Functional renormalization group for anisotropic depinning and relation to branching processes

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Using the functional renormalization group, we study the depinning of elastic objects in presence of anisotropy. We explicitly demonstrate how the KPZ-term is always generated, even in the limit of vanishing velocity, except where excluded by symmetry. This mechanism has two steps: First a non-analytic disorder-distribution is generated under renormalization beyond the Larkin-length. This non-analyticity then generates the KPZ-term. We compute the β-function to one loop taking properly into account the non-analyticity. This gives rise to additional terms, missed in earlier studies. A crucial question is whether the non-renormalization of the KPZ-coupling found at 1-loop order extends beyond the leading one. Using a Cole-Hopf-transformed theory we argue that it is indeed uncorrected to all orders. The resulting flow-equations describe a variety of physical situations: We study manifolds in periodic disorder, relevant for charge density waves, as well as in non-periodic disorder. Further the elasticity of the manifold can either be short-range (SR) or long-range (LR). A careful analysis of the flow yields several non-trivial fixed points. All these fixed points are transient since they possess one unstable direction towards a runaway flow, which leaves open the question of the upper critical dimension. The runaway flow is dominated by a Landau-ghost-mode. For LR elasticity, relevant for contact line depinning, we show that there are two phases depending on the strength of the KPZ coupling. For SR elasticity, using the Cole-Hopf transformed theory we identify a non-trivial 3-dimensional subspace which is *invariant to all orders* and contains all above fixed points as well as the Landau-mode. It belongs to a class of theories which describe branching and reaction-diffusion processes, of which some have been mapped onto directed percolation.

I. INTRODUCTION

The physics of systems driven through a random environment is by construction irreversible. The fluctuation dissipation relation does not hold and one expects the coarse grained description to exhibit signatures of this irreversibility. In driven manifolds it has indeed been shown that non-linear Kardar-Parisi-Zhang (KPZ) terms are generated in the equation of motion, except when forbidden by symmetry [1, 2]. A question which was debated for long time is whether at zero temperature these terms vanish as the velocity $v \to 0^+$. This is the limit which is relevant to describe depinning ($f \to f_c^+$). It was found some time ago that there are two main universality classes for interface depinning [3–5]. The conclusion was reached mainly on the basis of numerical simulations, which measure the interface velocity $v(\theta)$ as a function of an average imposed slope θ, as well as various arguments related to symmetry. In the first universality class, the isotropic depinning class (ID), the coefficient λ of the KPZ term vanishes as $v \to 0^+$ and the KPZ term is thus not needed in the field theoretic description. In the second class, the anisotropic depinning class (AD), $v(\theta)$ still depends on θ as $f \to f_c^+$ and the KPZ term is present even at $v \to 0^+$. For AD, numerical simulations based on cellular automaton models which are believed to be in the same universality class [6, 7], indicate a roughness exponent $\zeta \approx 0.63$ in $d = 1$ and $\zeta \approx 0.48$ in $d = 2$. On a phenomenological level it has been argued [6–8] that configurations at depinning can be mapped onto directed percolation in $d = 1 + 1$ dimensions, which yields indeed a roughness exponent $\zeta_{\rm DP} = \nu_\perp/\nu_{||} = 0.630 \pm 0.001$, a dynamical exponent $z = 1$, a velocity exponent $\beta_{\rm DP} = \nu_{||} - \nu_\perp \approx 0.636$ and a depinning correlation length exponent $\nu_{\rm DP} = \nu_{||} = 1.733 \pm 0.001$. Some higher dimensional extensions of these arguments in terms of blocking surfaces have been proposed [9–12], but there is, to our knowledge, no systematic field theoretical connection between these problems.

Recently we have reexamined the functional renormalization group (FRG) approach, introduced previously [13–17] to describe isotropic depinning to one loop within a $\epsilon = 4-d$ expansion. We constructed [18, 19] a consistent renormalizable field theoretical description up to two loops, taking into account the main important physical feature – and difficulty – of the problem, namely that the second cumulant $\Delta(u)$ of the random pinning force becomes non-analytic beyond the Larkin scale. The 2-loop result for the exponent ζ shows deviations from the conjecture[17] $\zeta = (4-d)/3$. The reason is the appearance of "anomalous" corrections caused by the non-analytic renormalized disorder correlator. The 2-loop corrections proved to be crucial to reconcile theory and numerical simulations[18, 19].

The aim of this paper is to extend this FRG analysis to the universality class of anisotropic depinning. We first show that beyond the Larkin length, the KPZ-term is indeed generated at $v = 0^+$, as long as it is not forbidden by symmetry. We explicitly compute the lowest order corrections for a simple model studied in recent simulations [20, 21]. Next we derive the FRG-flow equations for the second cumulant $\Delta(u)$ in a $4-\epsilon$ expansion. In a previous study, Stepanow[22] considered the model to one loop, but did not take properly into account the non-analyticity of the renormalized disorder. Since this is physically important, we reexamine the problem here. Indeed, we find several new important "anomalous" corrections, including the one which generates the KPZ term in the first

place, as well as terms correcting the β-function. We then introduce an equivalent description in terms of Cole-Hopf transformed fields. This description is not only much simpler to study in perturbation theory (e.g. to two loops it reduces the number of diagrams by an order of magnitude), but it allows us to obtain a number of results to *all orders*. We argue that the coefficient λ/c which measures the strength of the KPZ non-linearity is uncorrected to all orders. We also determine a non-trivial subspace of the disorder correlators in the form of simple exponentials which is an exact invariant of the FRG to all orders. In the Cole-Hopf variables it is reformulated as the field theory of a specific branching process, or equivalently reaction-diffusion process.

Our flow-equations allow to study both periodic disorder, relevant for charge density waves (CDW), and non-periodic disorder, relevant for lines or interfaces in a random environment. In both cases we find several non-trivial fixed points. All these fixed points possess at least one unstable direction and should thus be associated to transitions. It seems that perturbatively the large scale behaviour is dominated by a runaway-flow, as it is in the standard KPZ problem[23, 24]. The difference is that its direction is a non-trivial function $\Delta(u)$ in functional space. Analysis of the above mentioned invariant subspace suggests that the flow goes towards a specific branching process. The present RG analysis is however unable to attain the non-perturbative fixed point. Thus, it also does not allow to strictly decide whether $d = 4$ is the upper critical dimension of the anisotropic depinning problem, which is an open issue.

Finally, since there are indications that KPZ terms may be needed in the description of the motion of a contact line [25], we have studied manifolds with long range elasticity and the simplest KPZ term. We determine the critical dimension above which this KPZ term is irrelevant, as well as the roughness at crossover.

II. MODEL

We consider a d-dimensional interface (in $d+1$ embedding dimensions) with no overhangs parameterized by a single component height field $u(x)$. The case where the disorder is periodic corresponds to a single component CDW in d dimensions. The common starting point is the equation of motion

$$\eta\partial_t u_{xt} = c\partial_x^2 u_{xt} + \lambda(\partial_x u_{xt})^2 + F(x, u_{xt}) + f_{xt} \quad \text{(II.1)}$$

with friction η, a driving force $f_{xt} = f$ and in the case of long-range elasticity we replace (in Fourier) $q^2 u_q$ by $|q|^\alpha u_q$ (with mostly $\alpha = 1$) in the elastic force. The pinning force $F(x, u)$ is chosen Gaussian with second cumulant

$$\overline{F(x,u)F(x',u')} = \Delta(u-u')\delta^d(x-x') \ . \quad \text{(II.2)}$$

Temperature can be taken into account as an additional white noise $\eta(x,t)$ on the r.h.s. of (II.1) with $\langle\eta(x,t)\eta(x',t')\rangle = 2\eta T\delta(t-t')\delta(x-x')$, but we will focus here on $T = 0$.

Disorder averaged correlation functions $\overline{\langle A[u_{xt}]\rangle} = \langle A[u_{xt}]\rangle_S$ and response functions $\overline{\delta\langle A[u]\rangle/\delta f_{xt}} = \langle \hat{u}_{xt} A[u]\rangle_S$ can be computed from the dynamical action

$$S = \int_{xt} \hat{u}_{xt}(\eta\partial_t - c\partial_x^2)u_{xt} - \lambda\hat{u}_{xt}(\partial_x u_{xt})^2 \quad \text{(II.3)}$$
$$-\frac{1}{2}\int_{xtt'} \hat{u}_{xt}\hat{u}_{xt'}\Delta(u_{xt}-u_{xt'}) - \int_{xt}\hat{u}_{xt}f_{xt}$$

The uniform driving force $f_{xt} = f > 0$ (beyond threshold at $T = 0$) may produce a velocity $v = \overline{\partial_t\langle u_{xt}\rangle} > 0$, a situation which we study by going to the comoving frame (where $\overline{\langle u_{xt}\rangle} = 0$) shifting $u_{xt} \to u_{xt} + vt$, resulting in $f \to f - \eta v$. This is implied below: Each Δ is of the form $\Delta(u_{xt} - u_{xt'} + v(t-t'))$, and we always consider the quasi-static limit $v = 0^+$. Perturbation theory is performed both in KPZ and disorder terms, using the free response function

$$\langle \hat{u}_{q,t'} u_{-q,t}\rangle_0 = R_{q,t-t'} = \eta^{-1}\mathrm{e}^{-(t-t')q^2/\eta}\theta(t-t') . \quad \text{(II.4)}$$

III. GENERATION OF THE KPZ-TERM

In this section we show how the irreversible (non-potential) KPZ term is generated, even in the limit $v \to 0^+$, starting from a purely reversible equation of motion, where all forces are derivatives of a potential.

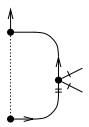
Let us first consider the model recently studied numerically by Rosso and Krauth [20, 21], where the elastic energy is $\int_x E(\nabla u_x)$, and e.g. $E(\theta) = \frac{c}{2}\theta^2 + \frac{c_4}{4}\theta^4$. The relevant continuum equation of motion is:

$$\eta\partial_t u_{xt} = E''(\partial_x u_{xt})\partial_x^2 u_{xt} + F(x, u_{xt}+vt) + f - \eta v \quad \text{(III.1)}$$

Note first that when $c_4 = 0$, which corresponds to the isotropic depinning class with $E(\theta) = \frac{c}{2}\theta^2$, the generation of the KPZ term is forbidden by the statistical tilt symmetry (STS), i.e. the invariance of the equation of motion under a shift $u_{xt} \to u_{xt} + f_x$ with $f_x = hx$ (or more generally the covariance under an arbitrary f_x) [29]. When $c_4 \neq 0$ the model does not obey STS and the KPZ term is not forbidden, and indeed it is generated at finite velocity $v > 0$. This consideration alone is insufficient to show that it is still generated as $v \to 0$ since in that limit the symmetry $u \to -u$ should forbid it. Indeed, if one performs conventional perturbation theory with an *analytic* disorder correlator $\Delta(u)$, one does immediately find that the KPZ term vanishes as $v \to 0^+$. However one needs a mechanism by which, as $v \to 0^+$, the symmetry $u \to -u$ remains broken.

As we now show, this mechanism is provided by the non-analytic nature of the disorder. We know from studies of isotropic depinning [14, 16, 18, 19] that at $T = 0$ the coarse grained disorder becomes *non-analytic* (NA) beyond the Larkin length [30]. We show below that this is also the case for the situation considered here.

Using the techniques developed in Ref. [18, 19] the corresponding perturbation theory, with a non-analytic $\Delta(u)$ becomes (see figure III.1 for notation)

$\delta\lambda =$ 

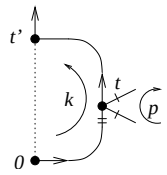


FIG. III.1: The diagram generating the irreversible nonlinear KPZ term with one disorder vertex (notations are as in Ref. [18, 19]) and one c_4 vertex (the bars denote spatial derivatives).

$$= -\frac{c_4}{p^2}\int_{t>0}\int_{t'>0}\int_k \mathrm{e}^{-(t+t')k^2}\left(k^2p^2+2(kp)^2\right) \times\Delta'(u_{x,t+t'}-u_{x,0}+v(t+t'))\ , \tag{III.2}$$

At $T=0$, $u_{x,t}$ has vanishing expectation value and the argument of Δ' becomes $v(t+t')$. Using that

$$\Delta(u)=\Delta(0)+\Delta'(0^+)|u|+\frac{1}{2}\Delta''(0^+)u^2+\dots \tag{III.3}$$

$$\Delta'(u)=\mathrm{sign}(u)+\Delta''(0^+)u+\dots \tag{III.4}$$

and observing that $t,t'>0$, (III.2) can be written as

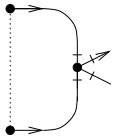
$$\delta\lambda=-\frac{c_4}{p^2}\int_t\int_{t'}\int_k \mathrm{e}^{-(t+t')k^2}\left(k^2p^2+2(kp)^2\right) \times\left(\Delta'(0^+)+\Delta''(0^+)v(t+t')+O(v^2)\right) \tag{III.5}$$

The leading term of this expansion, which is the only UV-diverging one for $4>d>2$, is obtained by setting $v=0$. Integrating over t,t' and using the radial symmetry in k gives

$$\delta\lambda=-c_4\left(1+\frac{2}{d}\right)\int_k\frac{\Delta'(0^+)}{k^2}+O(v)\ . \tag{III.6}$$

Similarly, there is a correction to c, which reads

$$\delta c = c_4 \quad = \int_k\frac{\Delta(0)}{k^2}\left(1+\frac{2}{d}\right) \tag{III.7}$$



leading to

$$\delta c=c_4\left(1+\frac{2}{d}\right)\Delta(0)\int_k\frac{1}{k^2} \tag{III.8}$$

As will become clear below, the natural coupling for the KPZ-term is not λ, but the ratio $\hat\lambda=\lambda/c$, which is corrected as [31]:

$$\delta\hat\lambda=-c_4\left(1+\frac{2}{d}\right)\left(\Delta'(0^+)+\hat\lambda\Delta(0)\right)\int_k\frac{1}{k^2} \tag{III.9}$$

Thus we have shown that the symmetry $u\to -u$ which forbids the KPZ term (e.g. in an analytic perturbation theory where $\Delta'(0)=0$), is broken here at $v=0^+$ by the non-analytic term, and that a KPZ term is indeed generated at depinning. As in our previous study [18, 19] the only assumption is that the interface always advances forward (or that backward motion can be neglected in the steady state), supported in this single component model by no passing theorems [16, 20, 21]. By providing a physical mechanism, this explicit calculation confirms the argument given in [4] based on a Larkin type estimate of the angle θ dependence of the critical force.

Note the sign of the generated KPZ term. Since $\Delta'(0^+)$ is negative, λ is positive as found in simulations [3, 4]. It is a bit counter-intuitive that the surface should become stiffer. Also it effectively corresponds to the generation of a positive average curvature. This is presumably through non-analytic coarse grained configurations of the string (in $d=1$) since otherwise $\int_0^L\nabla^2u=[\nabla u]_0^L$ would grow as L which is unphysical, while cusps in $u(x)$ allow for such a result.

This model is only a particular case, which shows that the anisotropic depinning class is rather broad and not limited to anisotropic disorder. In general, unless they are excluded by symmetry, KPZ-terms will appear. One such case, corresponding to a flux line in $1+1$ dimensions which moves perpendicular to itself was considered in [4]. There disorder is anisotropic with correlators Δ_x and Δ_u for the pinning force. In the case of isotropic disorder $\Delta_x=\Delta_u$, exact rotational invariance (which in infinitesimal form reads $u\to u+\theta x$, $x\to x-\theta u$) should suffice to exclude the KPZ term. We have indeed checked this by adding to the above MSR-action with $\lambda=0$ the non-linear terms of [4]

$$\delta S=-\int_{xt}\hat u_{xt}\left[A\nabla^2u_{xt}(\nabla u_{xt})^2+Bf(\nabla u_{xt}^2)\right] \tag{III.10}$$
$$-\int_{xtt'}\hat u_{xt}\hat u_{xt'}\left[C(\nabla u_{xt})^2+D\nabla u_{xt}\nabla u_{xt'}\right]\Delta(u_{xt}-u_{xt'})\ .$$

The generated KPZ term reads to lowest order

$$\delta\lambda=2(-A+C+D)\Delta'(0^+)\int_k\frac{1}{k^2}\ . \tag{III.11}$$

Since the equation of motion of Ref. [4] for $\Delta_x=\Delta_h$ corresponds to $A=-1$, $D=C=1/2$, one checks to lowest order that the KPZ term is indeed not generated. Although we have not checked it further, it is clear that this property should extend to all orders. In the anisotropic class λ, can a priori be of any sign. The argument given in [4] suggests that for the flux-line model λ is positive when $\Delta_x<\Delta_h$ and negative for $\Delta_x>\Delta_h$. Note that anisotropy by itself is not enough to generate the KPZ term, but that a non-linear and non-analytic disorder correlator is needed, and that this term will of course not be generated in a simple Larkin-type random force model, where Δ_x and Δ_h are constants.

IV. DIMENSIONAL FLORY ESTIMATES

Before using analytical methods, let us indicate a simple Flory, or dimensional, argument which indicates how exponents for ID and AD can differ. In the absence of a KPZ term and setting $u\sim x^\zeta$ the two static terms in the equation of

motion scale as

$$\nabla_x^2 u \sim x^{\zeta-2} \tag{IV.1}$$
$$F(u_x, x) \sim x^{-\frac{d+\zeta}{2}} \ . \tag{IV.2}$$

Using $\overline{F(u,x)F(u',x')} \sim \delta(u-u')\delta^d(x-x')$ for random field disorder gives the Imry-Ma value

$$\zeta_{\rm F} = \frac{4-d}{3} \tag{IV.3}$$

which can be argued to be exact for the statics and is corrected by $O(\epsilon^2)$ terms at depinning. These types of arguments typically give the exact result for LR correlated disorder, as the LR disorder part is not renormalized. It happens that this range is long enough for the statics but not for depinning; hence there is a correction at depinning which increases ζ. Note that it becomes again exact for depinning if the range of Δ in u or x is large enough (see e.g. the end of Section IV B in [26] and Appendix B).

In presence of a KPZ term the latter scales as

$$(\nabla_x u)^2 \sim x^{2\zeta-2} \ . \tag{IV.4}$$

Supposing that it is relevant, it dominates over the elastic term. Balancing KPZ-term against disorder gives the modified Flory estimate

$$\zeta_{\rm F} = \frac{4-d}{5} \ . \tag{IV.5}$$

For $d=1$ it yields $\zeta_{\rm F} = 0.6$ versus $\zeta = 0.63$ observed in simulations[20], which is not bad an estimate for such a simple argument. Again it is possible that if one increases the range of Δ the estimate (IV.5) becomes again exact, as is the case for standard KPZ (directed polymer) see Appendix B. Note however that it works with an upper critical dimension $d=4$, which is an open question, and is thus merely indicative.

V. FLOW-EQUATIONS IN PRESENCE OF A KPZ-TERM

Let us start by deriving the FRG flow of λ, c, η and Δ to one loop starting from (II.3). The KPZ and disorder terms are both marginal in $d=4$ and become relevant below. Simple dimensional arguments show that these are the only needed counter-terms. We have computed the effective action to lowest order. The corrections as given by the diagrams on figure V.1 are (for details see Appendix A):

$$\begin{aligned}
\frac{\delta\eta}{\eta} &= -\left[a_0 c^{-3}\lambda\Delta'(0^+) + c^{-2}\Delta''(0^+)\right] I \\
\frac{\delta c}{c} &= -\left[a_1 \lambda c^{-3}\Delta'(0^+) + a_2\lambda^2 c^{-4}\Delta(0)\right] I \\
\frac{\delta\lambda}{\lambda} &= -\left[a_3 \lambda c^{-3}\Delta'(0^+) + a_4\lambda^2 c^{-4}\Delta(0)\right] I \\
\delta\Delta &= \left[a_5\lambda^2 c^{-4}\Delta^2 + c^{-2}(\Delta''(\Delta(0)-\Delta) - (\Delta')^2)\right] I
\end{aligned} \tag{V.1}$$

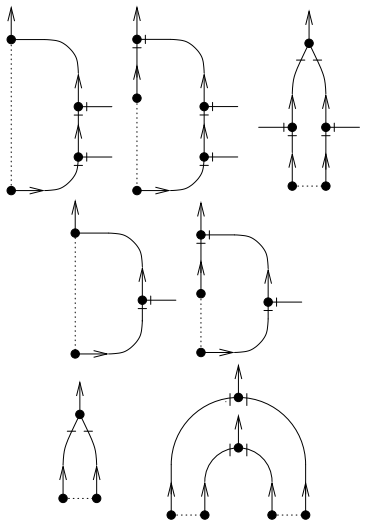


FIG. V.1: 1-loop-diagrams correcting λ (top), c (middle), and η and $\Delta(u)$ (bottom).

where $I = \int 1/q^4$ (integrated over the shell if using Wilson's scheme) and the coefficients are:

$$\begin{aligned}
&a_0 = 1 \ , \qquad a_1 = 2(d-2)/d \ , \qquad a_2 = 4/d \\
&a_3 = a_4 = 4/d \ , \qquad a_5 = 2 \ .
\end{aligned} \tag{V.2}$$

In the following we will set $d=4$ in these coefficients since they are universal only to this order. This gives

$$a_0 = a_1 = a_2 = a_3 = a_4 = 1 \ , \qquad a_5 = 2 \ . \tag{V.3}$$

One then notes that the quantity λ/c remains uncorrected to first order in $d=4$. In the next section we shall argue that this remains true to all orders. The corrections to the linear term in (II.3) can be interpreted as the correction to the critical force:

$$\delta f = -\delta f_c = (\lambda c^{-2}\Delta(0) + c^{-1}\Delta'(0^+))I_1 \tag{V.4}$$

where $I_1 = \int_q \frac{1}{q^2}$. It does not require an additional counter-term if we tune f to be exactly at depinning $f = f_c$.

In view of the non-renormalization of λ/c in (V.1) it is useful to denote the unrescaled coupling constants as

$$\hat\lambda = \frac{\lambda}{c} \ , \qquad \hat\eta = \frac{\eta}{c} \ , \qquad \hat\Delta = \frac{\Delta}{c^2} \ . \tag{V.5}$$

One should also notice that if one performs the change of variable in the initial model $u \to u/\hat\lambda$, $\hat u \to \hat u\hat\lambda$, then the free (quadratic) part of the action (proportional to c and η) remains invariant while disorder and KPZ terms become:

$$\begin{aligned}
\hat\lambda &\to 1 \\
\Delta(u) &\to \hat\lambda^2\Delta(u/\hat\lambda)
\end{aligned} \tag{V.6}$$

Thus the coefficient $\hat\lambda$ can be set to one upon appropriate redefinitions of disorder and displacements.

It is natural to start the study of the FRG flow and the search for fixed points as for $\lambda = 0$ by defining the following rescaled parameters

$$\tilde\lambda = \hat\lambda\Lambda_\ell^{-\zeta} \tag{V.7}$$
$$\tilde\Delta(u) = \Lambda_\ell^{2\zeta-\epsilon}\hat\Delta(u\Lambda_\ell^{-\zeta}) \tag{V.8}$$

within a Wilson scheme where $\Lambda_\ell = \Lambda \mathrm{e}^{-\ell}$ is the running UV cutoff. This yields two coupled equations for the couplings $\tilde{\lambda}$ and $\tilde{\Delta}(u)$

$$\partial_\ell \ln \tilde{\lambda} = \zeta \tag{V.9}$$

$$\begin{aligned}\partial_\ell \tilde{\Delta}(u) = &(\epsilon - 2\zeta)\tilde{\Delta}(u) + u\zeta\tilde{\Delta}'(u) \\ &+2\tilde{\lambda}^2\tilde{\Delta}(u)^2 + [2\tilde{\lambda}^2\tilde{\Delta}(0) + 2\tilde{\lambda}\tilde{\Delta}'(0^+)]\tilde{\Delta}(u) \\ &-\tilde{\Delta}'(u)^2 - \tilde{\Delta}''(u)(\tilde{\Delta}(u) - \tilde{\Delta}(0))\end{aligned} \tag{V.10}$$

where here and below we absorb $\epsilon I = S_4/(2\pi)^4$ in the couplings. One notes that if there is a fixed point for $\tilde{\Delta}(u)$, then ζ is the roughness exponent since

$$\begin{aligned}\overline{\langle u_q u_{-q}\rangle} &= \Delta(0)/c^2 q^4 \\ &= \Lambda_\ell^{\epsilon - 2\zeta}\tilde{\Delta}^*(0)/q^4 \sim \tilde{\Delta}^*(0)/q^{d-2\zeta}\end{aligned} \tag{V.11}$$

when evaluated at scale $\Lambda_\ell = q$. A more rigorous calculation uses the effective action[19] at non-zero momentum. but to one loop gives the same result. The dynamical exponent z in $t \sim x^z$ and the anomalous dimension of the elasticity can be determined from

$$\begin{aligned}-\psi = \partial_\ell \ln c &= -\tilde{\lambda}\tilde{\Delta}'(0^+) - \tilde{\lambda}^2\tilde{\Delta}(0) \\ z - 2 = \partial_\ell \ln(\eta/c) &= -\tilde{\Delta}''(0^+) + \tilde{\lambda}^2\tilde{\Delta}(0)\ .\end{aligned} \tag{V.12}$$

The correlation-length exponent ν in $\xi \sim (f - f_c)^{-\nu}$ and the velocity exponent β in $v \sim (f - f_c)^\beta$ are given by the scaling relations

$$\nu = \frac{1}{2 - \zeta + \psi} \tag{V.13}$$

$$\beta = \nu(z - \zeta) = \frac{z - \zeta}{2 - \zeta + \psi}\ . \tag{V.14}$$

This can be seen by noting that the action (II.3) is invariant under $x = \mathrm{e}^\ell x'$, $t = \mathrm{e}^{z\ell} t'$, $u = \mathrm{e}^{\zeta\ell} u'$, $\hat{u} = \hat{u}' \mathrm{e}^{(2-z-\zeta-d+\psi)\ell}$ provided $\eta = \eta'\,\mathrm{e}^{(2-z+\psi)\ell}$, $c = c'\,\mathrm{e}^{\psi\ell}$, $\lambda = \lambda'\,\mathrm{e}^{(\psi+\zeta)\ell}$, $f = f'\,\mathrm{e}^{(2-\zeta+\psi)\ell}$ and $\Delta = \Delta'\,\mathrm{e}^{(\epsilon-2\zeta+2\psi)\ell}$ as well as $T = T'\,\mathrm{e}^{(2-d-2\zeta+\psi)\ell}$. While in presence of STS one has $\psi = 0$, this is not the case here. In a Wilson formulation, the critical force is obtained by integration over scales of

$$\partial_\ell f_c = -c_\ell(\tilde{\lambda}_\ell\tilde{\Delta}_\ell(0) + \tilde{\Delta}'_\ell(0^+))\Lambda_\ell^{2-\zeta}\ , \tag{V.15}$$

a quantity which physically is likely to remain positive.

A salient feature of the AP class is that the critical force depends on the angle by which the interface is tilted. From the arguments of [3, 4] the characteristic slope θ should scale like the ratio of the characteristic lengths orthogonal and parallel to the interface, $\theta \sim \xi_\perp/\xi_{||} \sim (f - f_c)^{\nu(1-\zeta)}$ and more generally the velocity should behave as

$$v(f,\theta) = (f - f_c(0))^\beta\, g\left(\frac{\theta}{(f - f_c(0))^{\nu(1-\zeta)}}\right)\ . \tag{V.16}$$

Defining $\lambda_{\rm eff}$ by[4] $v(f,\theta) = \lambda_{\rm eff}\theta^2 + \ldots$, the small θ expansion of $v(f,\theta)$ gives he effective $\lambda_{\rm eff}$ as

$$\lambda_{\rm eff} \sim (f - f_c(0))^{\beta - 2\nu(1-\zeta)} = (f - f_c(0))^{-\nu(2-\zeta-z)}\ . \tag{V.17}$$

Performing the redefinition $u = \tilde{u} + \theta x$, we can compute the critical force as a function of the angle θ to lowest order in disorder

$$\begin{aligned}\delta f_c(\theta) &= -\theta^2\lambda\left(1 - \frac{4}{d}I\left(\tilde{\lambda}^2\tilde{\Delta}(0) + \tilde{\lambda}\tilde{\Delta}'(0^+)\right)\right) \\ &= -\theta^2\lambda\left(1 + \frac{\delta\lambda}{\lambda}\right)\end{aligned} \tag{V.18}$$

and thus we find an angular dependence, which is increased under renormalization.

The notable feature of the above FRG equation is the absence of corrections to $\hat{\lambda}$ to this order in eqs. (V.1). It is crucial to determine whether this persists beyond one loop. If there were corrections to higher order this might allow for a non-trivial fixed point of $\hat{\lambda}$ and thus to fix ζ. On the other hand, absence of corrections would imply that for $\zeta > 0$, $\tilde{\lambda}$ flows to infinity, which makes the existence of a perturbative fixed point doubtful. In the next section, we present a different approach, which allows to clarify this question.

It is worth noting, that since KPZ-terms are only generated above the Larkin length, the FRG flow below the Larkin length (as well as the value of this length) is identical to the case $\lambda = 0$. It is however instructive to artificially consider the above FRG flow for an analytic function and with a given imposed bare value of $\hat{\lambda}$ (setting $\zeta = 0$). One gets

$$\partial_\ell\tilde{\Delta}(0) = \epsilon\tilde{\Delta}(0) + 4\hat{\lambda}^2\tilde{\Delta}(0)^2 \tag{V.19}$$

$$\partial_\ell\Delta''(0) = \epsilon\Delta''(0) - 3\Delta''(0)^2 + 6\hat{\lambda}^2\Delta(0)\Delta''(0)\ . \tag{V.20}$$

The bare disorder has $\Delta(0) > 0$ and $\Delta''(0) < 0$. Since all terms on the r.h.s. of (V.20) have the same sign, $|\Delta''(0)|$ diverges faster if $\hat{\lambda} \neq 0$, meaning that the KPZ-term cannot prevent $\Delta(u)$ from becoming non-analytic. Note that the first equation exhibits a runaway at $L_{\Delta(0)}$ which can shorten the Larkin length. In $d = 4 + \epsilon$ at $\lambda = 0$ there is an unstable fixed point at $\Delta''(0) = -\epsilon/3$ separating a Gaussian weak-disorder phase with the bare unrescaled Larkin force producing finite displacements, and a phase where disorder seems to become non-analytic, only to become irrelevant at larger scales as can be seen by examining the flow in the non-analytic space beyond the Larkin length. At $\lambda > 0$ there is a fixed line at $\Delta(0) = -\epsilon/(4\hat{\lambda}^2) > 0$ which separates a phase where $\Delta(0)$ grows from a phase where it decays to zero. On the transition line the flow is towards a non-analytic disorder.

VI. COLE-HOPF TRANSFORMED THEORY

We now introduce the Cole-Hopf transformed theory which has a lot of interesting properties.

Starting from (II.1) we first divide by c. This gives

$$\hat{\eta}\partial_t u_{xt} = \partial_x^2 u_{xt} + \hat{\lambda}(\partial_x u_{xt})^2 + \frac{1}{c}F(x, u_{xt}) + \frac{f}{c} \tag{VI.1}$$

We then define the Cole-Hopf transformed fields

$$Z_{xt} := \mathrm{e}^{\hat{\lambda} u_{xt}} \qquad \Leftrightarrow \qquad u_{xt} = \frac{\ln(Z_{xt})}{\hat{\lambda}}\ . \tag{VI.2}$$

The equation of motion becomes after multiplying with $\hat{\lambda} Z_{xt}$

$$\hat{\eta}\partial_t Z_{xt} = \partial_x^2 Z_{xt} + \frac{\hat{\lambda}}{c} F\left(x, \frac{\ln(Z_{xt})}{\hat{\lambda}}\right) Z_{xt} + \frac{\hat{\lambda} f}{c} Z_{xt} \quad \text{(VI.3)}$$

and the dynamical action

$$\begin{aligned}\mathcal{S} = &\int_{xt} \hat{Z}_{xt}\left(\hat{\eta}\partial_t - \partial_x^2\right) Z_{xt} \\ &-\frac{\hat{\lambda}^2}{2}\int_{xtt'} \hat{Z}_{xt} Z_{xt}\, \hat{\Delta}\left(\frac{\ln Z_{xt} - \ln Z_{xt'}}{\hat{\lambda}}\right) \hat{Z}_{xt'} Z_{xt'} \\ &-\frac{\hat{\lambda}}{c} f \int_{xt} \hat{Z}_{xt} Z_{xt} \end{aligned} \quad \text{(VI.4)}$$

It is important to note that the above formal manipulations are only valid in the mid-point (Stratonovich) discretization. The strategy therefore is to start from the original equation of motion, which is interpreted in the Itô discretization, switch to Stratonovich, make the change of variables, and then switch back to Itô. Note the identification:

$$\hat{u}_{xt} \equiv \frac{\hat{\lambda}}{c} \hat{Z}_{xt} Z_{xt} \quad \text{(VI.5)}$$

and that in this formalism the force (or the distance to the critical force) corresponds to a mass:

$$m^2 = \frac{\hat{\lambda}}{c}(f - f_c) \quad \text{(VI.6)}$$

Let us first illustrate how perturbation theory works in this new formulation and how one can easily recover the 1-loop FRG equation obtained in the previous section. Perturbation theory is performed with the standard response-function. We note a very important property: To contract $\hat{Z}_{00}$ with a disorder-insertion $\hat{Z}_{xt} Z_{xt} \hat{\lambda}^2\, \hat{\Delta}\left(\frac{\ln Z_{xt} - \ln Z_{xt'}}{\hat{\lambda}}\right) \hat{Z}_{xt'} Z_{xt'}$ and focusing on Z_{xt} (not $Z_{xt'}$), one can decide to either contract Z_{xt} standing outside the $\hat{\Delta}$ or inside. In the first place, this eliminates the factor Z_{xt}, but leaves $\hat{\Delta}$ underived. In the second case, deriving the argument of $\hat{\Delta}$, gives $\hat{\Delta}'/\hat{\lambda}$, together with a factor of $1/Z_{xt}$ from the inner derivative. The latter also cancels the Z_{xt} standing outside the $\hat{\Delta}$. So independently of where one derives, one always looses the factor of Z_{xt} outside $\hat{\Delta}$. Contracting n times towards the vertex at x, t thus gives a factor of Z_{xt}^{1-n}. This observation shows that the diagrammatics are a very simple generalization of the case without the KPZ-term which was detailed up to two loops in [19]. One easily verifies that the latter case is reproduced upon contracting only the argument of $\hat{\Delta}$. To see this, one performs the perturbation theory and finally takes the limit of $\hat{\lambda} \to 0$. Each time, one has contracted a Z_{xt} outside of $\hat{\Delta}$, one is missing a factor of $1/\hat{\lambda}$, and the term vanishes in the limit of $\hat{\lambda} \to 0$. Further remark that for $\hat{\lambda} \to 0$, the argument of $\hat{\Delta}$ becomes

$$\frac{Z_{xt} - Z_{xt'}}{\hat{\lambda}} = u_{xt} - u_{xt'} + O(\hat{\lambda}) \quad \text{(VI.7)}$$

This shows that the perturbation theory for isotropic depinning is reproduced.

Thus the new diagrams, in the presence of the KPZ-term, can be deduced from those for $\lambda = 0$ by allowing additional contractions of a Z_{xt} outside the $\hat{\Delta}$. Compared to performing calculations using (II.3) this yields a much simpler perturbation theory, with far less distinct diagrams. E.g. to two loops, the number of diagrams is reduced by at least a factor of ten.

Note that now a renormalization of the term $\hat{Z}\hat{\Delta}Z$ is allowed, since it is no longer forbidden by STS. Indeed shifting $u_{xt} \to u_{xt} + \alpha x/\hat{\lambda}$ and $\hat{Z}_{xt} \to \hat{Z}_{xt} \mathrm{e}^{-\alpha x}$, we find that the action changes by

$$\delta S = \int_{xt} \hat{Z}_{xt}\left(\alpha^2 + \alpha\nabla\right) Z_{xt} \ . \quad \text{(VI.8)}$$

However, since the action (VI.4) is still translationally invariant, it remains unchanged under

$$\begin{aligned} Z_{xt} &\to \mu Z_{xt} \\ \hat{Z}_{xt} &\to \frac{1}{\mu}\hat{Z}_{xt} \ . \end{aligned} \quad \text{(VI.9)}$$

Transforming only $Z_{xt} \to \mu Z_{xt}$ without changing $\hat{Z}_{xt}$ will allows us later to fix the coefficient of the Laplacian to unity and transfer all its corrections into corrections to $\hat{\Delta}$ and $\hat{\eta}$.

We now present the calculations at 1-loop order. We start with the corrections to $\hat{\eta}$. Contracting one disorder vertex once with itself, we obtain

$$\hat{\lambda}^2 \hat{Z}_{xt'} Z_{xt} \left[\hat{\Delta}\left(\frac{\ln Z_{xt'} - \ln Z_{xt}}{\hat{\lambda}}\right) + \frac{1}{\hat{\lambda}}\hat{\Delta}'\left(\frac{\ln Z_{xt'} - \ln Z_{xt}}{\hat{\lambda}}\right)\right] \times R_{0,t'-t} \quad \text{(VI.10)}$$

Expanding $\ln Z_{xt'} - \ln Z_{xt}$ for small times yields

$$\ln Z_{xt'} - \ln Z_{xt} = \frac{(t'-t)\partial_t Z_{xt}}{Z_{xt}} + O(t-t')^2 \quad \text{(VI.11)}$$

One also has to expand Z_{xt} around xt':

$$Z_{xt} = -(t'-t)\partial_t Z_{xt'} \quad \text{(VI.12)}$$

Since the manifold only jumps ahead, the arguments of $\hat{\Delta}$ and $\hat{\Delta}'$ are always positive. Putting all terms together, we obtain:

$$\hat{Z}_{xt'}\partial_t Z_{xt'}(t'-t) R_{0,t'-t} \times \left[\left(\hat{\lambda}\hat{\Delta}'(0^+) + \hat{\Delta}''(0^+)\right) - \left(\hat{\lambda}^2\hat{\Delta}(0) + \hat{\lambda}\hat{\Delta}'(0^+)\right)\right] \quad \text{(VI.13)}$$

Integrating over $t' - t$ yields

$$\hat{Z}_{xt'}\partial_t Z_{xt'} I \times \left[\left(\hat{\lambda}\hat{\Delta}'(0^+) + \hat{\Delta}''(0^+)\right) - \left(\hat{\lambda}^2\hat{\Delta}(0) + \hat{\lambda}\hat{\Delta}'(0^+)\right)\right] \quad \text{(VI.14)}$$

We have grouped terms such that in the first bracket there appear the corrections to $-\frac{\delta\eta}{\eta}$ and in the second those to $\frac{\delta c}{c}$. Here they appear all together in one diagram. In the absence of the KPZ-term only the term independent of $\hat{\lambda}$ survives. Noting the cancellation between the two terms, we finally arrive at

$$\frac{\delta\hat{\eta}}{\hat{\eta}} = \left[\hat{\Delta}''(0^+) - \hat{\lambda}^2\hat{\Delta}(0)\right] I \quad \text{(VI.15)}$$

We now turn to corrections to disorder. Reminding that the arrows can either enter into the argument of Δ or into the single Z-field, we get the following contributions (plus some odd terms, which we do not write):

$$\begin{aligned}
\delta\hat{\Delta}(u)^{\rm a} &= \left[-\hat{\Delta}''(u)\hat{\Delta}(u) + \hat{\lambda}^2\hat{\Delta}(u)^2\right] I \\
\delta\hat{\Delta}(u)^{\rm b} &= \left[-\hat{\Delta}'(u)^2 + \hat{\lambda}^2\hat{\Delta}(u)^2\right] I \\
\delta\hat{\Delta}(u)^{\rm c} &= \left[\hat{\Delta}''(u)\hat{\Delta}(0)\right] I \\
\delta\hat{\Delta}(u)^{\rm d} &= 2\left[\hat{\lambda}\hat{\Delta}(u)\hat{\Delta}'(0^+) + \hat{\lambda}^2\hat{\Delta}(u)\hat{\Delta}(0)\right] I
\end{aligned} \tag{VI.16}$$

These reproduce the corrections obtained in the previous section, but quite differently.

The Cole-Hopf transformed theory suggests that

$$\delta\hat{\lambda} = 0 \tag{VI.17}$$

to all orders. To prove this one has to show that the following terms are not generated in the effective action

$$\hat{Z}_{xt}\frac{1}{Z_{xt}}(\nabla Z_{xt})^2 \ . \tag{VI.18}$$

It is easy to see that these terms result from a change of $\hat{\lambda}$ (keeping u_{xt} and $\hat{u}_{xt}$ fixed):

$$Z_{xt} \to Z_{xt}\left(1 + \frac{\delta\hat{\lambda}}{\hat{\lambda}}\ln Z_{xt}\right) \tag{VI.19}$$

$$\hat{Z}_{xt} \to \hat{Z}_{xt}\left(1 - \frac{\delta\hat{\lambda}}{\hat{\lambda}}\ln Z_{xt}\right) \tag{VI.20}$$

and thus the Laplacian generates (VI.18). One can also again consider a term like c_4 which is known to produce a shift in $\hat{\lambda}$ (see (III.9)), and does produce (VI.18) above together with other irrelevant terms with more gradients. In fact (VI.18) is by power counting the only term marginal in $d = 4$ which can appear. This term could in principle come from vertices with several derivatives acting on $\hat{\Delta}$ at point x. As previously discussed, it is always compensated, but the compensating factor could be on a different vertex at position x' and hence produce (VI.18) via a gradient expansion. We have shown in Fig. VI.2 the 2-loop diagrams correcting terms with a single response field in the effective action and the Z and $1/Z$ fields which appear at each vertex. All terms contribute to $\hat{\eta}$. Graphs b, c and d each give a term of the form (VI.18) by expanding the Z^2 on the lower disorder, but the sum of them cancels. As we will discuss below this is graphically achieved by moving the ends of the arrows around on the upper vertex, suggesting

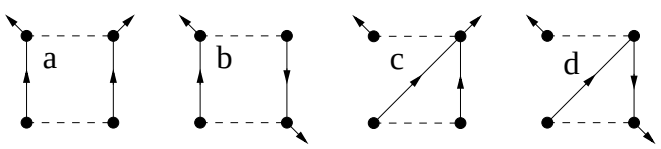


FIG. VI.1: 1-loop dynamical diagrams correcting $\hat{\Delta}$

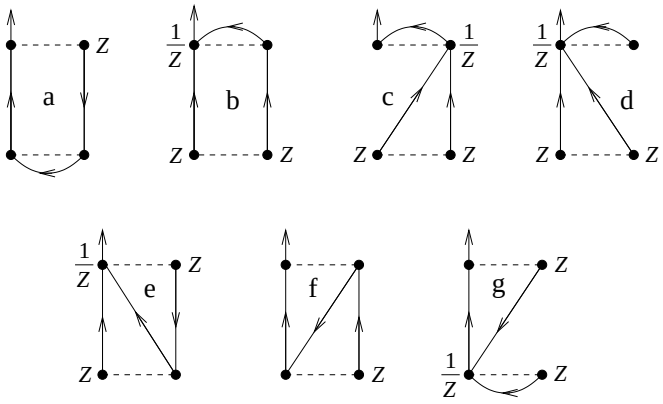


FIG. VI.2: 2-loop dynamical diagrams correcting the single $\hat{Z}$-component. Diagrams a – g correct the friction. Only diagrams e and f have a sufficiently strong divergence in space (after time-integration) that they can produce spatial gradients. In fact they both correct $\hat{Z}\Delta Z$. (The diagram is the well-known sun-set diagram from ϕ^4-theory.)

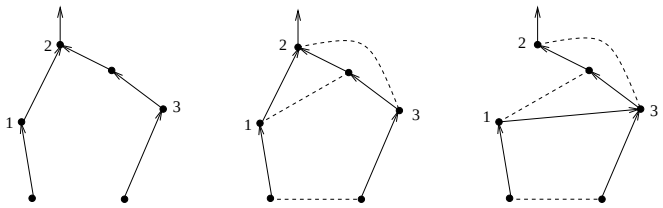


FIG. VI.3: Figure explaining the non-renormalization of $\hat{\lambda}$, see main text.

a more general cancelation. Another argument is that the divergence in space between the upper and lower vertex is not strong enough in order to contribute to (VI.18) or $\int \hat{Z}\Delta Z$. For this to happen, one needs three response-functions between upper and lower disorder, as is the case for diagrams e and f. They thus both contribute to $\int \hat{Z}\Delta Z$, but since they have only a single Z on the lower disorder, they do not contribute to (VI.18).

We now argue that to all orders in perturbation theory no diagram proportional to a single $\hat{Z}$ (one connected component) can be generated, which contains a factor of $(\nabla Z)^2\frac{1}{Z}$. We believe these arguments to be conclusive; especially we have not been able to construct any counter-example at 3- or 4-loop order. However the structure of the theory is sufficiently complicated that some caution is advised.

Look at figure VI.3. The response-functions (arrows) in an arbitrary diagram correcting a single-time vertex have a tree-structure (left). This diagram can be completed by adding the disorder-interactions between arbitrary pairs of points (middle). A potentially dangerous factor of $\frac{1}{Z}$ appears at point 2. Point 2 has a "brother" 3, to which it is connected by a disorder correlator $\hat{\Delta}$ (dashed line).

Then, two cases have to be distinguished: Either there is no line entering point 3, then point 3 can contribute his factor of Z to point 2: Since it is at the same point in space, the difference can be expanded in a series in time, giving time-

derivatives of Z which do not spoil the argument.

On the other hand, there may be a line entering point 3. This is drawn on figure VI.3 (middle). By construction (at least) two branches (of response-functions) enter at point 2. At least one of them does not contain the brother of 2 (here point 3). Here it is the left branch, containing point 1. Now consider the diagram where the response-function from 1 to 2 is replaced by a response-function from 1 to 3 (right). Since one can always contract last the response-field at point 1, leading to either the response-function from 1 to 2 or the one from 1 to 3, these diagrams have the same combinatorial factor, but differ by a a factor of -1, due to the derivative of $\Delta(\frac{\ln Z - \ln Z'}{\hat{\lambda}})$ on either the first or the second argument. This comes in both cases with a factor of $\frac{1}{Z}$, at the *same* position in space but at different positions in time. However, due to the tree-structure, the time-integration can always be done freely, and the two vertices finally cancel. This argument is sufficient before reaching the Larkin-length. However after reaching the Larking length, the non-analyticity of the disorder may yield additional sign-functions in time between both ends of the vertex, as has been observed in [19]. Then the proof gets more involved. There is another very powerfull constraint on the generation of terms like (VI.18): One has to construct a diagram with a strong spatial ultraviolet divergence, such that after Taylor-expanding Z in space the additional factor of x^2 together with this strong ultraviolet divergence gives a pole in $1/\epsilon$, i.e. a logarithmic divergence at $d = 4$. This is the situation for diagrams e and f in figure VI.2. It arises if and only if there are $2n + 1$ response-functions connecting n points in space (this may well be a sub-diagram), but where response-functions that connect the same point in space are not counted. In all examples which we considered up to 4-loop order, which had sufficiently many factors of $1/Z$, and which had the correct UV-structure, the $(2n + 1)$ response-functions where enough to enforce an ordering of times, such that the mounting proof sketched on figure VI.3 went through. We have to leave it as a challenge to the reader to either find a counter-example or to make the above arguments rigorous.

Let us now return to the analysis of the RG-equations. We introduce rescaled variables according to

$$\hat{\Delta}(u) = \Lambda_\ell^{\epsilon-2\zeta} \tilde{\Delta}(u\Lambda_\ell^\zeta) \tag{VI.21}$$
$$\tilde{\lambda} = \hat{\lambda}\Lambda_\ell^{-\zeta} \tag{VI.22}$$

with $\Lambda_\ell = \Lambda \mathrm{e}^{-\ell}$. Because we have defined $Z = \mathrm{e}^{\hat{\lambda}u}$, in order not to generate additional terms, a rescaling of u demands a (compensating) rescaling of $\hat{\lambda}$ such that the product remains unchanged. Even though this may not be the best choice corresponding to the existence of a fixed point, it is the only way to preserve the Cole-Hopf-transformation, leaving Z and $\ln Z$ unchanged. The rescaling of $\hat{\Delta}$ comes from the rescaling of $\hat{\lambda}$, which appears as a factor of $\hat{\lambda}^2$ in front of $\hat{\Delta}$ in the action and as a factor of $1/\hat{\lambda}$ in the argument of $\hat{\Delta}$.

This leads again to the FRG flow equation given in (V.10):

$$\begin{aligned}\partial_\ell \tilde{\Delta}(u) = (\epsilon - 2\zeta)\tilde{\Delta}(u) + \zeta u \tilde{\Delta}'(u) \\ -\tilde{\Delta}''(u)\left(\tilde{\Delta}(u) - \tilde{\Delta}(0)\right) - \tilde{\Delta}'(u)^2 \\ +2\tilde{\lambda}\tilde{\Delta}(u)\tilde{\Delta}'(0^+) \\ +2\tilde{\lambda}^2\left(\tilde{\Delta}(u)^2 + \tilde{\Delta}(u)\tilde{\Delta}(0)\right)\end{aligned} \tag{VI.23}$$

Further remarkable properties of the Cole-Hopf transformed theory will be shown below. We now turn to the study of the FRG flow.

VII. PERIODIC CASE

We now consider the case, where $\hat{\Delta}(u)$ is a periodic function with period 1. The starting point is (VI.23) with $\zeta = 0$, thus $\tilde{\lambda} = \hat{\lambda}$ remains constant under renormalization (to all orders). Since the period is fixed, $\tilde{\lambda}$ cannot be scaled away using (V.6). It is thus a continuously varying parameter and we must study the flow as a function of it.

In eq. (VI.23) there is a tendency for a runaway flow, as can be seen by analyzing the flow-equation (VI.23) with the trivial solution $\tilde{\Delta}(u) = \Delta$

$$\partial_\ell \Delta = \epsilon\Delta + 4\hat{\lambda}^2\Delta^2 \ . \tag{VII.1}$$

This corresponds to the localization - or self attracting chain - problem studied in [27] and we expect on physical grounds the full functional form of $\Delta(u)$ to be important, which may lead to other fixed points.

For $\hat{\lambda} = 0$ we already know that there is an unstable fixed point

$$\Delta_\ell(u) = \Delta^*(u) + c\,\mathrm{e}^{\epsilon\ell} \tag{VII.2}$$
$$\Delta^*(u) = \frac{1}{36} - \frac{1}{6}u(1-u) \ , \tag{VII.3}$$

which describes isotropic depinning for CDW. This fixed point survives for small λ as can be seen from a series expansion in powers of λ. Moreover at each order in λ, $\Delta^*(u)$ remains polynomial in $u(1-u)$. We do not reproduce this expansion here, since we have succeeded in obtaining the fixed point *analytically*. Equation (VI.23) possesses the following remarkable property:

> *A three parameter subspace of exponential functions forms an exactly invariant subspace.*

Even more strikingly, this is true *to all orders* in perturbation theory. This property, which is quite non-trivial, is understood in the Cole Hopf theory, as discussed below.

For our purposes, it is more convenient to write

$$\Delta(u) = \frac{1}{\hat{\lambda}^2}\epsilon f(u\hat{\lambda}) \ , \tag{VII.4}$$

such that f satisfies the same FRG equation (VI.23) with $\lambda = \epsilon = 1$, but with period λ. This allows to make an ansatz for a family of exponential functions

$$f(u) = a + b\,\mathrm{e}^{-u} + c\,\mathrm{e}^{u} \ . \tag{VII.5}$$

The FRG-flow (VI.23) closes in this subspace, leading to the

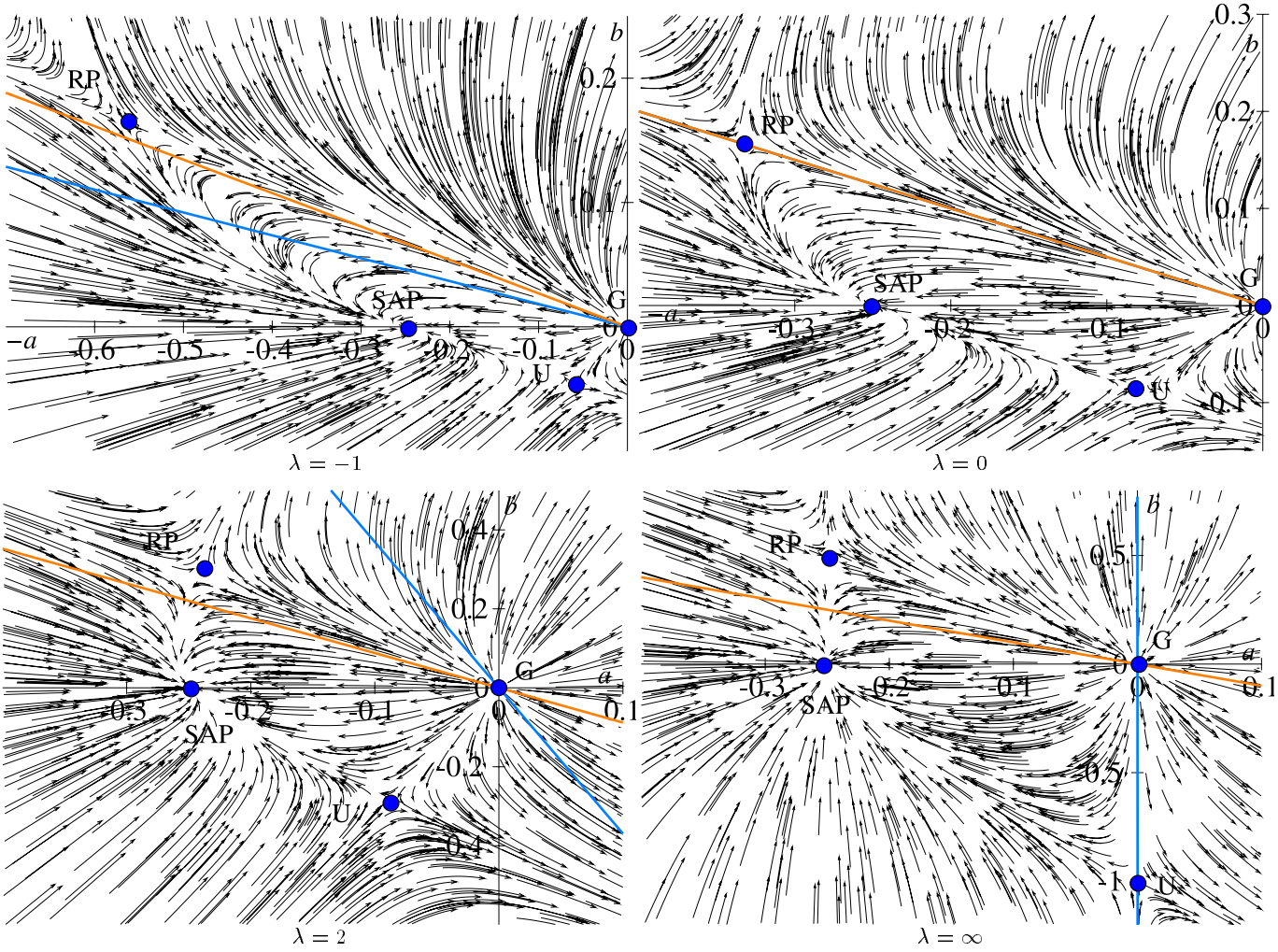


FIG. VII.1: Fixed point structure for different values of λ. The coordinate system is such that a grows to the right and b to the top. The both separatrices are $b = -\frac{1}{2}a\mathrm{e}^{\lambda}$ (blue/dark) and $b = -a/(1+\mathrm{e}^{-\lambda})$ (orange/bright).

simpler 3-dimensional flow:

$$\partial_\ell a = a + 4a^2 + 4ac + 4bc \qquad \text{(VII.6)}$$
$$\partial_\ell b = b(1 + 6a + b + 5c) \qquad \text{(VII.7)}$$
$$\partial_\ell c = c(1 + 6a + b + 5c) \qquad \text{(VII.8)}$$

This works only for amplitude one in the exponential; otherwise higher modes are generated. Also note that these equations are not symmetric under the exchange of b and c, as one might expect from the interpretation we will present later.

Requiring periodicity, or equivalently $f(u) = f(\lambda - u)$ imposes

$$c = b\,\mathrm{e}^{-\lambda} \qquad \text{(VII.9)}$$

and one checks that b/c is indeed unrenormalized. Thus one can study the simpler 2-dimensional flow

$$\partial_\ell a = a + 4a^2 + 4ab\mathrm{e}^{-\lambda} + 4b^2\mathrm{e}^{-\lambda} \qquad \text{(VII.10)}$$
$$\partial_\ell b = b(1 + 6a + b + 5b\mathrm{e}^{-\lambda}) \qquad \text{(VII.11)}$$

as a function of λ. A physical requirement is that

$$\Delta(0) = a + b(1 + \mathrm{e}^{-\lambda}) > 0 \qquad \text{(VII.12)}$$

$$f_c \sim -(\Delta'(0) + \lambda\Delta(0))$$
$$= -\frac{1}{\lambda}(a + 2b\mathrm{e}^{-\lambda}) > 0\ . \qquad \text{(VII.13)}$$

For $a < 0$ this is possible only if

$$-\frac{a}{1+\mathrm{e}^{-\lambda}} < b < -\frac{a}{2}\mathrm{e}^{\lambda} \qquad \text{(VII.14)}$$

On the other hand, for $a > 0$ the flow for a is always $a \to \infty$ in a finite time. Indeed the r.h.s. of (VII.10) is always positive for $a > 0$. For $b > 0$ this is trivial; for $b < 0$ this can be seen from

$$a + 4a^2 + 4ab\mathrm{e}^{-\lambda} + 4b^2\mathrm{e}^{-\lambda}$$
$$= a + 4\,(a+b)^2\,\mathrm{e}^{-\lambda} - 4ab\mathrm{e}^{-\lambda} + 4a^2(1-\mathrm{e}^{-\lambda})$$
$$> 0 \qquad \text{(VII.15)}$$

The flow given in (VII.10) and (VII.11) is shown in figure VII.1. There are four fixed points for $d = 4-\epsilon$. In the original variables they are

(i) Gaussian fixed point G (repulsive in all directions) with $\Delta(u) = 0$.

(ii) Self-avoiding polymer fixed point SAP, where the correlator is a negative constant:

$$\Delta(u) = -\frac{\epsilon}{4\lambda^2} \tag{VII.16}$$

It is the problem of localization in an imaginary random potential, i.e. the Edwards version of the better known self-avoiding polymer. It is attractive in all directions, even those not drawn here. Writing $f(u) = -1/4 + \phi(u)$ and linearizing (VI.23) gives

$$\partial_\ell \phi(0) = -\phi(0) - \frac{1}{2}\phi'(0^+) \tag{VII.17}$$

$$\partial_\ell \phi'(u) = -\frac{1}{2}\phi'(u) \tag{VII.18}$$

This self-avoiding polymer fixed point will not play a role in the following since for the disordered problem $\Delta(0) > 0$. However it is interesting in other contexts, as discussed below.

(iii) Fixed point U, with one attractive and one repulsive direction.

$$\Delta(u) = \frac{1}{\lambda^2}\Bigg[-\frac{1 + 54\mathrm{e}^{-\lambda} + 5\mathrm{e}^{-2\lambda} - (1+5\mathrm{e}^{-\lambda})\sqrt{1+\mathrm{e}^{-\lambda}(34+\mathrm{e}^{-\lambda})}}{8(1-5\mathrm{e}^{-\lambda}(\mathrm{e}^{-\lambda}-8)))} + \frac{2}{(1-7\mathrm{e}^{-\lambda}-3\sqrt{1+\mathrm{e}^{-\lambda}(34+\mathrm{e}^{-\lambda})}}(\mathrm{e}^{-\lambda u}+\mathrm{e}^{-\lambda(1-u)})\Bigg] \tag{VII.19}$$

The value at zero

$$\Delta(0) = -\frac{3 + \mathrm{e}^{\lambda}(3+\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}})}{2\lambda^2(7+\mathrm{e}^{\lambda}(3\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}}-1))} \tag{VII.20}$$

is always negative for $\lambda \geq 0$, thus the FP is unphysical for our problem in $d = 4 - \epsilon$. The combination yielding the corrections to the critical force

$$f_c = \frac{-1 + \mathrm{e}^{\lambda}(7+\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}})}{2\lambda(7+\mathrm{e}^{\lambda}(3\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}}-1))} \tag{VII.21}$$

is always positive for $\lambda \geq 0$.

(iv) The random periodic fixed point RP has:

$$\Delta(u) = \frac{1}{\lambda^2}\Bigg[-\frac{1 + 54\mathrm{e}^{-\lambda} + 5\mathrm{e}^{-2\lambda} + (1+5\mathrm{e}^{-\lambda})\sqrt{1+\mathrm{e}^{-\lambda}(34+\mathrm{e}^{-\lambda})}}{8(1-5\mathrm{e}^{-\lambda}(\mathrm{e}^{-\lambda}-8)))} + \frac{2}{(1-7\mathrm{e}^{-\lambda}+3\sqrt{1+\mathrm{e}^{-\lambda}(34+\mathrm{e}^{-\lambda})}}(\mathrm{e}^{-\lambda u}+\mathrm{e}^{-\lambda(1-u)})\Bigg] \tag{VII.22}$$

$$\Delta(0) = \frac{3 - \mathrm{e}^{\lambda}(-3+\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}})}{2\lambda^2(-7+\mathrm{e}^{\lambda}(1+3\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}}))} \tag{VII.23}$$

$$\begin{aligned} f_c &\sim -(\Delta'(0^+) + \lambda\Delta(0)) \\ &= \frac{-7 + \mathrm{e}^{\lambda}(1+\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}})}{2\lambda(-7+\mathrm{e}^{\lambda}(1+3\sqrt{1+\mathrm{e}^{-2\lambda}+34\mathrm{e}^{-\lambda}}))} \end{aligned} \tag{VII.24}$$

Both quantities $\Delta(0)$ and f_c are positive for all $\lambda \geq 0$, thus this fixed point is physical.

The fixed point RP is the continuation of the fixed point (VII.3) at $\lambda = 0$: Note that apart from a constant only the term $u(1-u)$ survives from the exponential functions. Like the fixed point at $\lambda = 0$, it is attractive in one direction (towards the fixed point SAP) and repulsive in another (towards large $\Delta(u)$). It is thus a critical fixed point. One can argue that any perturbation which leads to SAP is unphysical, since at some scale $\Delta(0)$ becomes negative. Since we did not find any strong reason why the system would be exactly on this critical surface, it is more likely that this FP represents a critical regime which lies on the boundary of the physical domain. It is however interesting that its analytic form can be obtained. In particular one can compute correlation functions exactly at RP.

An important question is whether there are fixed points outside of the exponential subspace considered above. Let us give a few general properties. First the flow equations and fixed

point conditions near $u = 0$

$$\partial_\ell \tilde{\Delta}(0) = \epsilon\tilde{\Delta}(0) + 4\hat{\lambda}^2\tilde{\Delta}(0)^2 - \tilde{\Delta}'(0)^2 + 2\hat{\lambda}\tilde{\Delta}(0)\tilde{\Delta}'(0^+)$$
$$\partial_\ell \tilde{\Delta}'(0^+) = \tilde{\Delta}'(0^+)(\epsilon + 2\hat{\lambda}\tilde{\Delta}'(0^+) + 6\hat{\lambda}^2\tilde{\Delta}(0) - 3\Delta''(0^+)) \quad \text{(VII.25)}$$

and the flow equation for $\int \tilde{\Delta}$

$$\partial_\ell \int_0^1 \mathrm{d}u\, \tilde{\Delta}(u) = \Big(\epsilon + 2\hat{\lambda}\tilde{\Delta}'(0^+) + 2\hat{\lambda}^2\tilde{\Delta}(0)\Big)\int_0^1 \mathrm{d}u\, \tilde{\Delta}(u) + 2\hat{\lambda}^2\int_0^1 \mathrm{d}u\, \tilde{\Delta}(u)^2 \quad \text{(VII.26)}$$

shows that starting from $\int \tilde{\Delta} = 0$, a positive value for $\int \tilde{\Delta}$ is generated in the early stage of the RG. If there is a fixed point value for $\int \tilde{\Delta}$ it must be equal to

$$\int_0^1 \mathrm{d}u\, \tilde{\Delta}^*(u) = -\frac{2\hat{\lambda}^2\int_0^1 \mathrm{d}u\, \tilde{\Delta}^*(u)^2}{\epsilon + 2\hat{\lambda}\tilde{\Delta}^{*\prime}(0^+) + 2\hat{\lambda}^2\tilde{\Delta}^*(0)} \ . \quad \text{(VII.27)}$$

For small λ at least this appears to be negative and of $O(\lambda^2)$. From the flow-equation for $\Delta'(u)$

$$\partial_\ell \tilde{\Delta}'(u) = -\tilde{\Delta}'''(u)\Big(\tilde{\Delta}(u) - \tilde{\Delta}(0)\Big) + \tilde{\Delta}'(u)\Big[\epsilon + 2\hat{\lambda}\tilde{\Delta}'(0^+) + 2\hat{\lambda}^2\tilde{\Delta}(0) + 4\hat{\lambda}^2\tilde{\Delta}(u) - 3\Delta''(u)\Big]$$

one sees that the behaviour at large u/λ must be exponential. It seems that there are no non-exponential fixed points.

The runaway flow will be discussed in the next section.

VIII. RANDOM FIELD DISORDER

Let us now consider non-periodic functions. The main problem with the natural rescaling of $u = u'\mathrm{e}^{\zeta l}$ as in (VI.22) is that $\tilde{\lambda}$ grows exponentially, and no fixed point can be found. Let us therefore study (VII.6)–(VII.8) setting the rescaling factor $\zeta = 0$. Again we consider the invariant subspace of exponential functions, parameterized by

$$\tilde{\Delta}(u) = \frac{1}{\hat{\lambda}^2}\epsilon f(u\hat{\lambda}) \quad \text{(VIII.1)}$$
$$f = a + b\mathrm{e}^{-u} \quad \text{(VIII.2)}$$

for $u > 0$. Note that we have put the coefficient $c = 0$, since we are not interested in solutions growing exponentially in u. The flow is

$$\partial a = a + 4a^2 \quad \text{(VIII.3)}$$
$$\partial b = b(1 + 6a + b) \ . \quad \text{(VIII.4)}$$

The physical requirements now read

$$\Delta(0) \sim a + b > 0 \quad \text{(VIII.5)}$$
$$f_c \sim -a > 0 \ . \quad \text{(VIII.6)}$$

So it is natural to look in the regime

$$b > -a \geq 0 \quad \text{(VIII.7)}$$

There is again the fixed point

$$f(x) = -\frac{1}{4} + \frac{1}{2}\mathrm{e}^{-x} \quad \text{(VIII.8)}$$

which is the infinite $\hat{\lambda}$ limit of the fixed point RP of the previous section. Since $f(x)$ does not go to zero at infinity as is expected for random field disorder, and since it is unstable along the line $a = -1/4$ it is unlikely to have any physical relevance for the anisotropic depinning class. The other fixed point is

$$f(x) = -\mathrm{e}^{-x} \quad \text{(VIII.9)}$$

which has the wrong sign. One clearly has runaway-flows within the exponential subspace.

We have examined the flow of the FRG numerically. For all initial conditions considered, which were not exactly at one of the fixed points mentioned above, we found the solution to explode at some finite scale, a phenomenon which is known as the *Landau pole*. One issue is to identify the corresponding direction in functional space. This issue is related to fixed points in $d = 4 + \epsilon$ dimensions which we now briefly address. The diagram for $4+\epsilon$ is obtained by changing $\Delta \to -\Delta$ and $\partial_\ell \to -\partial_\ell$. This means to replace $a \to -a$ and $b \to -b$ on figure VII.1 as well as inverting the direction of all arrows. U then controls the boundary between the strong coupling regime of KPZ and the Gaussian fixed point G; SAP between localization (attractive polymers); the Gaussian fixed point is multi-critical and RP between branched polymers and Gaussian. For the random field case we now have

$$\Delta(u) = \frac{1}{\hat{\lambda}^2}\epsilon f(u\hat{\lambda}) \quad \text{(VIII.10)}$$

The fixed point RP gives

$$f(x) = \frac{1}{4} - \frac{1}{2}\mathrm{e}^{-x} \quad \text{(VIII.11)}$$

and the fixed point U is

$$f(x) = \mathrm{e}^{-x} \ , \quad \text{(VIII.12)}$$

which has the correct sign. It has a vanishing critical force, but is a good candidate for the critical behaviour between the Gaussian phase and the strong coupling KPZ phase.

Let us now study the runaway flow for $d = 4 - \epsilon$. Suppose that $\Delta_1(u)$ is the solution of the $(4 + \epsilon)$-dimensional flow equation at $\epsilon = 1$. Then

$$\Delta_\ell(u) := g_\ell\Delta_1(u) \quad \text{(VIII.13)}$$

leads to the flow-equation for the amplitude g_ℓ

$$\partial g_\ell = \epsilon g_\ell + g_\ell^2 \ . \quad \text{(VIII.14)}$$

For the RF-case one has one such point at the boundary of the physical domain, as can be seen from the flow-equations

$$a = 0 \quad \text{(VIII.15)}$$
$$\partial b = b + b^2 \quad \text{(VIII.16)}$$

Also not that since this mode explodes after a finite renormalization time, it is difficult to avoid. However, we have not yet completely ruled out another scenario, where at least some trajectories have exponential growth. Making the ansatz

$$\Delta_l(u) = \epsilon(\mathrm{e}^{2\zeta l} f(u) + g(u)) \ , \qquad \text{(VIII.17)}$$

this requires to find a solution to the β-function at $\epsilon = 0$, which we write symbolically

$$\beta(f, f) = 0 \ . \qquad \text{(VIII.18)}$$

One can check that near zero such a solution is in principle possible. There is a solution, which vanishes at $u = u^* = 1.39895$ (for $\lambda = 1$) and becomes negative beyond. One can argue that one needs it only up to $u = u_0 < u^*$, since the linear term can no longer be neglected when $f(u)$ approaches zero. Noting $r = (1+\sqrt{5})/4$ one has $f'(0) = -1/r$, $f''(0) = (1+2r^2)/(3r^2)$. In this scenario ζ is determined together with g. It is unclear how this carries to higher orders, since it seems to require that $f(u)$ is also solution of the β-function at $\epsilon = 0$. This is however exactly what happens in the case $\lambda = 0$ with the constant shift $\Delta(0)$. Although numerics does not seem to confirm it, it is hard to disprove. A question which remains to be answered is what the basin of attraction of runaway growth and eventually of exponential growth are.

IX. GENERAL ARGUMENTS FROM THE COLE-HOPF REPRESENTATION AND BRANCHING PROCESSES

In the Cole-Hopf representation, it is easy to see why the exponential manifold is preserved to all orders. Let us insert

$$\Delta(u) = \frac{1}{\lambda^2}\left(a + b\,\mathrm{e}^{-\lambda u} + c\,\mathrm{e}^{\lambda u}\right) \qquad \text{(IX.1)}$$

in (VI.4). The complicated functional disorder takes a very simple polynomial form

$$\mathcal{S} = \int_{xt} \hat{Z}_{xt}\left(\hat{\eta}\partial_t - \partial_x^2\right) Z_{xt} - \int_x \int_{t<t'} \hat{Z}_{xt}\hat{Z}_{xt'}\left(aZ_{xt}Z_{xt'} + bZ_{xt}^2 + cZ_{xt'}^2\right) \ . \qquad \text{(IX.2)}$$

Note that we have ordered the vertices in time to distinguish between b and c taking correctly into account that the full correlator for the present non-analytic e.g. random field, problem

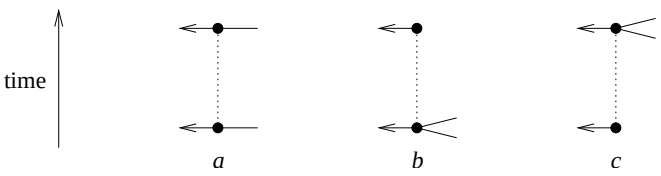


FIG. IX.1: The three vertices proportional to a, b and c in $\int_{x,t<t'} \hat{Z}_{xt}\hat{Z}_{xt'}\left(aZ_{xt}Z_{xt'} + bZ_{xt}^2 + cZ_{xt'}^2\right)$.

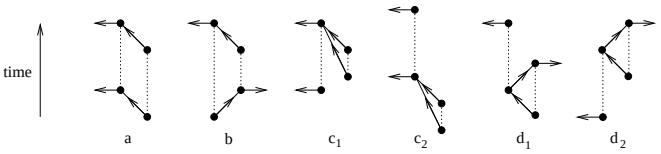


FIG. IX.2: Diagrams correcting the disorder in the branching-representation.

is (IX.1) with u replaced by $|u|$ (if (IX.1 holded as an analytic function there would be no distinction between b and c, "thus no arrow of time").

The vertices presented on figure IX.1 can be interpreted as branching processes, and we shall thus call this form *branching representation*. Let us show how one reproduces the flow-equations (VII.6)- (VII.8). In the time-ordered representation, diagrams a to d of figure VI.1 have the form given on figure IX.2. To simplify notations, we set $\lambda = 1$. Then

$$\Delta(u) = a + b\mathrm{e}^{-u} + c\mathrm{e}^{u} \qquad \text{(IX.3)}$$
$$\Delta'(u) = -b\mathrm{e}^{-u} + c\mathrm{e}^{u} \qquad \text{(IX.4)}$$
$$\Delta''(u) = b\mathrm{e}^{-u} + c\mathrm{e}^{u} \qquad \text{(IX.5)}$$
$$\Delta(0) = a + b + c \qquad \text{(IX.6)}$$
$$\Delta'(0^+) = c - b \qquad \text{(IX.7)}$$
$$\Delta''(0^+) = b + c \qquad \text{(IX.8)}$$

The diagrams have the following contributions

$$\delta\Delta^{\mathrm{a}}(u) \longrightarrow \begin{cases} \delta a = aaI \\ \delta b = abI \\ \delta c = acI \end{cases} \qquad \text{(IX.9)}$$

$$\delta\Delta^{\mathrm{b}}(u) \longrightarrow \begin{cases} \delta a = (aa + 4bc)I \\ \delta b = 2baI \\ \delta c = 2acI \end{cases} \qquad \text{(IX.10)}$$

$$\delta\Delta^{\mathrm{c}}(u) \longrightarrow \begin{cases} \delta a = 0 \\ \delta b = 2b(a+b+c)\frac{I}{2} & (\text{from } \mathrm{c}_2) \\ \delta c = 2c(a+b+c)\frac{I}{2} & (\text{from } \mathrm{c}_1) \end{cases} \qquad \text{(IX.11)}$$

$$\delta\Delta^{\mathrm{c}}(u) \longrightarrow \begin{cases} \delta a = 2a(a+2b) & (\text{from } \mathrm{d}_1 \text{ and } \mathrm{d}_2) \\ \delta b = 2b(a+b+c)I & (\text{from } \mathrm{d}_2) \\ \delta c = 2c(a+b+c)I & (\text{from } \mathrm{d}_1) \end{cases} \qquad \text{(IX.12)}$$

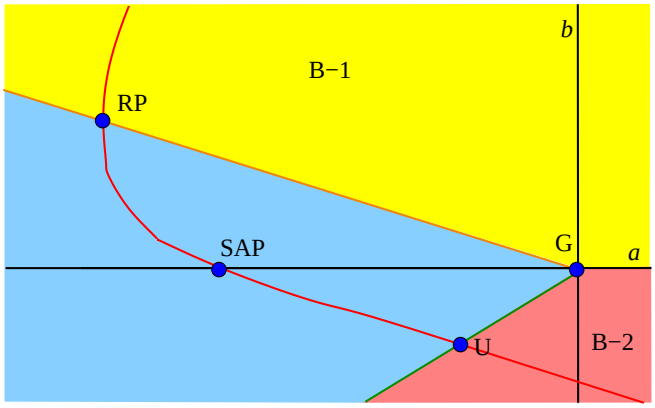
Note that the factors of 2 come in general from contracting Z_{xt}^2. The non-trivial factor of $\frac{1}{2}$ is due to the fact that the two right-most points in c_1 and c_2 are time-ordered. To relate the integral to I, one can first symmetrize (yielding the factor of $\frac{1}{2}$) and then freely integrate over time. Also note that only the last diagram, $\mathrm{d}_1 + \mathrm{d}_2$ contributes to the asymmetry between b and c.

In the same way, one can reproduce the corrections to η. The only vertex in (IX.2) which contributes at leading order is the one proportional to a: b does not allow for a contraction and c will have both $\hat{Z}$ and Z at the same point, thus only corrects the critical force. a leads to

$$\hat{Z_{xt}} Z_{xt'} R_{x,t-t'} \qquad \text{(IX.13)}$$

and after a gradient-expansion following the procedure described after (VI.10) we have

$$\hat{Z}_{xt}\dot{Z}_{xt}(t'-t)R_{x,t'-t} \ . \qquad \text{(IX.14)}$$



[t]

FIG. IX.3: The three phases of the flow-diagrams on figure VII.1.

Integration over t' leads to the correction to $\hat{\eta}$

$$\frac{\delta\hat{\eta}}{\hat{\eta}} = -aI \ , \tag{IX.15}$$

which is the same one obtained from (VI.15) using (IX.6) and (IX.8).

Let us now exploit this representation further: It is immediately clear, that one cannot generate $\mathrm{e}^{-2\lambda u}$ which corresponds to

$$\int_x \int_{t<t'} \tilde{Z}_{xt}\tilde{Z}_{xt'}\frac{Z^3_{xt'}}{Z_{xt}} \tag{IX.16}$$

or any other such fractions. This shows that the space of functions spanned by (IX.1) is indeed closed to *all orders in perturbation theory*. Also there is no renormalization to $\hat{\lambda}$, whereas a correction to the elasticity $\int \hat{Z}\partial_x^2 Z$ is allowed, and indeed shows up at 2-loop order.

Finally, note that the domain of variation of u, in the periodic case yields an action with multiplicative periodicity in Z, but this does not seem to be important here.

Let us now discuss the relation of our findings with self-avoiding polymers, branching processes and directed percolation.

First, on figure IX.4 we have drawn a diagram corresponding to the perturbation expansion of fixpoint SAP, which is the only fully attractive fixed point in the phase-diagram IX.3. One easily checks that by integrating over times, one recovers a standard ϕ^4-perturbation theory, as depicted on figure IX.4.

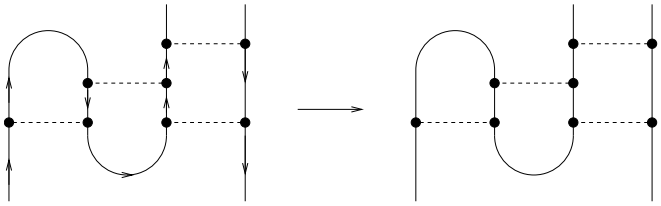


FIG. IX.4: A self-avoiding polymer.

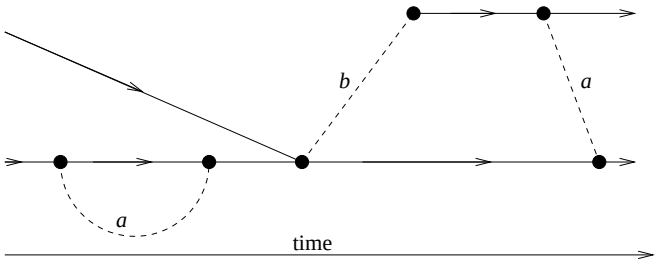


FIG. IX.5: Self-avoidance plus branching.

By first integrating over the momenta, one recovers the perturbation expansion of self-avoiding polymers. It is well known, that this fixed point is stable. In terms of particles, it can be interpreted as the world-lines of diffusing particles, which are not allowed to visit twice the same point in space. Let us now add some terms b and c. In interesting limit is $\lambda = \infty$, since there c can be set to zero. Adding a term propotional to b, the diffusing particle is allowed to branch. More precisely, two particles can meet at a time t. Then one of the particles becomes inactive, before reappearing at some later time $t' > t$. One can interprete this as

$$\mathrm{A} + \mathrm{A} \longrightarrow \mathrm{A} + \mathrm{B} \tag{IX.17}$$
$$\mathrm{B} \longrightarrow \mathrm{A} \ . \tag{IX.18}$$

Particle B is completely inert, and does not diffuse away from its position of creation, before it decays into A again. However note that any point in the future is equally likely to see B change back to A. This is very different from e.g. a spontaneous decay. This process is depicted on figure IX.5. b can either come with a positive sign, or with a negative sign. If the sign is positive, this can be interpreted as the two particles attracting to make the branching-process. It is clear that after some critical threshold, the process and such the phase SAP becomes unstable, since the induced attraction between particles tends to make them collapse at the same point in space and then annihilate. This leads to the runaway-flow in phase B-1 on figure IX.3. On the other hand, for negative b, even a large $|b|$ does not lead to a collapse. This is why on figure VII.1 in the case of $\lambda = \infty$ the SAP-phase with $a < 0$ extends to $b \to -\infty$. This remains valid for finite λ if in the full flow-equations (VII.6) to (VII.8) $c = 0$ is set from the beginning. However the situation for finite λ discussed in (VII.10)–(VII.11) maps in the language of branching processes to a finite initial ratio between a and b, parameterized by $c = b\mathrm{e}^{-\lambda}$, which remains uncorrected under renormalization. The second branching-process c being present, it can render the phase SAP unstable to B-2. The vertex c is interpreted as

$$\mathrm{A} \longrightarrow \mathrm{C} \tag{IX.19}$$
$$\mathrm{A} + \mathrm{C} \longrightarrow \mathrm{A} \ . \tag{IX.20}$$

This means that a particle A becomes spontaneously inactive at some time t. It remains at position x until at some time $t' > t$ another particle A comes by to free it. The reduced flow-equations for the combined situation are given in (VII.10) and

(VII.11), and lead to the instability of the phase SAP induced by the branching process c.

X. LONG-RANGE ELASTICITY

Let us now study anisotropic depinning in the case of a manifold with long range (LR) elasticity, the elastic force in (II.1) being, in Fourier:

$$cq^2 u_{q,t} \to (c_\alpha |q|^\alpha + cq^2) u_{q,t} \tag{X.1}$$

There are now two elastic constants, the LR one c_α and the short range (SR) one c_2, and we thus define the two dimensional regularization-parameters,

$$\epsilon = 2\alpha - d \tag{X.2}$$
$$\kappa = 2 - \alpha \ . \tag{X.3}$$

The case of most interest corresponds to the parameters for the contact line depinning, $d = 1$, $\alpha = 1$, i.e. $\epsilon = \kappa = 1$.

Power counting shows that disorder is perturbatively relevant below the critical dimension $d < d_c = 2\alpha$. Disorder is thus relevant for the contact line case but the crucial question we investigate here is whether the KPZ terms are important there. Study of the contact line depinning is usually performed within a $d = 2\alpha - \epsilon$ expansion (see Ref. [19]) at fixed α. This is the solid line in figure X.1. However as soon as elasticity is long range ($\kappa > 0$) simple power counting shows that the KPZ terms are perturbatively irrelevant for d near d_c. Working at fixed α as e.g. $\alpha = 1$ is thus not the best method. One alternative is to study the vicinity of the point $d = 4$, $\alpha = 2$ and perform a *double expansion* both for ϵ and κ small. The idea is to determine a line $d_{\rm KPZ}(\alpha)$ in the (α, d)-plane below which the KPZ terms are important and must be included. One can determine this line near the point $d = 4$, $\alpha = 2$ and, by extrapolation, find on which side of the line lies the interesting case $\epsilon = \kappa = 1$ (see figure X.1.).

Through the replacement $q^2 \to q^\alpha$ in the propagators of the 1-loop diagrams of section V, it is easy to derive the 1-loop FRG equations for a general α, in presence of a KPZ term as in (II.1). First one obtains as usual that c_α is uncorrected to all orders, and thus we set $c_\alpha = 1$ in the following. Defining the dimensionless couplings

$$\tilde\lambda = \lambda \Lambda_\ell^{\kappa - \zeta} \tag{X.4}$$
$$\tilde\Delta(u) = \Lambda_\ell^{2\zeta - \epsilon} \Delta(u \Lambda_\ell^{-\zeta}) \tag{X.5}$$

within a Wilson scheme where $\Lambda_\ell = \Lambda {\rm e}^{-\ell}$ is the running UV cutoff, we find the flow equations:

$$\partial_\ell \ln \tilde\lambda = \zeta - \kappa - \tilde\lambda^2 \tilde\Delta(0) - \tilde\lambda \tilde\Delta'(0^+) \tag{X.6}$$
$$\partial_\ell \tilde\Delta(u) = (\epsilon - 2\zeta)\tilde\Delta(u) + u\zeta \tilde\Delta'(u) + 2\tilde\lambda^2 \tilde\Delta(u)^2 - \tilde\Delta'(u)^2 - \tilde\Delta''(u)(\tilde\Delta(u) - \tilde\Delta(0)) \ . \tag{X.7}$$

We work to lowest order in both ϵ and κ (and thus neglect the small changes in the coefficients of order κ) and define the ratios

$$\hat\kappa = \frac{\kappa}{\epsilon} \ , \qquad \zeta_1 = \frac{\zeta}{\epsilon} \ . \tag{X.8}$$

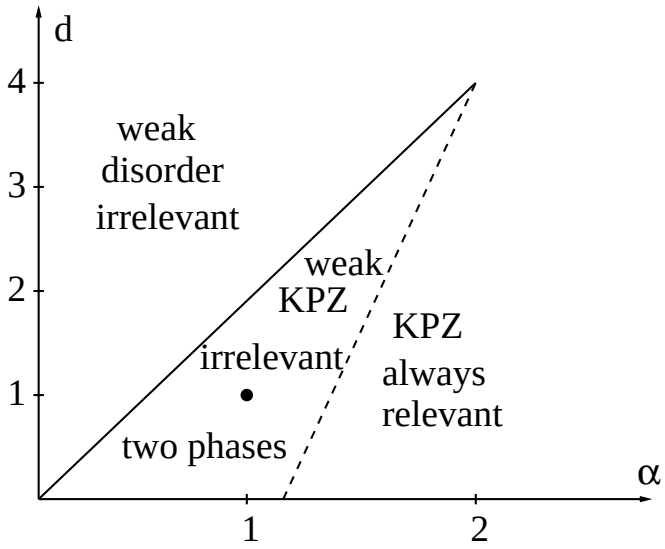


FIG. X.1: Phase-diagram in the (α, d)-plane. The solid line is $\epsilon = 0$. The dashed line corresponding to (X.11) separates the domain where an infinitesimal KPZ-term is relevant from those where it is irrelevant. At order ϵ^2 this line will bend to the left, but should not cross the point $d = 1$ and $\alpha = 1$.

Of course the SR part of the elasticity is corrected:

$$\partial_\ell \ln \left(\frac{c_2}{c_\alpha} \right) = -\kappa - \tilde\lambda \tilde\Delta'(0^+) - \tilde\lambda^2 \tilde\Delta(0) \tag{X.9}$$

and we will focus on situations where it is irrelevant (a condition which must be checked a posteriori).

Note that since the LR-elasticity is uncorrected, the dimensionless variables, contrary to (V.5), are not divided by c_2 but by $c_\alpha = 1$ and their RG-equations thus do not contain additional contributions from the corrections to c_2. As a result $\tilde\lambda$ has now non-trivial corrections and the Cole-Hopf mapping no longer works, or has to be defined with a flowing $\tilde\lambda$.

Before embarking on a more detailed analysis let us indicate the main behaviour we expect from Eqs. (X.6) and (X.7). For $\tilde\lambda = 0$ one has the usual anisotropic depinning fixed point studied in Ref. [19]. One can perform a linear stability analysis of this FP for small $\tilde\lambda$. From (X.6) one finds that linear stability holds provided

$$\kappa > \zeta_{\rm iso} = \frac{\epsilon}{3} + O(\epsilon^2) \tag{X.10}$$

for the non-periodic problem, and $\zeta_{\rm iso} = 0$ for the periodic case. This is the dashed line

$$d_{\rm KPZ}(\alpha) = 5\alpha - 6 \tag{X.11}$$

represented in Fig. X.1. For $d > d_{\rm KPZ}(\alpha)$ the isotropic FP is stable. This is the case for the contact line depinning. On the other hand, one expects from Eqs. (X.6) and (X.7) that even then, if the value of $\tilde\lambda$ is large enough, the RG may flow again to KPZ strong coupling. This is the same run-away flow as for SR elasticity. Both fixed points should be separated by an instable fixed point, of which we will show that it is attainable

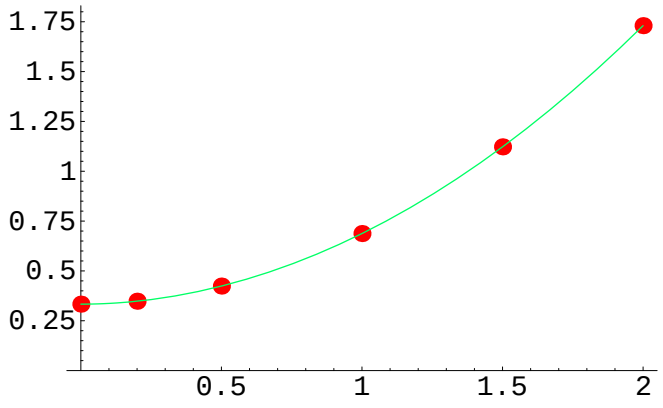


FIG. X.2: $\zeta_1 = \zeta/\epsilon$ as a function of $\tilde{\lambda}$ for LR elasticity. Note that $\zeta_1(-\tilde{\lambda}) = \zeta_1(\tilde{\lambda})$.

perturbatively. Thus for $d > d_{\rm KPZ}(\alpha)$ we expect, and find below, *two phases*: one where $\tilde{\lambda}$ flows to zero (denoted the ID phase) and one where the KPZ terms are important (the AD phase). The question is thus to determine the basin of attraction of each phase and the critical (i.e. repulsive) fixed point which separates the two phases. Quite generally one expects a critical value $\tilde{\lambda}^*$ below which $\tilde{\lambda}$ flows to zero and above which it runs away.

A simple argument, confirmed by the more detailed analysis presented below, allows to estimate $\tilde{\lambda}^*$ for small values of ϵ, i.e. near $d \approx d_{\rm KPZ}(\alpha)$. Since $\tilde{\Delta}(u)$ changes by order $\tilde{\lambda}^2$ for small $\tilde{\lambda}$, the Eq (X.6) gives the critical value:

$$\tilde{\lambda}^*_{\hat{\kappa}} = \frac{\kappa - \zeta_{\rm iso}}{|\tilde{\Delta}'(0^+)|} \qquad \text{(X.12)}$$

for small $\hat{\kappa} - (\zeta_{\rm iso}/\epsilon)$, where $\tilde{\Delta}'(0^+) = O(\epsilon)$ takes its (negative) value for the isotropic depinning fixed point.

Although analysis of the full FRG-flow requires numerics, one can obtain some analytical information on the transition between the isotropic phase and the anisotropic strong-KPZ-coupling phase.

A. Non-periodic systems

Let us start with non-periodic systems and search for a perturbative fixed point of the system (X.6), (X.7). Interestingly in that case, there is one, whose properties depend continuously on $\hat{\kappa} = \kappa/\epsilon$.

For each value of $\hat{\kappa}$ we can determine the FP through the following construction. Given the reparametrization invariance (V.6) of (X.7), we can always set

$$\Delta(0) = \epsilon \ , \qquad \text{(X.13)}$$

and for each fixed value of $\tilde{\lambda}$ search numerically for a fixed point function of Eq (X.7) which decreases at infinity (short range pinning force correlations of the random field type). Interestingly we find, through explicit numerical integration, that there is always one such solution, denoted by $\Delta^*_{\tilde{\lambda}}(u)$, if one tunes ζ to a value noted $\zeta(\tilde{\lambda})$. The resulting curve $\zeta_1(\tilde{\lambda}) := \zeta(\tilde{\lambda})/\epsilon$ is plotted in Fig. X.2. It starts at $\zeta(\tilde{\lambda} = 0) = 1/3$ (the isotropic value) and increases as $\tilde{\lambda}$ increases.

Considering the fixed point equation of (X.7) at $u = 0$ using (X.13) shows that the value of $\Delta'(0^+)$ is a simple expression:

$$\Delta^{*\prime}_{\tilde{\lambda}}(0^+) = -\epsilon\sqrt{1 - 2\zeta_1(\tilde{\lambda}) + 2\tilde{\lambda}^2} \ . \qquad \text{(X.14)}$$

Thus, reporting this value, as well as $\Delta(0) = \epsilon$ in Eq. (X.6) we see that for each value of $\hat{\kappa}$ we can determine the value of $\tilde{\lambda}$ by solving the equation

$$\hat{\kappa} = f(\tilde{\lambda}) \equiv \zeta_1(\tilde{\lambda}) - \tilde{\lambda}^2 + \tilde{\lambda}\sqrt{1 - 2\zeta_1(\tilde{\lambda}) + 2\tilde{\lambda}^2} \ . \qquad \text{(X.15)}$$

Denoting $\tilde{\lambda}^*_{\hat{\kappa}}$ this solution, we obtain the FP function $\Delta^*_{\tilde{\lambda}^*_{\hat{\kappa}}}(u)$ and the value of the roughness exponent $\zeta_1(\hat{\kappa}) := \zeta_1(\tilde{\lambda}^*_{\hat{\kappa}})$. Comparing (X.6) and (X.9) we note that the SR elastic part is indeed irrelevant as soon as $\zeta > 0$, and thus the above analysis is consistent.

The curves $f(\tilde{\lambda})$, $f^{-1}(\hat{\kappa})$ and the resulting $\zeta_1(\hat{\kappa}) = \zeta(\tilde{\lambda}^*_{\hat{\kappa}}) = \zeta(f^{-1}(\hat{\kappa}))$ are plotted in Fig. X.3 and X.4 respectively.

One sees that there is a solution with a positive $\tilde{\lambda}^*_{\hat{\kappa}}$ only if $\hat{\kappa} > \kappa_c = \frac{1}{3}$ consistent with the linear stability analysis given above. The roughness exponent associated to this FP then increases continuously, as shown on figure X.4, from $\zeta_1 = \frac{1}{3}$ to larger values as $\hat{\kappa}$ increases beyond $\hat{\kappa}_c$. In particular, since we are interested in the point $\kappa = 1, \epsilon = 1$ of the (α, d)-plane (see figure X.1) it is worthwile to give the extrapolation:

$$\zeta(\hat{\kappa} = 1) = 0.7 \ , \qquad \text{(X.16)}$$

and $\tilde{\lambda}_{\hat{\kappa}=1} = 1.037$, values which give the simplest extrapolation for the contact-line depinning. One should however not expect too high a precision from this crude estimate.

Thus we have found a non-trivial FP for this problem. It continuously depends on κ/ϵ and exists only for $\kappa/\epsilon > 1/3$.

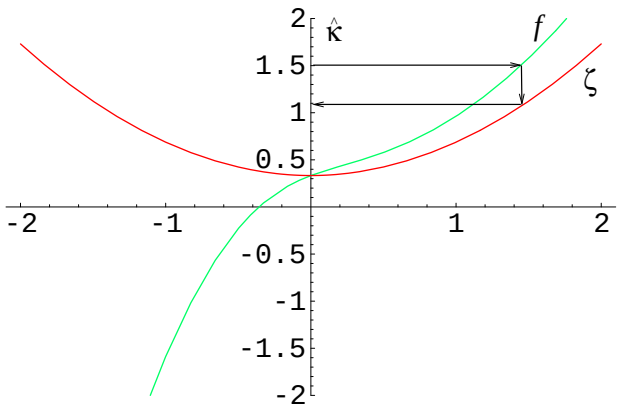


FIG. X.3: The function $f(x)$ (green/light) and $\zeta_1(x)$ (red/dark) defined in the text. One can read off ζ_1 as a function of $\hat{\kappa}$ as follows: The curve $y = f(x)$ yields $\tilde{\lambda}_{\hat{\kappa}}$ (x axis) from $\hat{\kappa}$ (y axis) and in turn, one reads $\zeta_1(\hat{\kappa})$ (y axis) from $\tilde{\lambda}_{\hat{\kappa}}$ (x axis) using the curve $y = \zeta_1(x)$. This is indicated by the arrows. The result is plotted in Fig. X.4

The simplest scenario is that this FP is associated with the critical behaviour at the transition between the phase where KPZ is irrelevant (isotropic depinning) and the phase where KPZ grows (anisotropic depinning). To confirm it and check that this FP has only one unstable direction one needs a more detailed numerical analysis. Note that this is also indicated by an adiabatic approximation considering (X.6) alone and assuming that the disorder does not vary, which yields that the FP is repulsive if $f'(x) > 0$ and attractive if $f'(x) < 0$.

B. Periodic systems

In the periodic case, since $\zeta = 0$ is requested at any FP, we see that we cannot enforce the SR-elasticity coefficient c_2 to scale to 0 under renormalization, since the FP condition on $\tilde{\lambda}$ implies that (X.9) vanishes. However if we start with a small ratio of c_2/c_α or if the flow is such that this ratio gets small before we reach the fixed point, then it is legitimate to neglect the effect of c_2. We restrict our analysis to that case, and study equations (X.6), (X.7) searching for a FP. A more detailed numerical analysis of the flow equation is feasable.

It can easily be seen that the form

$$\Delta(u) = \frac{\epsilon}{\tilde{\lambda}^2}(a + be^{-u\tilde{\lambda}} + ce^{u\tilde{\lambda}}) \tag{X.17}$$

is not exactly preserved by the flow anymore (e.g. $\partial_l \Delta(u)$ yields a term proportional to $ue^{-u\tilde{\lambda}}\partial_l\tilde{\lambda}_l$ through variations of $\tilde{\lambda}_l$ which here flows). One can still however search for exponential fixed points since then $\tilde{\lambda}_l$ does not flow. (X.7) yields the conditions

$$a + 2a^2 + 4bc = 0 \tag{X.18}$$
$$b + 4ab + b^2 + bc = 0 \tag{X.19}$$
$$c + 4ac + c^2 + bc = 0 \tag{X.20}$$

and we can set $c = be^{-\tilde{\lambda}}$ to ensure periodicity $\Delta(u) = \Delta(1-u)$. We obtain the following fixed points:

$$b = \frac{\pm e^{\tilde{\lambda}}}{\sqrt{1 + 34e^{\tilde{\lambda}} + e^{2\tilde{\lambda}}}} \tag{X.21}$$
$$a = -\frac{1}{4} \mp \frac{1 + e^{\tilde{\lambda}}}{4\sqrt{1 + 34e^{\tilde{\lambda}} + e^{2\tilde{\lambda}}}} \tag{X.22}$$

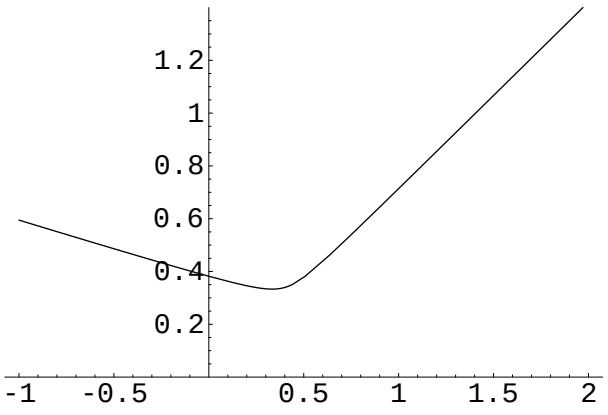


FIG. X.4: $\zeta_1(\hat{\kappa})$, using $\zeta(\tilde{\lambda})$ and $\tilde{\lambda} = f^{-1}(\hat{\kappa})$

as well as two others $(a = -\frac{1}{2}, b = 0)$ and $(a = 0, b = 0)$. The corresponding FP condition for $\tilde{\lambda}$ gives

$$0 = -\hat{\kappa} - a - 2be^{-\tilde{\lambda}} \ . \tag{X.23}$$

The FP with a positive b is the one of interest. It is again presumably the boundary between the zero and strong KPZ phases. The value of $\tilde{\lambda}_{\hat{\kappa}}$ is given by the positive root of

$$4\hat{\kappa} = 1 + \frac{e^{\tilde{\lambda}} - 7}{\sqrt{1 + 34e^{\tilde{\lambda}} + e^{2\tilde{\lambda}}}} \ . \tag{X.24}$$

which reproduces (X.12), $\tilde{\lambda}_{\hat{\kappa}} \sim 6\hat{\kappa}$, to lowest order in $\hat{\kappa}$. One finds that $\tilde{\lambda}_{\hat{\kappa}}$ increases monotonically with $\hat{\kappa}$ and diverges $\tilde{\lambda}_{\hat{\kappa}} \to +\infty$ as $\hat{\kappa} \to \frac{1}{2}$. This suggests that for $\hat{\kappa} \geq \frac{1}{2}$ only the ID phase exist.

XI. CONCLUSION

In this paper we have reexamined the functional renormalization group approach to anisotropic depinning. This was mandatory since non-analytic renormalized disorder correlators were found to be crucial already for isotropic depinning and were neglected in previous approaches of AD.

Indeed we have shown that the non-analyticity of disorder arising beyond the Larkin length is crucial to generate the KPZ-term, a first explicit field theoretic demonstration of how these terms appear at depinning. The resulting anomalous terms in the β-function modify the flow compared to previous approaches in interesting ways. We found several non-trivial fixed points and for SR elasticity a Cole Hopf transformed theory which allows to simplify considerably perturbation theory and indicates that the KPZ coupling λ/c is uncorrected to all orders.

For LR-elasticity we have found the domains of parameters belonging to ID and AD respectively. We found that for the experimentally interesting case of contact-line depinning, two phases exist, ID and AD, and that the KPZ-coupling (i.e. the anisotropy) should be large enough for the AD class to apply (otherwise the ID exponents is expected [18, 26]). At the transition a larger value of $\zeta \approx 0.7\epsilon$ (with $\epsilon = 1$ for the contact line) is obtained. This scenario could be checked in a numerical simulation. To make the comparison with experiments more accurate one should consider the more involved structure for the KPZ terms unveiled in [25] but this can be done by methods similar to the one introduced here.

For SR-elasticity we have found interesting new fixed points. A bit disappointingly, they possess one unstable direction and thus correspond to transient or critical behaviour, and not to the asymptotic behaviour which instead is controlled by a runaway flow to a regime not perturbatively accessible by the present method. On the other hand, an encouraging result is that we found a class of disorder correlators (in the form of exponentials) which should be invariant to all orders. These correspond to a set of branching processes which look tantalizingly close to the ones introduced to describe reaction diffusion and directed percolation. More work is necessary to understand this simpler equivalent class of theories at strong

coupling, as they may contain the key to this conjectured connection between anisotropic depinning and directed percolation (in $d = 1+1$) and its generalizations in terms of blocking surfaces (in higher d) and ultimately an understanding of the upper critical dimension for this problem.

A posteriori, it is not suprising that the present approach yields again a flow to strong coupling KPZ, as it does in the thermal version of the problem [23, 24]. It is possible that as in the thermal problem another representation, as e.g. the directed polymer, better exposes the physics and in particular what is missed in the present approach. The corresponding formulation would be

$$Z(x,t) = \int_{y(t)=x} \mathcal{D}[y(\tau)] \exp\left[-\int_{t'}^{t} \mathrm{d}\tau \frac{1}{4T}\left(\frac{dy}{d\tau}\right)^2 + \frac{1}{T}V(y(\tau),\tau)\right] \quad \text{(XI.1)}$$

i.e. a directed polymer in a random potential but with the choice $T = \frac{1}{\hat{\eta}}$ and the additional self consistency condition:

$$V(y,\tau) = \frac{\hat{\lambda}}{c\hat{\eta}^2} F\left(y, \frac{1}{\hat{\lambda}} \ln Z(y,\tau)\right) \ , \quad \text{(XI.2)}$$

which relates the random potential to the pinning force and to the free energy of the directed polymer and makes the problem analytically far more complex. It may possess similar physics and thus be amenable to some extended FRG approach which would better account (as it does for the thermal problem) for the coarse grained correlations in the y direction a property clearly not taken into account by the present method, which treats correctly only correlations in the $\ln Z$ space.

Acknowledgments

It is a pleasure to thank M. Kardar, W. Krauth, A. Ludwig and A. Rosso for stimulating discussions.

APPENDIX A: DIAGRAMS

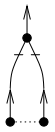
We use the following model setting $c = \eta = 1$ to simplify notations

$$S = \int_{x,t} \eta\hat{u}\dot{u} - c\hat{u}\Delta u - \lambda\hat{u}(\nabla u)^2 \quad \text{(A.1)}$$

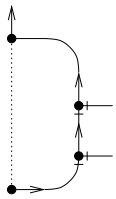
$$-\frac{1}{2}\int_{x,t,t'} \hat{u}_{xt}\hat{u}_{xt'}\Delta(u_{xt} - u_{xt'}) \quad \text{(A.2)}$$

One also has to specify a cut-off procedure. For convenience, we chose to put a mass-term. This is justified at 1-loop order since the results are universal, i.e. cut-off independent. At second order, one would have to be more careful and use, e.g. an external momentum IR cutoff.

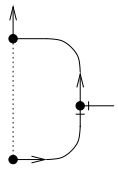
Many of the diagrams which we need are identical to the driven manifold problem at $\lambda = 0$. These diagrams are detailed in [19]. The *new* diagrams are



$$\begin{aligned} &= 2\int_{0<t<t'}\int_k \mathrm{e}^{-t(k^2+m^2)}\mathrm{e}^{-t'(k^2+m^2)}|t-t'|k^2 \\ &\qquad \times \Delta'(0^+)\hat{u}\dot{u} \\ &= 2\lambda\Delta'(0^+)\hat{u}\dot{u}\int_k\int_{t'} \mathrm{e}^{-t'(k^2+m^2)}\frac{k^2}{(k^2+m^2)^2} \\ &\qquad \times\left(\mathrm{e}^{-t'(k^2+m^2)} - 1 - (k^2+m^2)t'\right) \\ &= 2\lambda\Delta'(0^+)\hat{u}\dot{u} \\ &\quad \times\int_k \frac{k^2}{2(k^2+m^2)^3} - \frac{k^2}{(k^2+m^2)^3} + \frac{k^2}{(k^2+m^2)^3} \\ &= \lambda\Delta'(0^+)\hat{u}\dot{u}\int_k \frac{k^2}{(k^2+m^2)^3} \\ &= \lambda\Delta'(0^+)\hat{u}\dot{u}\int_k \frac{k^2}{(k^2+m^2)^3} \end{aligned} \quad \text{(A.3)}$$

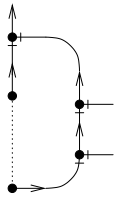


$$= -\frac{4}{d}\Delta'(0^+)\lambda^2\int \frac{k^2}{(k^2+m^2)^3}\,\hat{u}(\nabla u)^2 \quad \text{(A.4)}$$

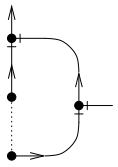


$$\begin{aligned} &= 2\Delta'(0^+)\lambda\int_k \frac{(k_\alpha+p_\alpha)p_\alpha}{((k+p)^2+m^2)(k^2+m^2)} \\ &\approx 2\Delta'(0^+)\lambda\int_k \frac{p^2}{(k^2+m^2)^2} - \frac{2(kp)^2}{(k^2+m^2)^3} \\ &= \frac{2(2-d)}{d}\Delta'(0^+)\lambda\int_k \frac{1}{(k^2+m^2)^2}\hat{u}\Delta u \end{aligned} \quad \text{(A.5)}$$

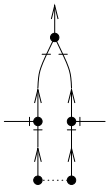
(Note that $\Delta \leftrightarrow -p^2$.) Dots indicate omitted subleading terms.



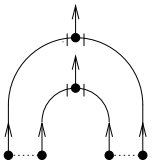
$$= -\frac{8}{d}\Delta(0)\lambda^3\int_k \left[\frac{k^4}{(k^2+m^2)^4} + \dots\right]\hat{u}(\nabla u)^2 \quad \text{(A.6)}$$



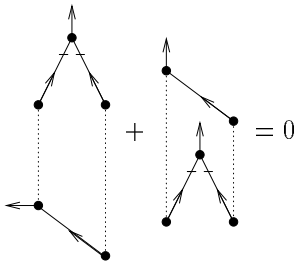
$$\begin{aligned} &= -4\Delta(0)\lambda^2\int_k \frac{k(k+p)\,(kp)}{(k^2+m^2)^2((k+p)^2+m^2)} \\ &= -4\Delta(0)\lambda^2\int_k \frac{(kp)^2}{(k^2+m^2)^3} - 2\frac{k^2(kp)^2}{(k^2+m^2)^4} \\ &= -\frac{4}{d}\Delta(0)\lambda^2\left(\int_k \frac{1}{(k^2+m^2)^2} + \dots\right)(\hat{u}\Delta u) \end{aligned} \quad \text{(A.7)}$$



$$= \frac{4}{d}\Delta(0)\lambda^3 \int_k \frac{k^4}{(k^2+m^2)^4}\, \hat{u}(\nabla u)^2 \qquad \text{(A.8)}$$



$$= \Delta(u_{xt}-u_{xt'})^2\lambda^2 \int_k \frac{(k^2)^2}{(k^2+m^2)^4}\hat{u}_{xt}\hat{u}_{xt'} \qquad \text{(A.9)}$$



$$+ \qquad = 0 \qquad \text{(A.10)}$$

Therefore, we have the following corrections to c, η, λ and δ (setting $I := \int_k \frac{1}{(k^2+m^2)^2}$, dropping finite terms in ϵ, but for the moment keeping the explicit d-dependence)

$$\delta c/c = \left[\frac{2(2-d)}{d}\Delta'(0^+)\lambda - \frac{4}{d}\Delta(0)\lambda^2\right] I \qquad \text{(A.11)}$$

$$\delta\eta/\eta = -\lambda\Delta'(0^+) I \qquad \text{(A.12)}$$

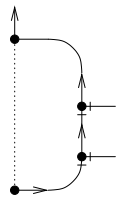
$$\delta\lambda/\lambda = \left[-\frac{4}{d}\lambda\Delta'(0^+) - \frac{8}{d}\Delta(0)\lambda^2 + \frac{4}{d}\Delta(0)\lambda^2\right] I \qquad \text{(A.13)}$$

$$\delta\Delta(u) = 2\Delta(u)^2\lambda^2 I \qquad \text{(A.14)}$$

The coupling-constant is $\hat{\lambda} := \lambda/c$. Note that its flow vanishes at leading order in $1/\epsilon$. We now check cancellations beyond the leading order. We use

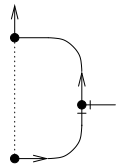
$$\int_k = A_d \int_0^\infty \mathrm{d}k k^{d-1} \qquad \text{(A.15)}$$

The two diagrams proportional to $\Delta'(0^+)$ are:



$$\longrightarrow -\int_k \frac{4}{d}\frac{k^2}{(k^2+m^2)^3}$$

$$= A_d \frac{(-2+d)\ \pi\ \csc(\frac{d\,\pi}{2})}{4\,m^\epsilon} \qquad \text{(A.16)}$$

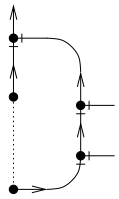


$$\longrightarrow \int_k \frac{-4\,k^2}{d\,(k^2+m^2)^3} + \frac{2}{(k^2+m^2)^2}$$

$$= A_d \frac{(2-d)\ \pi\ \csc(\frac{d\,\pi}{2})}{4\,m^\epsilon} \qquad \text{(A.17)}$$

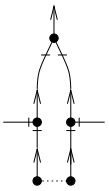
The sum (which gives the renormalization of $\hat{\lambda}$) exactly vanishes.

The corrections proportional to $\Delta(0)$ are



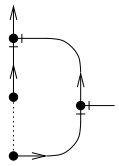
$$\longrightarrow \int_k \frac{-8\,k^4}{d\,(k^2+m^2)^4}$$

$$= A_d \frac{(-4+d^2)\ \pi\ \csc(\frac{d\,\pi}{2})}{12\,m^\epsilon} \qquad \text{(A.18)}$$



$$\longrightarrow \int_k \frac{4\,k^4}{d\,(k^2+m^2)^4}$$

$$= A_d \frac{(4-d^2)\ \pi\ \csc(\frac{d\,\pi}{2})}{24\,m^\epsilon} \qquad \text{(A.19)}$$



$$\longrightarrow \int_k \frac{8\,k^4}{d\,(k^2+m^2)^4} - \frac{4\,k^2}{d\,(k^2+m^2)^3}$$

$$= A_d \frac{(d-2)\ (d-1)\ \pi\ \csc(\frac{d\,\pi}{2})}{12\,m^\epsilon} \qquad \text{(A.20)}$$

The sum of the above three terms is

$$A_d \frac{(4-d)\ (d-2)\ \pi\ \csc(\frac{d\,\pi}{2})}{24\,m^\epsilon} \qquad \text{(A.21)}$$

Note that

$$\pi\ \csc(\frac{d\,\pi}{2}) = \frac{2}{d-4} + \frac{\pi^2\ (d-4)}{12} + \frac{7\,\pi^4\,(d-4)^3}{2880} + \dots \qquad \text{(A.22)}$$

So, working in a massive scheme, there are corrections at order ϵ, compared to the leading term which would be $1/\epsilon$. We see that the fixed point of Stepanow [22] is –even if one would accept his scheme– incorrect. However, as we have already stated above, one should do the calculations in a massless scheme.

APPENDIX B: LONG RANGE DISORDER

In this Appendix we give a quick study of the case with long range disorder in internal space x. We show that one recovers the Flory estimate of Section IV in the case of isotropic depinning. For anisotropic depinning we find a runaway flow and cannot conclude.

We study

$$S_{\rm DO} = \frac{1}{2}\int_{xtx't'} \hat{Z}_{xt}\hat{Z}_{x't'}\hat{\lambda}^2\Delta(u_{xt}-u_{x't'})f(x-x') \qquad \text{(B.1)}$$

$$f(x) \sim x^{-\alpha} \qquad \text{(B.2)}$$

We find the FRG equation for the LR disorder:

$$\partial\Delta = \epsilon\Delta + \Delta(0)\Delta'' + (2-\mu)(\lambda\Delta'(0^+) + \lambda^2\Delta(0))\Delta \quad \text{(B.3)}$$

with $\epsilon = 4 - \alpha$, d large enough ($d > \alpha$ or more). We have absorbed ϵA in Δ with:

$$A = \int_q C(q)^2 f(q) \quad \text{(B.4)}$$

This is because the graphs leading to two $\Delta(u)^2$ functions or more do not contribute. This remains true to all orders, inspection for $\lambda = 0$ shows that to two or three loops no corrections arise, except anomalous terms (which, as we will see are not needed as we find analytic fixed points). So for $\lambda = 0$ the one loop result is probably exact to all orders.

The coefficient μ comes from the corrections to the gradient term:

$$(\lambda\Delta'(0^+) + \lambda^2\Delta(0))B \quad \text{(B.5)}$$

$$B = \frac{1}{2d}\int_x x^2 f(x)C(x) \quad \text{(B.6)}$$

$$\mu = 2B/A = \frac{2(d-4)}{d} \quad \text{(B.7)}$$

with $\alpha = 4$, $d > 4$ (note that it goes to 2 when d goes to infinity).

One easily finds the fixed points for $\lambda = 0$. For periodic disorder one has:

$$\Delta(u) = g\cos(2\pi u) \quad \text{(B.8)}$$

$$\partial g = \epsilon g - (2\pi)^2 g^2 \quad \text{(B.9)}$$

The correlations are:

$$\langle uu\rangle = \hat{\Delta}(0)\frac{f(q)}{q^4} \sim q^{-(d+2\zeta)} \quad \text{(B.10)}$$

with $\zeta = \epsilon/2$ as if $\Delta(0)$ was uncorrected.

For non-periodic disorder, rescaling Δ gives:

$$\tilde{\Delta}(u) = \tilde{\Delta}(0)\mathrm{e}^{-\epsilon u^2/(6\tilde{\Delta}(0))} \quad \text{(B.11)}$$

$$\zeta = \epsilon/3 \quad \text{(B.12)}$$

Thus the Flory estimate is exact. Note that the fact that the LR correlator $\tilde{\Delta}(u)$ is analytic is not puzzling, since it generates in turn a SR part which should be nonanalytic in order to e.g. successfully generate a depinning threshold force.

On the other hand for $\lambda > 0$ we find that:

$$\Delta(u) = g\cos(2\pi u) \quad \text{(B.13)}$$

$$\partial g = \epsilon g - (2\pi)^2 g^2 + \gamma\lambda^2 g^2 \quad \text{(B.14)}$$

and there is thus a critical λ beyond which there is no fixed point. This seems also to be the case for RF. Because of this runaway flow we cannot conclude.

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[29] If STS is an *exact* symmetry, e.g. in the continuum limit for a pure $\delta^d(x-x')$ correlator in II.2, or as will be discussed below for models with exact rotational invariance, the KPZ term is forbidden for any velocity v. If the correlations have a finite range it may not be exact, leading to a small KPZ term, estimated at finite velocity in [2].

[30] A non-perturbative proof of this was recently obtained from the exact solution at large N [28].

[31] The equivalence of the combinatorial factors for $\delta\lambda$ and δc can be deduced by imagining that Δ be two independent vertices of the statics and observing that both diagrams correct the same vertex after using FDT.