j+1} - p_j} + j\, e^{\,p_{j-1} - p_j} - K \right)$$

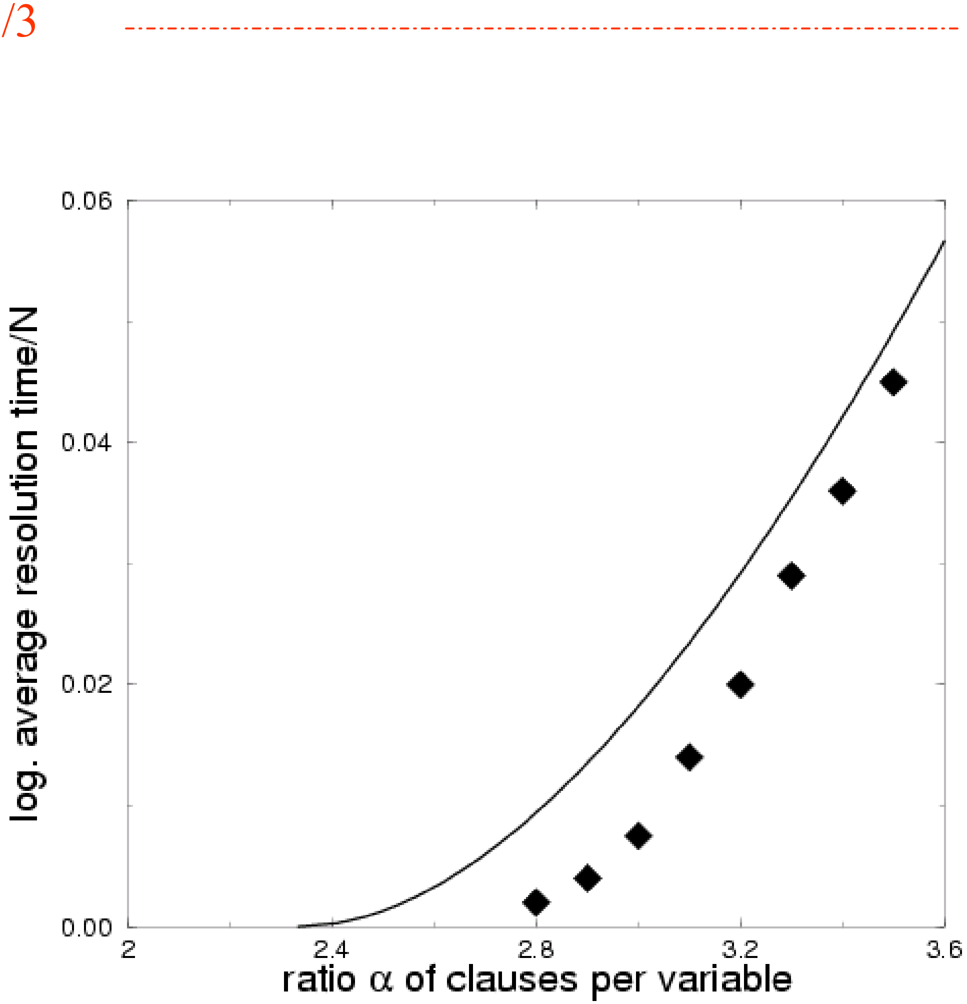
with initial condition:

$$p(\mathbf{j}, t=0) = \overset{K}{\underset{j=0}{S}} \mathbf{j}_j \ln\left(\mathbf{j}_j / \mathbf{j}_j^{0}\right) \quad ; \quad \mathbf{j}_j^{0} = \frac{1}{2^K}\binom{K}{j}$$

Large deviations and escape from plateau (III)

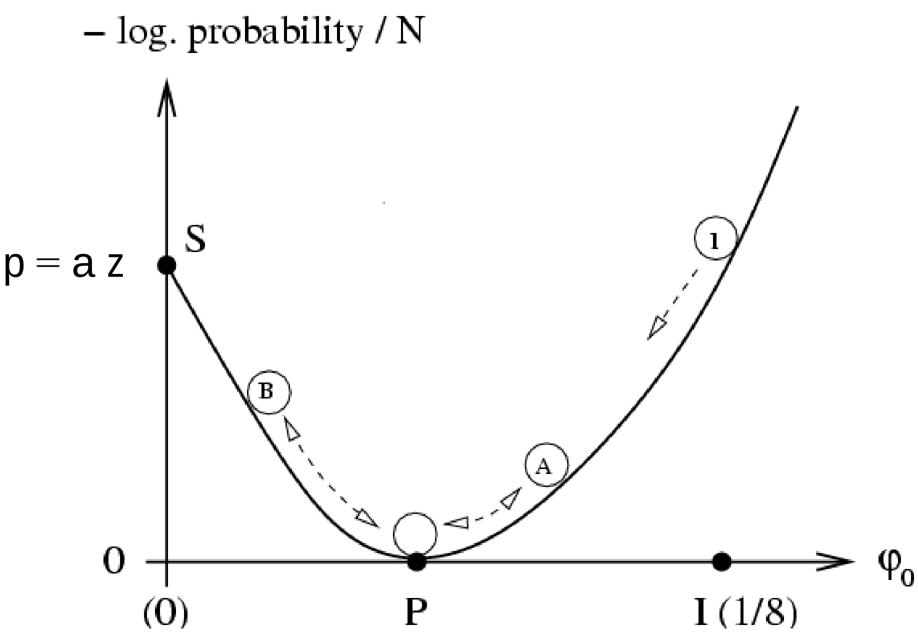
upper bound = ln 4/3

z = p(0)/a

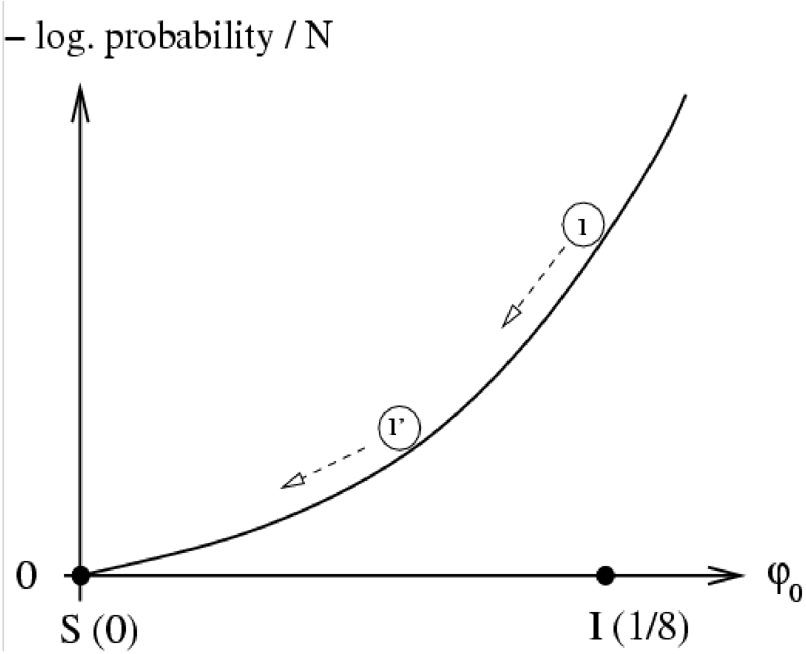


Intuitive summary: the rolling ball picture

On the plateau: $P_N(j_0)$ o exp(-M p(j_0))



Exponential regime



Polynomial regime

Conclusions (I)

1. « Approximate » analysis of Pure Random Walk for random SAT problems (also XOR-SAT)

- Annealed approximation may be justified for infinite a,K at fixed $a^* = aK/2^K$

$a_d^* = 1$ (and $a_c^* = ¶$)

$$\boxed{t^*_{res}(\alpha^*) \quad = \quad -\frac{1}{\alpha^*}\ln\left(1-\alpha^*\right) = 1 + \frac{1}{2}\,\alpha^* + \frac{1}{3}\,(\alpha^*)^2 + O\left((\alpha^*)^3\right)}$$

Approximate result

Exact expansion

- Finite K corrections ? Much tougher, under way …

- What can be said from a rigorous point of view?

Conclusions (II)

2. Exponentially rare events (large deviations) matter for analyzing algorithms

- notion of metastability
- case of DPLL and random restarts

3. Technical set-up:

Average behaviour	Large deviations
Ordinary Differential Equations	Partial Differential Equations

http://www.lpt.ens.fr/~monasson