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Replica structure of one-dimensional disordered Ising models

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Abstract. – We analyse the eigenvalue structure of the replicated transfer matrix of one-dimensional disordered Ising models. In the limit of $n \to 0$ replicas, an infinite sequence of transfer matrices is found, each corresponding to a different irreducible representation (labelled by a positive integer ρ) of the permutation group. We show that the free energy can be calculated from the replica-symmetric subspace ($\rho = 0$). The other "replica symmetry broken" representations ($\rho \neq 0$) are physically meaningful, since their largest eigenvalues $\lambda^{(\rho)}$ control the disorder-averaged moments $\langle\langle (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^\rho \rangle\rangle \propto (\lambda^{(\rho)})^{|i-j|}$ of the connected two-point correlations.

In spite of extensive studies during the last twenty years, randomly disordered systems are not yet fully understood [1]. One of the main obstacles to the theoretical analysis of such systems is of course the lack of translational invariance of the Hamiltonian. The replica approach, which consists in replacing the original system and its randomly disordered Hamiltonian by the study of n ($\to 0$) identical and coupled systems with translationally invariant interactions, has been proven to be successful in providing a highly interesting mean-field theory of spin glasses [2], [3]. The latter relies on a fascinating, but mathematically obscure aspect of the replica mean-field theory, *i.e.* that the spin glass transition coincides with the so-called replica symmetry breaking (RSB), or, in other words, with the breaking of the permutation group symmetry of $n \to 0$ elements. In this context, the crucial question is to find the correct scheme of breaking, allowing an analytical continuation of the theory when the number n of replicas tends to zero. To what extent the physical picture originated in the resolution of long-range models [4] applies to finite-dimensional systems and the occurrence itself of replica symmetry breaking are still under question [5].

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In this note, we present some preliminary remarks about this important issue by analysing one-dimensional spin models with quenched random couplings and/or fields. As in the case of one-dimensional ferromagnetism, no transition is expected at finite temperature [6]. However, since one-dimensional disordered systems may be solvable to a large extent with rigorous techniques, they constitute interesting examples on which the replica symmetry-breaking approach can be tested before being applied to more realistic systems.

To start with, let us consider the Hamiltonian

$$H = -\sum_{i=1}^{N} (J_i S_i S_{i+1} + h_i S_i) \tag{1}$$

with Ising spins $S_i = \pm 1$ and periodic boundary conditions. The coupling constants and the external fields are randomly drawn with the probability distribution $\prod_{i=1}^{N} \mathcal{P}(J_i) p(h_i)$. In general the 2×2 transfer matrices

$$T_1(J_i, h_i) = \begin{pmatrix} e^{+\beta J_i + \beta h_i} & e^{-\beta J_i + \beta h_i} \\ e^{-\beta J_i - \beta h_i} & e^{+\beta J_i - \beta h_i} \end{pmatrix} \tag{2}$$

at temperature $1/\beta$ are not simultaneously diagonalizable. The calculation of the thermodynamic properties, which requires the knowledge of the asymptotic properties of the product $\prod_i T_1(J_i, h_i)$, is rather involved but has been achieved for some particular choices of disorder distribution, using Dyson's method [6]-[9].

Hereafter, we shall follow a different route by noticing that, since the disorder is independently distributed from site to site, the free energy may be computed through the knowledge of the replicated and disorder-averaged transfer matrix $T_n = \langle\langle T_1(J,h)^{\otimes n} \rangle\rangle$, where $\langle\langle (.) \rangle\rangle$ denotes $\int \mathrm{d}J \mathrm{d}h \cdot \mathcal{P}(J) p(h) (.)$. The entries of this $2^n \times 2^n$ matrix are determined by the replicated spins $\{S^a = \pm 1, a = 1, \ldots, n\}$. We introduce the 2^n vectors $|a_1, a_2, \ldots, a_\rho\rangle = \bigotimes_a |S^a\rangle$, $a_i \neq a_j \; \forall i \neq j$, with up-spins $S^a = +1$ at and only at the sites $a \in \{a_1, \ldots, a_\rho\}$. They constitute an orthonormal basis of the underlying replicated space V_n. The transfer matrix elements are now given by

$$\langle a_1, \ldots, a_\rho | T_n | b_1, \ldots, b_\sigma \rangle = \left\langle\!\!\left\langle \exp \left[\beta J \sum_{a=1}^{n} R^a S^a + \beta h \sum_{a=1}^{n} R^a \right] \right\rangle\!\!\right\rangle \tag{3}$$

with $\{R^a\}$, $\{S^b\}$ corresponding to $|a_1, \ldots, a_\rho\rangle$, $|b_1, \ldots, b_\sigma\rangle$. They are obviously invariant under replica renumbering, *i.e.* under all transformations of the permutation group $\mathcal{S}_n$ given by its 2^n-dimensional representation $D(\pi)|a_1, \ldots, a_\rho\rangle = |\pi(a_1), \ldots, \pi(a_\rho)\rangle$, $\forall \pi \in \mathcal{S}_n$. Every irreducible decomposition of this representation D is isomorphous to the most general eigenspace decomposition of V_n. D already appears in the theory of atomic spectra, for a detailed presentation see [10]. The number of up-spins in the basis vectors of V_n remains clearly invariant under replica permutations. So we find $n+1$ subrepresentations Δ_ρ which are carried by the spaces spanned by $\{|a_1, \ldots, a_\rho\rangle, 1 \leq a_1 < \ldots < a_\rho \leq n\}$, $\rho = 0, \ldots, n$. But for $\rho \neq 0, n$ these Δ_ρ are still reducible. Consider, *e.g.*, the vector $\sum_{a \neq a_i} |a, a_1, \ldots, a_{\rho-1}\rangle$; under permutations it behaves like $|a_1, \ldots, a_{\rho-1}\rangle$, so we find $\Delta_\rho \cong \Delta_{\rho-1} \oplus D_\rho$, $\forall \rho \leq n/2$. Using in addition the symmetry $\rho \mapsto n - \rho$, $|\pm\rangle \mapsto |\mp\rangle$, we obtain the complete decomposition of D: $\Delta_0 \cong D_0$, $\Delta_1 \cong D_0 \oplus D_1$, …, $\Delta_\rho \cong D_0 \oplus D_1 \oplus \ldots \oplus D_{\min(\rho, n-\rho)}$, …, $\Delta_n \cong D_0$, whose irreducibility has been proven in [10]. By changing the basis of V_n with respect to this decomposition and taking at first the vectors of all D_0-spaces, then those of all D_1-spaces and so on, we can block-diagonalize the transfer matrix T_n.

Replica symmetry (RS) corresponds to the restriction to the first block, since all D_0-vectors are invariant under permutations. Due to the representation structure, the RS transfer matrix has $n+1$ non-degenerate eigenvalues and reads, cf. [11],

$$T_n^{(0)}(\sigma,\tau) = \sum_{\mu=\mu_-}^{\mu_+} \binom{\sigma}{\mu}\binom{n-\sigma}{\tau-\mu} \langle\langle \exp\big[\beta J(n+4\mu-2\tau-2\sigma)+\beta h(2\sigma-n)\big]\rangle\rangle\,, \tag{4}$$

where $\mu_- = \max(0,\sigma+\tau-n)$ and $\mu_+ = \min(\sigma,\tau)$ and the indices σ,τ run from 0 to n. The RS site-dependent partition function is given by the iterative description $Z_{i+1}(\tau) = \sum_{\sigma=0}^{n} T_n^{(0)}(\sigma,\tau) Z_i(\sigma)$. Introducing the generating function $Z_i[x] = \sum_\sigma Z_i(\sigma)x^\sigma$, the latter reads

$$Z_{i+1}[x] = \int_0^\infty \mathrm{d}y \langle\langle e^{-\beta hn}(e^{\beta J}+xe^{-\beta J})^n \delta\left(y-f(x)\right)\rangle\rangle Z_i[y]\,, \tag{5}$$

where

$$f(x) = e^{2\beta h}\frac{e^{-\beta J}+xe^{+\beta J}}{e^{+\beta J}+xe^{-\beta J}}\,. \tag{6}$$

In the thermodynamic limit, we call $\Phi(x)$ the right eigenfunction of $T_{n\to 0}^{(0)}$ which has the (maximal) eigenvalue unity and an integral normalized to one,

$$\Phi(x) = \int_0^\infty \mathrm{d}y\, \langle\langle \delta(x-f(y))\rangle\rangle \Phi(y)\,. \tag{7}$$

The free-energy density is given by the O(n)-corrections in (5),

$$f = \langle\langle h\rangle\rangle - \frac{1}{\beta}\int_0^\infty \mathrm{d}x\, \langle\langle \log(e^{\beta J}+xe^{-\beta J})\rangle\rangle \Phi(x)\,. \tag{8}$$

Equations (7) and (8) for the random field Ising model are precisely the results derived in [12] by analysing the Lyapunov exponent of the infinite product of disordered transfer matrices $T_1(J_i,h_i)$ given in (2). RS gives therefore the correct result for the free energy, and (7) is usually interpreted as the Dyson-Schmidt equation for the invariant density Φ of the local fields [6]. A similar result was already established by Lin [13] who showed the equivalence of an early replica method developed by Kac with Dyson's approach for the phonon spectrum of a chain of random masses and springs.

To be sure that replica symmetry is not violated, we have to check that the eigenvalue unity is not degenerate, *i.e.* reached in another eigenspace of $T_{n\to 0}$. In the following, we shall therefore analyse the transfer matrix blocks corresponding to the non-trivial representations D_ρ, $\rho \ge 1$. Each has $n+1-2\rho$ different eigenvalues of degeneracy $\binom{n}{\rho}-\binom{n}{\rho-1}$. By ordering the basis vectors according to their permutation properties, one can achieve a further block-diagonalization of these blocks into $\binom{n}{\rho}-\binom{n}{\rho-1}$ identical blocks of size $(n+1-2\rho)\times(n+1-2\rho)$, each one containing every eigenvalue of the D_ρ-block exactly once. The transfer matrix blocks read

$$T_n^{(\rho)}(\sigma,\tau) = \langle\langle (2\sinh 2\beta J)^\rho\, T_{n-2\rho}^{(0)}(\sigma-\rho,\tau-\rho;J,h)\rangle\rangle\,, \tag{9}$$

$\rho \le \sigma,\tau \le n-\rho$, where $T_n^{(0)}(\sigma,\tau;J,h)$ is given by (4) without averaging over the quenched disorder. In the limit $n\to 0$ we obtain for every positive number ρ a different eigenvalue equation

$$\lambda^{(\rho)}\,\Phi^{(\rho)}(x) = \int_0^\infty \mathrm{d}y \langle\langle \delta(x-f(y))\,(f'(y))^\rho\rangle\rangle\,\Phi^{(\rho)}(y)\,, \tag{10}$$

where we again used the function $f(x)$ defined in (6). For every eigenfunction $\Phi^{(1)}(x)$ of $T^{(1)}_{n\to 0}$ the function $\frac{\mathrm{d}}{\mathrm{d}x}\Phi^{(1)}(x)$ is an eigenfunction of the RS transfer matrix with the same eigenvalue. Only the largest eigenvalue (equal to one) of $T^{(0)}_{n\to 0}$, corresponding to the density (7), cannot be reached by this procedure. Therefore the largest eigenvalue of $T^{(1)}_{n\to 0}$ equals the second of $T^{(0)}_{n\to 0}$ and so on.

To unveil the physical meaning of the maximal eigenvalues of all irreducible representations $\rho \geq 1$, one has to consider the disorder-averaged moments of the spin correlation functions $\chi^{(\rho)}_{ij} = \langle\langle(\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle)^\rho\rangle\rangle$. More precisely, we shall see that they are dominated by

$$\chi^{(\rho)}_{ij} \propto \left(\lambda^{(\rho)}\right)^{|i-j|} , \tag{11}$$

up to a multiplicative factor which depends on ρ. To show this, we consider the operator $\Sigma^{(\rho)} = (\Sigma \otimes \mathbf{1} - \mathbf{1} \otimes \Sigma)^{\otimes\rho}$, where Σ is the usual spin operator : $\Sigma|S\rangle = S|S\rangle$. The operator $\Sigma^{(\rho)}$ is defined on a set of at least 2ρ identical real replicas of the original disordered system. One can realize that it maps the spaces carrying D_0 onto those carrying D_ρ. Since the ground state was found to be replica symmetric for any kind of disorder and is *a priori* non-orthogonal to $\Sigma^{(\rho)}(D_\rho)$, the correlations of two $\Sigma^{(\rho)}$ situated at sites i and j of the replicated Ising chain decay exponentially as $(\lambda^{(\rho)})^{|j-i|}$. These correlations are in addition equal to the left-hand side of (11) as follows from the identity

$$\chi^{(\rho)}_{ij} = \frac{1}{2^\rho}\lim_{n\to 0} \mathrm{Tr}\left[\Sigma^{(\rho)}_i (T_n)^{j-i}\Sigma^{(\rho)}_j (T_n)^{N-j}\right] . \tag{12}$$

Therefore, the criterion $\lambda^{(\rho)} \to 1$ for replica symmetry breaking is equivalent to the divergence of the correlation length $-(\log \lambda^{(\rho)})^{-1}$ of the ρ-th moment of the connected two-point correlation function. The case $\rho = 2$ would correspond to the divergence of the spin glass susceptibility $\chi^{(2)} = \frac{1}{N}\sum_{i,j}(\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle)^2$, whereas the linear susceptibility $\chi^{(1)} = \frac{1}{N}\sum_{i,j}(\langle S_i S_j\rangle - -\langle S_i\rangle\langle S_j\rangle)$ remains finite. This behaviour can be found in spin glass experiments as well as in mean-field theory [1].

As a result of a replica calculation, the validity of (10), (11) is questionable. It is a simple exercise to check its correctness for the spin glass chain without external fields: $\Phi^{(\rho)}(x) = \delta(x-1)$ and $\lambda^{(\rho)} = \langle\langle(\tanh(\beta J))^\rho\rangle\rangle$, for any $\rho > 0$, in agreement with known results [6]. For a more generic distribution of the disorder, exact calculations of the $\chi^{(\rho)}$'s are much harder and, to our knowledge, have been performed for $\rho = 1$ only [6]. We show now that our replica calculation is exact in this case. Reproducing the approach of [14], we compute the Fourier transform of $\chi^{(1)}$ by adding a small sinusoidal field of wave number q in the Hamiltonian (1) and expanding the free energy at second order in the field to obtain

$$\chi^{(1)}(q) = \int \mathrm{d}x\,\mathrm{d}a\,\mathrm{d}\tilde{a}\,\mathrm{d}u\,\Phi_q(x,a,\tilde{a},u)\cdot \cdot\left\langle\!\!\left\langle 2u\left(\frac{e^{-\beta J}}{e^{\beta J}+xe^{-\beta J}}\right) - (a^2+\tilde{a}^2)\left(\frac{e^{-\beta J}}{e^{\beta J}+xe^{-\beta J}}\right)^2\right\rangle\!\!\right\rangle , \tag{13}$$

where the invariant measure Φ_q fulfils

$$\Phi_q(x,a,\tilde{a},u) = \int \mathrm{d}y\,\mathrm{d}b\,\mathrm{d}\tilde{b}\,\mathrm{d}v\,\Phi_q(y,b,\tilde{b},v)\,\langle\langle\delta(x-f(y))\,\delta(a-2x-(b\cos q+\tilde{b}\sin q)f'(y))\cdot \cdot\delta(\tilde{a}-(\tilde{b}\cos q-b\sin q)f'(y))\,\delta(u-2a+2x-vf'(y)-\frac{1}{2}(b^2+\tilde{b}^2)f''(y))\rangle\rangle . \tag{14}$$

The susceptibility (13) depends on Φ_q through the average values $[u]_q(x)$, $[a^2]_q(x)$ and $[\tilde{a}^2]_q(x)$ only, where $[.]_q(x) = \int \mathrm{d}a\,\mathrm{d}\tilde{a}\,\mathrm{d}u\,\Phi_q(x,a,\tilde{a},u)(.)$ (note that $[1]_q(x) = \Phi(x)$, see (7)). These three functions obey three self-consistent equations obtained from (14) whose integral kernels are regular. But their calculation still contains the averages $[a]_q(x)$ and $[\tilde{a}]_q(x)$ fulfilling the complex equation

$$\int_0^\infty \mathrm{d}y \ \left(\delta(x-y) - e^{-iq}\langle\langle\delta(x-f(y))f'(y)\rangle\rangle\right)\left([a]_q(y) + i[\tilde{a}]_q(y)\right) = 2x\Phi(x) \ , \qquad (15)$$

cf. (14). It becomes now clear that the pole Q of $\chi^{(1)}(q)$, closest to the real axis, is reached when $e^{iQ} = \lambda^{(1)}$, see (10). The asymptotic scaling (11) coincides therefore with exactly known results for $\chi^{(1)}$. It would be of high interest to extend the method of [14] (if possible) to test our replica predictions for larger values of $\rho \geq 2$.

In this letter, we have studied the replica structure of one-dimensional Ising systems. From a technical point of view, we have first block-diagonalized the transfer matrix according to the irreducible representations of the permutation group of (finite) n elements and then sent n to zero in the eigenvalue equation for each representation. This procedure has allowed us to understand the physical meaning of the non-trivial, "replica symmmetry broken" representations in terms of the moments of spin-spin correlations. Though one-dimensional Ising models do not display any replica symmetry-breaking transition, we hope that the present approach will be of interest in the future to understand if such a phase transition could occur in more realistic two- or three-dimensional systems.

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