Supplementary Material: Theory of spike timing based neural classifiers

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EXTREME VALUE THEORY

Here we cite (without proof) the necessary theorems required to derive the distributions of U_{max} and N_{spikes}.

We consider a stationary continuous Gaussian process $U(t)$ with zero mean and an auto-correlation function $C(t) = \langle U(t') U(t'+t) \rangle$ that satisfies

$$C(t) \approx 1 - \frac{1}{2} \left| \frac{d^2 C(0)}{dt^2} \right| t^2 \tag{1}$$

as $t \to 0$.

The average number of times $U(t)$ crosses a given threshold level, U_{th}, from below in a time interval of length T is given by [1]:

$$\nu(U_{\mathrm{th}}, T) = K \frac{\exp\left(-\frac{1}{2} U_{\mathrm{th}}^2\right)}{2\pi} \tag{2}$$

where we define the effective duration of the time interval, K, as

$$K = T \sqrt{\left| \frac{d^2 C(0)}{dt^2} \right|} \tag{3}$$

We now consider the $K \to \infty$ limit while keeping ν constant, that is, taking the threshold level to be

$$U_{\mathrm{th}}(K, \nu) = \sqrt{2 \ln \left(\frac{K}{2\pi\nu} \right)} \tag{4}$$

We are interested in the probability distribution of the number of crossing of the threshold level and the distribution of the maximal value of $U(t)$.

Given that the following condition holds

$$\lim_{K \to \infty} C(T) \ln K = 0 \tag{5}$$

it can be shown [2, 3] that the probability distribution of the number of crossings of the threshold level from bellow, N_s, is given by

$$\lim_{K \to \infty} P(N_s(K, \nu) = m) = \frac{e^{-\nu} \nu^m}{m!} \tag{6}$$

which is a Poisson distribution with mean ν.

To evaluate the distribution of the maximal value of $U(t)$ we note that

$$P\left(\max_{0<t<T} U(t) < U_{\rm th}\left(K,\nu\right)\right) = P\left(N_s\left(K,\nu\right)=0\right) = e^{-\nu} \tag{7}$$

We now consider $\nu = e^{-x}$ and obtain

$$\lim_{K\to\infty} P\left(\max_{0<t<T} U(t) < U_{\rm th}\left(K,e^{-x}\right)\right) = e^{-e^{-x}} \tag{8}$$

where the right-hand side is the cumulative probability function of the Gumbel distribution. From Eq. (4) we have

$$\left[U_{\rm th}\left(K,e^{-x}\right)\right]^2 = 2\left(\ln K + x - \ln 2\pi\right) \tag{9}$$

which in the large K limit entails

$$U_{\rm th}\left(K,e^{-x}\right) \approx \sqrt{2\ln K} + \frac{x-\ln 2\pi}{\sqrt{2\ln K}} \tag{10}$$

Finally, the distribution of the maximum of $U(t)$ is given by

$$\lim_{K\to\infty} P\left[\max_{0<t<T} \sqrt{2\ln K}\left(U(t) - \sqrt{2\ln K} + \frac{\ln 2\pi}{\sqrt{2\ln K}}\right) < x\right] = e^{-e^{-x}} \tag{11}$$

and therefore the random variable

$$X = \max_{0<t<T} \sqrt{2\ln K}\left(U(t) - \sqrt{2\ln K} + \frac{\ln 2\pi}{\sqrt{2\ln K}}\right) \tag{12}$$

is asymptotically distributed according to a Gumbel distribution. Equations (10) and (11) lead to the result presented in Eq. (4) in the main text.

For further details and proofs regarding the asymptotic properties of Gaussian processes we refer the reader to [4].

SIMULATION OF A HODGKIN-HUXLEY NEURON

In Fig. 4b in the main text we present measurements of the probability of two Hodgkin-Huxley neurons to agree on the classification of a random pattern as a function of the overlap between their weight vectors. Here we give the details of the model neurons used in the simulations. We use the model neuron described in [5]. This model consists of a Hodgkin-Huxley neuron with an additional slow A-current. The membrane potential follows

$$C_m\frac{dV}{dt} = -g_{\rm L}\left(V-E_{\rm L}\right) - \bar{g}_{\rm Na}m_\infty^3 h\left(V-E_{\rm Na}\right) - \bar{g}_{\rm K}n^4\left(V-E_{\rm K}\right) - \bar{g}_A a_\infty^3 b\left(V-E_{\rm K}\right) + I^{syn} \tag{13}$$

with $m_\infty = \frac{\alpha_m}{\alpha_m+\beta_m}$, $\alpha_m = \frac{-0.1(V+30)}{\exp(-0.1(V+30))-1}$, $\beta_m = 4\exp\left(-\frac{V+55}{18}\right)$ and $a_\infty = \frac{1}{\exp\left(-\frac{V+50}{20}\right)+1}$.

The additional dynamical variables are h, n and b which follow the equations:

$$\frac{dh}{dt} = \frac{h_\infty - h}{\tau_h} \tag{14}$$

$$\frac{dn}{dt} = \frac{n_\infty - n}{\tau_n} \tag{15}$$

$$\frac{db}{dt} = \tau_A^{-1}\left(\frac{1}{\exp\left(\frac{V+80}{6}\right)+1} - b\right) \tag{16}$$

with $h_\infty = \frac{\alpha_h}{\alpha_h+\beta_h}$, $\tau_h = \frac{\phi}{\alpha_h+\beta_h}$, $\alpha_h = 0.07\exp\left(\frac{V+44}{20}\right)$, $\beta_h = \frac{1}{1+\exp(-0.1(V+14))}$, $n_\infty = \frac{\alpha_n}{\alpha_n+\beta_n}$, $\tau_n = \frac{\phi}{\alpha_n+\beta_n}$, $\alpha_n = \frac{-0.01(V+34)}{\exp(-0.1(V+34))-1}$, $\beta_n = 0.125\exp\left(-\frac{V+44}{80}\right)$.~ The additional parameters are: $C_m = 1\ \mu F/\mathrm{cm}^2$, $\phi = 0.1$, $\bar{g}_{\mathrm{Na}} = 100\ \mathrm{mS/cm}^2$, $\bar{g}_{\mathrm{K}} = 40\ \mathrm{mS/cm}^2$, $g_{\mathrm{L}} = 0.05\ \mathrm{mS/cm}^2$, $\bar{g}_{\mathrm{A}} = 20\ \mathrm{mS/cm}^2$, $\tau_{\mathrm{A}} = 20$ ms, $E_{\mathrm{Na}} = 55$ mV, $E_{\mathrm{K}} = -80$ mV, $E_L = -65$ mV .

The synaptic input to the neuron, I^{syn}, is modeled with N conductance based synapses with random synaptic efficacies, ω_i, drawn from a Gaussian distribution with mean zero. The sign of the synaptic efficacy indicates whether the synapse is excitetory or inhibitory and the total synaptic input is given by

$$I^{app} = G^e\left(E_e - V\right) + G^{in}\left(E_{in} - V\right) \tag{17}$$

with $E_e = 0$ mV, $E_{in} = -80$ mV and

$$G^e = \sum_{i=1}^{N} |\omega_i|\,\Theta\left(\omega_i\right)\sum_{t_i<t} e^{-\frac{t-t_i}{\tau_{\mathrm{synapse}}}} \tag{18}$$

$$G^{in} = \sum_{i=1}^{N} |\omega_i|\,\Theta\left(-\omega_i\right)\sum_{t_i<t} e^{-\frac{t-t_i}{\tau_{\mathrm{synapse}}}} \tag{19}$$

where $\tau_{\mathrm{synapse}} = 5$ ms and $\Theta\left(x\right)$ is the Heaviside step function. Figure 1 presents an example of a voltage trace of the neuron driven with random synaptic inputs and the excitetory and inhibitory post synaptic potential induced by one synaptic input.

We simulated two neurons with correlated weights and measured P_{equal} as a function of the overlap between their weight vectors. We varied the total duration of the input patterns, T, and adjusted the standard deviation of the synaptic weights to ensure that a random

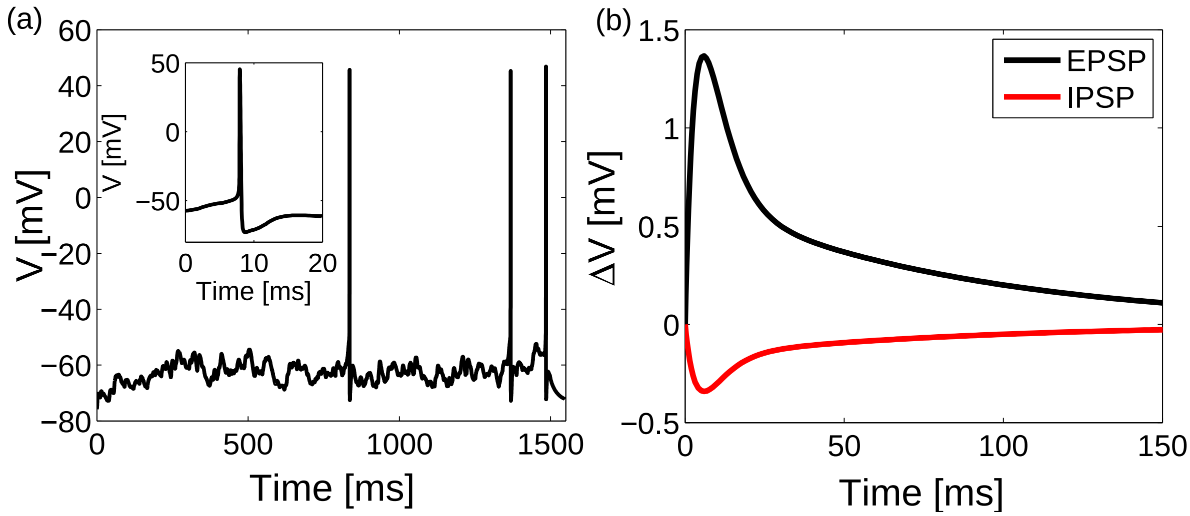

Figure 1. (a) An example of a voltage trace of the neuron driven with randomly chosen spikes of $N = 500$ neurons each firing with a Poisson rate of 1 Hz. The inset shows the shape of one spike generated by the neuron. (b) The excitatory post synaptic potential (EPSP) and the inhibitory post synaptic potential (IPSP) induced by a single input spike with $|\omega_i| = 1$ measured from the membrane resting potential. The half width of the PSP is $\tau_{1/2} \sim 20$ ms.

pattern is classified by each neuron ± 1 with equal probabilities. To produce the two curves shown in Fig. 4b of the main text we first fitted T to find the best agreement between the measurement of P_{equal} for the Tempotron with $K_1 = 100$ to the measurement of P_{equal} for the Hodgkin-Huxley neuron and then multiplied this T by 4 to produce the second curve. Notice that this curve fits very well the curve of P_{equal} for the Tempotron with $K_2 = 4K_1 = 400$.

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[5] O. Shriki, D. Hansel, and H. Sompolinsky, Neural computation **15**, 1809 (2003).