Some remarks on hierarchical replica symmetry breaking in finite–connectivity systems *

Rémi Monasson †

CNRS - Laboratoire de Physique Théorique de l'ENS,

24 rue Lhomond, 75231 Paris cedex 05, France

Abstract

A functional formalism capable of handling the first step of Parisi's hierarchical symmetry breaking for finite–connectivity systems is introduced. The order parameter is proposed to be a probability distribution over the set of all cavity fields probability distributions inside each states. As an application, the ground state properties of the 2-Satisfiability model are investigated and some evidence for the occurence of replica symmetry breaking above the percolation threshold is exposed.

I. INTRODUCTION

While infinitely-connected (IC) spin glasses have been extensively studied in the last twenty years [Mézard et al. 1987], much less progress has been made on disordered spins systems with a finite connectivity (FC). Such models are however of importance for at least two reasons. First, spin-glasses models with FC are expected to share common properties with finite-dimensional physical systems that IC models cannot display [Bray and Huifang 1989]. Secondly, it is now well–known that optimization or decision problems can

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†Email: Remi.Monasson@lpt.ens.fr ; preprint 97/12

be made in one-to-one correspondence with zero temperature properties of spin-glasses models with FC [Mézard et al. 1987]. One recent example we shall come back to in the following is the 2-Satisfiability problem [Gray and Johnson 1979,Monasson and Zecchina 1996] that can be mapped onto the ground-state of the Viana-Bray model [Viana and Bray 1985,Kanter and Sompolinsky 1987].

The latter model is defined as follows. Consider N spins $S_i = \pm 1$, $i = 1, \ldots, N$ interacting according to Hamiltonian

$$H = -\sum_{i<j} J_{ij} S_i S_j \qquad , \tag{1}$$

where the J_{ij}'s are infinitely-ranged random couplings drawn independently from each other and with probability distribution

$$\rho(J_{ij}) = (1 - \frac{\alpha}{N})\delta(J_{ij}) + \frac{\alpha}{2N}\delta(J_{ij}+1) + \frac{\alpha}{2N}\delta(J_{ij}-1) \qquad . \tag{2}$$

In the above equation, α accounts for the average connectivity of each spin and is of order one. The underlying bonds network is therefore highly diluted but, above a percolation threshold $\alpha_c = 1$, the system enters a spin-glass phase due to the presence of frustrated loops with strong $O(1)$ couplings [Kanter and Sompolinsky 1987]. Understanding the nature of this phase, as well as knowing the exact value of the ground state energy would be of high interest for the 2-Satisfiability decision problem [Monasson and Zecchina 1996].

With respect to its IC counterpart, that is the Sherrington-Kirkpatrick model [Mézard et al. 1987], the Viana-Bray model therefore exhibit strong "structural" fluctuations from site to site. As a result of the finite number of neighbors, the field incoming onto each spin does not obey a Gaussian law anymore and its distribution is non trivial. This complexity may be immediately seen when working out the replica theory of FC spin-glasses. A single overlaps matrix q^{ab} cannot take into account all effective interactions between replicas and there emerge infinitely numerous order parameters $q^{abc}, q^{abcd}, \ldots$.. This leads to additional technical difficulties which have been solved within the replica symmetric assumption (RS) [Bray and Huifang 1989,Kanter and Sompolinsky 1987,Orland 1985, Mézard and Parisi 1985]. As far as low temperature properties are concerned, replica symmetry breaking effects have to be taken into consideration [Viana and Bray 1985] but calculations become intolerably heavy. The purpose of this note is to present a formalism, inspired from previous works by Mottishaw and De Dominicis [De Dominicis and Mottishaw 1987] to make such calculations easier. In Section 2, a new order parameter is defined and we show how to find back RS theory in Section 3. Section 4 is devoted to one step RSB within Parisi's hierarchical Ansatz [Mézard et al. 1987]. Finally, we give in Section 5 some new results for the 2-Satisfiability model above the unsatisfiability threshold.

II. THE MULTI–LEVEL "GAS" PICTURE

The models we consider here are mean-field models in that, once they have been replicated, their effective Hamiltonian depend on global fields rather than local ones. Usually, these global fields are chosen to be the mean overlaps between replicas

$$q^{a_1,a_2,\ldots,a_r} = \frac{1}{N}\sum_{i=1}^{N} S_i^{a_1} S_i^{a_2} \ldots S_i^{a_r} \qquad , \tag{3}$$

whose total number equals $2^n - 1$ (some of them maybe eventually absent in the free–energy functional, e.g. in IC models). Such a choice is not necessarily convenient since it may requires (as in the case of neural networks or models with p-spins interactions) the introduction of some conjugated Lagrange multipliers, the meaning of which is not clear. In this Section, we shall adopt another representation that will make disordered spins models rather look like "gas" of 2^n interacting levels.

As an illustration, consider the Viana-Bray model (1). The n^{th} moment of the partition function at temperature $1/\beta$ reads, after having performed the average $\overline{(.)}$ over the disorder (2),

$$\overline{Z(\{J_{ij}\})^n} = \overline{\left(\sum_{\{S_i=\pm 1\}} e^{\beta \sum_{i<j} J_{ij} S_i S_j} \right)^n} = \sum_{\{S_i^a=\pm 1\}} e^{\sum_{i<j} \mathcal{H}_{VB}[\vec{S}_i;\vec{S}_j]} \qquad , \tag{4}$$

where

$$\mathcal{H}_{VB}[\vec{\sigma},\vec{\tau}] \equiv \mathcal{H}_{VB}[\{\sigma^1,\sigma^2,\ldots,\sigma^n\};\{\tau^1,\tau^2,\ldots,\tau^n\}] = -\alpha + \alpha \cosh\left(\beta\, \vec{\sigma}.\vec{\tau}\right) \qquad . \tag{5}$$

Whereas the effective Hamiltonian $\mathcal{H}$ depends upon the particular model under study, e.g. $\mathcal{H}_{SK}[\vec{\sigma},\vec{\tau}] = \beta^2(\vec{\sigma}.\vec{\tau})^2/2$ for the Sherrington-Kirkpatrick model, expression (4) is quite generic for FC or IC mean–field systems, provided $\mathcal{H}$ is allowed to include multi-spins interactions as in p-spins or K-Satisfiability models. The mean-field nature of (4) is indeed transparent; call $c(\vec{\sigma})$ the fraction of sites i such that every attached spins S_i^a equal σ^a, for all $a = 1,\ldots,n$. There are 2^n vectors $\vec{\sigma}$ with ± 1 components and the normalization constraint reads

$$\sum_{\vec{\sigma}} c(\vec{\sigma}) = 1 \qquad . \tag{6}$$

The energy contribution to (4) may be easily written in terms of the new order parameter $c(\vec{\sigma})$ and, taking into account the entropy stemming from different choices of the sites, we end up with

$$\overline{Z(\{J_{ij}\})^n} = \int_0^1 \prod_{\vec{\sigma}} dc(\vec{\sigma}) \;\exp\; N\left(-\sum_{\vec{\sigma}} c(\vec{\sigma}) \ln c(\vec{\sigma}) + \frac{1}{2}\sum_{\vec{\sigma},\vec{\tau}} c(\vec{\sigma})c(\vec{\tau})\mathcal{H}[\vec{\sigma},\vec{\tau}]\right) \qquad , \tag{7}$$

where the integration measure is restricted to $c(\vec{\sigma})$'s fulfilling constraint (6). In the large N limit, the integrand of the above expression has to be extremized, leading to the saddle–point equation

$$c(\vec{\sigma}) = \lambda(n)\;\exp\left(\sum_{\vec{\tau}} c(\vec{\tau})\mathcal{H}[\vec{\sigma},\vec{\tau}]\right) \qquad ,\forall\ \vec{\sigma} \qquad , \tag{8}$$

where the normalization constant $\lambda(n)$ may be omitted in the small n limit : $\lambda(0) = 1$. The above formulation may be interpreted as follows. Consider a "gas" of 2^n levels $\vec{\sigma}$ to be filled in with N particles. Call $-\frac{1}{N}\mathcal{H}[\vec{\sigma},\vec{\tau}]$ the interaction energy between two particles located on levels $\vec{\sigma}$ and $\vec{\tau}$ respectively. Define then $c(\vec{\sigma})$ the fraction of the N particles living on level $\vec{\sigma}$. The question arising is : what is the stationary distribution of level populations c's in the thermodynamical limit $N \to \infty$? The answer is given by the solution of equation (8). For completeness, note that another generating function of all overlaps was already introduced by Mottishaw and De Dominicis for technical reasons. Their order parameter however includes some disorder dependence (see equation (3) in [De Dominicis and Mottishaw 1987]) and its interpretation is therefore less clear than $c(\vec{\sigma})$'s one.

III. REPLICA SYMMETRIC ANSATZ

At high temperatures - small connectivities, the system is expected to be in a RS phase [Viana and Bray 1985,Kanter and Sompolinsky 1987,Mottishaw and De Dominicis 1987]. From a mathematical point of view, we have : $c(\sigma^1, \sigma^2, \dots, \sigma^n) = c(\sigma^{\pi(1)}, \sigma^{\pi(2)}, \dots, \sigma^{\pi(n)})$ for any permutation π acting on n elements. Therefore, $c(\vec{\sigma})$ depends on $\vec{\sigma}$ through its magnetization $s = \sum_a \sigma^a$ only. It results convenient to introduce a generating function of all $n+1$ remaining order parameters [Orland 1985, Mézard and Parisi 1985]. Drawing our intuition from the cavity approach [Mézard et al. 1987], we choose $P_{rs}(h)$ that is the distribution of cavity fields acting on single spins. We then obtain, see Appendix A in [Monasson and Zecchina 1996],

$$c(\vec{\sigma}) \equiv c(s) = \int_{-\infty}^{\infty} dh P_{rs}(h) \prod_{a=1}^{n} \left(\frac{e^{-\beta h \sigma^a}}{e^{-\beta h} + e^{\beta h}} \right) \underset{n\to 0}{=} \int_{-\infty}^{\infty} dh P_{rs}(h) e^{-\beta h s} \quad . \tag{9}$$

The saddle–point equations (8) can then be transformed into self-consistent equations for the field distributions $P(h)$, which read

$$P_{rs}(h) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \cos(sh) \exp\left\{ -\alpha + \alpha \int_{-\infty}^{\infty} dx P_{rs}(x) \cos\left[\frac{s}{2\beta} \ln\left(\frac{\cosh \beta(x+1)}{\cosh \beta(x-1)} \right) \right] \right\} \tag{10}$$

$$= \int_{-\infty}^{\infty} \frac{ds}{2\pi} \cos(sh) \exp\left\{ -\frac{s^2}{2} \int_{-\infty}^{\infty} dx P_{rs}(x) \tanh^2(\beta x) \right\} \quad , \tag{11}$$

for the Viana-Bray and the Sherrington-Kirkpatrick models respectively. In the latter case (11), P_{rs} is found to be a Gaussian distribution; its variance equals the RS overlap q and satisfies the usual self-consistency equation for q. Expression (10) is identical to the RS saddle-point equation derived when considering all overlaps (3) in [Kanter and Sompolinsky 1987]. As a consequence, the present procedure allows to derive RS theory in a simple way. We shall now apply it to the more involved case of one-step replica symmetry breaking.

IV. REPLICA SYMMETRY BREAKING ANSATZ

We now consider the first step of Parisi's hierarchical RSB scheme, regardless of the possible existence of any new RSB pattern in FC spin-glasses. Each replica a is labelled by a couple of integers $(b, \tilde{b})$ according to the number b of the block it belongs to $(1 \leq b \leq n/m)$ and its position $\tilde{b}$ inside this block $(1 \leq \tilde{b} \leq m)$. In a first stage, we shall focus on the symmetry and combinatorial properties of $c(\vec{\sigma})$. A functional representation of c allowing to have n and m real-valued will then be introduced.

A. Symmetries and multiplicities of the order parameter

Following De Dominicis and Mottishaw [De Dominicis and Mottishaw 1987], we see that $c(\vec{\sigma})$ is independent of any permutation inside the blocks. Thus, it depends only on the n/m block magnetizations $s_b = \sum_c \sigma^{(b,\tilde{b})}$. Among the $(m+1)^{n/m}$ different sets $(s_1, s_2, \ldots, s_{n/m})$, sequences that differ only by a permutation of their indices b correspond to the same value of $c(\vec{\sigma})$. In other words, if we call ν_j the number of blocks b with magnetizations $s_b = 2j - m$ (for $0 \leq j \leq m$), $c(\vec{\sigma})$ depends only on the set of ν_j's. The one-step RSB order parameters c's are in one-to-one correspondence with the partitions of n/m into a sum of at most $m+1$ integers. Their number therefore equals

$$\mathcal{N}_{RSB} = \binom{m + \frac{n}{m}}{m} \qquad . \tag{12}$$

With respect to the RS case recovered when $m = n$ or $m = 1$ and corresponding to

$$\mathcal{N}_{RS} = n + 1 \qquad , \tag{13}$$

the number of different order parameters grows faster with n, indicating that a larger subspace of the 2^n-dimensional $c(\vec{\sigma})$-space is spanned by the one step Ansatz. Consequently, a more complicated generating function, depending not only on a single variable as $P_{rs}(h)$ - see Section 2 - has to be introduced to account for all these order parameters.

B. Functional order parameter

Inside each state, RS hypothesis is valid and the cavity fields distribution is entirely defined through a single $P_s(h)$, see Section 2, independent of the state under consideration [Mézard et al. 1987]. Once P_s is known, we can easily reconstruct the order parameter $c(\vec{\sigma};P_s) = c(\nu_0,\nu_1,\ldots,\nu_m;P_s)$ using (9) :

$$c(\vec{\sigma};P_s) = \prod_{b=1}^{n/m}\int_{-\infty}^{\infty} dh P_s(h) \prod_{\tilde{b}=1}^{m}\left(\frac{e^{-\beta h \sigma^{(b,\tilde{b})}}}{e^{-\beta h}+e^{\beta h}}\right) = \prod_{j=0}^{m}\left(\int_{-\infty}^{\infty} dh P_s(h) \frac{e^{-\beta h(2j-m)}}{(2\cosh \beta h)^m}\right)^{\nu_j} \quad . \tag{14}$$

Still, one has to determine P_s. While this task is implicitly performed in IC spin-glasses calculations (see next paragraph), no *a priori* information is available for FC systems. Indeed, the extremization procedure over c in (7) has a clear meaning : we have to find the optimal distribution of P_s, that is maximizing (when $n \to 0$) free–energy. Call $\mathcal{V}$ the space of all probability distributions $P_s(h)$ of the cavity fields h. Then, consider the functionals $\mathcal{P}_{rsb}[P_s]$, which are normalized distributions over $\mathcal{V}$: $\int_{\mathcal{V}} \mathcal{D}P_s \; \mathcal{P}_{rsb}[P_s] = 1$. In the most general case, we have to find $\mathcal{P}_{rsb}$ such that $c(\vec{\sigma}) = \int_{\mathcal{V}} \mathcal{D}P_s \; \mathcal{P}_{rsb}[P_s] c(\vec{\sigma};P_s)$ (see (14)) satisfies the saddle-point equations (8).

Replica symmetric theory is recovered when states reduce to single configurations. This happens if, inside a state, h cannot fluctuate around its mean value, that is if

$$\mathcal{P}_{rs}[P_s] = \int_{-\infty}^{\infty} dx P_{rs}(x) \; \delta[P_s(h) - \delta(h-x)] \qquad , \tag{15}$$

where P_{rs} has been defined in Section 3 and the outer δ is a functional Dirac function (*i.e.* the product over all values of h is omitted for sake of simplicity).

A natural generalization of the above identity to RSB case is obtained by letting fields fluctuate inside the states [Wong and Sherrington 1988]. In the present formalism, this amounts to consider a functional order parameter of the form

$$\mathcal{P}_{rsb}[P_s] = \int_{-\infty}^{\infty} dx T_0(x) \; \delta[P_s(h) - T_1(h,x)] \qquad , \tag{16}$$

where the optimization has to be done over all T_0, T_1 functions. Let us emphasize that, though being already quite complex, expression (16) is not the most general one. Expression (16) is indeed exact if fonctions P_s with a non-zero weight can be labelled by (*i.e.* are not more numerous than) real numbers.

V. APPLICATION : THE 2-SATISFIABILITY MODEL

We now focus on the ground state properties of the 2-Satisfiability model. Even RS theory is far from being trivial [Kanter and Sompolinsky 1987,Monasson and Zecchina 1996]. Following the procedure exposed in Section 3, we find that the simplest RS order parameter reads (when $\beta \to \infty$)

$$P_{rs}(h) = \sum_{\ell=\infty}^{\infty} e^{-\gamma} I_\ell(\gamma) \delta\left(h - \frac{\ell}{2}\right) \qquad , \tag{17}$$

with

$$\gamma = \alpha \, \left(1 - e^{-\gamma} I_0(\gamma)\right) \qquad , \tag{18}$$

where I_ℓ denotes the ℓ^{th} modified Bessel function. That P_{rs} reduce to a distribution over half-integer fields is not surprising due to the discrete nature of both spins and couplings [Kanter and Sompolinsky 1987] (note that the fields take half-integer values since the number of violated clauses has to be an integer [Monasson and Zecchina 1996]). However, since P_{rs} is an average order parameter, there are more complicated RS solutions including all rational fields [Katsura 1986, Wong et al. 1998, de Oliveira 1988,Monasson and Zecchina 1996] with a lower free-energy. As far as the transition threshold is concerned, solution (18) already correctly predicts that unsatisfied clauses ($\gamma > 0$) emerge above $\alpha_c = 1$. One can however expect, both from numerical results [Monasson and Zecchina 1996] and related analytical studies [Mottishaw and De Dominicis 1987] that RS solutions are unstable above α_c.

The one step RSB order parameter $\mathcal{P}_{rsb}[P_s]$ satisfies a complicated self-consistent equation depending on α and $\mu = \lim_{\beta\to\infty} \beta m$ at zero temperature. For sake of simplicity, we shall not write below all the details of the calculation. We focus upon the transition region, taking α close to one. We assume that the transition is of second order with respect to the order parameter (as suggested by the RS theory and stability analysis performed for the Viana-Bray model). Consequently, μ vanishes at the transition and we can expand the free-energy functional in powers of μ. To zeroth order, we find back the RS free-energy $\mathcal{F}_0$ provided that

$$\int_{\mathcal{V}} \mathcal{D}P_s \; \mathcal{P}_{rsb}[P_s] \; P_s(h) = P_{rs}(h) \qquad , \forall \; h \qquad , \tag{19}$$

with P_{rs} given by (17). The μ expansion indeed coincides with an expansion of free-energy in higher and higher moments of $\mathcal{P}_{rsb}$. The first non trivial contribution is given by the linear term $\mathcal{F}_1$ in μ which involves the two-functions correlation

$$\int_{\mathcal{V}} \mathcal{D}P_s \; \mathcal{P}_{rsb}[P_s] \; P_s(h) \; P_s(h') = C(h,h') \qquad . \tag{20}$$

Note that, within RS hypothesis (15), fields cannot fluctuate and $C(h,h') = 0$ if $h \neq h'$. RSB effects therefore manifest themselves through off-diagonal contributions to $C(h,h')$.

A natural extension of (17) leads to the following Ansatz for C,

$$C(h,h') = \sum_{\ell,a,a'=-\infty}^{\infty} e^{-(\gamma+\sigma)} I_\ell(\gamma-\sigma) I_a(\sigma) I_{a'}(\sigma) \delta\left(h - \frac{\ell+a}{2}\right) \delta\left(h' - \frac{\ell+a'}{2}\right) \qquad . \tag{21}$$

The above choice is indeed a solution of the saddle-point equation $\partial \mathcal{F}_1 / \partial C = 0$ if σ is self-consistently determined by

$$\gamma - \sigma = 2\alpha \sum_{j=-\infty}^{\infty} e^{-(\gamma-\sigma)} I_j(\gamma-\sigma) \left[\sum_{k=-\infty}^{j-1} e^{-\sigma} I_k(\sigma) \right]^2 \qquad , \tag{22}$$

where γ is given by (18). In addition to the RS solution $\sigma = 0$, $\mathcal{F}_1 = 0$, there exists another solution for $\alpha > 1$ which read to lowest orders in $\alpha - 1$,

$$\begin{aligned} \sigma &= \frac{4}{9}(\alpha - 1) \\ \mathcal{F}_1 &= \frac{10}{729}(\alpha-1)^3 \qquad . \end{aligned} \tag{23}$$

As expected, the free-energy is increased when replica symmetry is broken, indicating that the $\sigma > 0$ solution has to be preferred. The above RSB solution is incomplete since the value of μ is still to be computed. To do so, one would have to expand free-energy to the second order in μ. This is however not compulsory to assert that RS solution is valid below and up to the threshold $\alpha_c = 1$.

In future [Monasson 1997], we plan to apply the present approach to the much more discussed problem of 3-Satisfiability [Gray and Johnson 1979,Monasson and Zecchina 1996]. While RS theory predicts a critical threshold $\alpha_c = 4.6$, there is numerical evidence that the correct value is close to 4.2. The current question is now to see whether the one-step Ansatz will allow to find back this precise value or not. In addition to its intrinsic interest, 3-Satisfiability can therefore be viewed as a valuable model to test our theoretical understanding of FC spin-glasses.

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