Finite temperature correlations in massive integrable QFT

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4. March 2010.

Introduction

1+1 dimensional relativistic integrable QFT

- Particle spectrum: $m_a \qquad a = 1..n$

$$\begin{pmatrix} E \\ p \end{pmatrix} = m_a \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix}$$

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- Only particle type in the spectrum: Sinh-Gordon model

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- No chemical potential:

$$T = 0 \quad \Leftrightarrow \quad \text{Fock-vacuum} \quad \Leftrightarrow \quad \text{"empty space"}$$

Form factors approach

The spectral expansion at zero temperature:

$$\langle \mathcal{O}(0,0)\mathcal{O}(0,i\tau)\rangle = \sum_{N=0}^{\infty} \frac{1}{N!} \int \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi} \left| F_N^{\mathcal{O}}(\theta_1, \dots, \theta_n) \right|^2 e^{-m\tau \sum_i \cosh \theta_i}$$

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Crossing relation:

$$F_{k,l}^{O}(\theta_1,\dots,\theta_k|\theta_1',\theta_2',\dots,\theta_l') = F_{k+l}^{O}(\theta_k+i\pi,\dots,\theta_1+i\pi,\theta_1',\theta_2',\dots,\theta_l') + \Big(\dots\Big)$$

Turning on a finite temperature

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- The thermal average is ill-defined in infinite volume
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- Regularization based on thermodynamics

The LeClair-Mussardo formalism

Thermodynamics:

$$\varepsilon(\theta) = \frac{m\cosh\theta}{T} - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\varphi(\theta-\theta')\log(1+e^{-\varepsilon(\theta')})$$

$$Z \approx e^{-fL/T}, \qquad f = -T\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} mc\cosh\theta\log\left(1+e^{-\varepsilon(\theta)}\right)$$

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The LeClair-Mussardo series for expectation values: (hep-th/9902075)

$$\langle\mathcal{O}\rangle_T = \sum_N \frac{1}{N!}\int \frac{d\theta_1}{2\pi}\dots\frac{d\theta_N}{2\pi}\left(\prod_j \frac{1}{1+e^{\varepsilon(\theta_j)}}\right) F^{\mathcal{O}}_{2N,c}(\theta_1,\dots,\theta_N)$$

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Finite volume regularization

The idea

- Put the system in a finite volume L (with periodic bc.)
- Evaluate the thermal average
- Take the limit $L \to \infty$

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An essential ingredient:

Finite volume form factors

(BP and G. Takács, arXiv:0706.1445)

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First approach

- A low-temperature expansion with parameter $e^{-m/T}$

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Result

LM series checked up to 3rd order

(BP and G. Takács, `arxiv:0706.3605`)

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- Evaluating the “thermodynamic limit” directly

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Results

- LM series checked to all orders

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Results

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- Even if $\mu \neq 0$

The two-point function

- Evaluate

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- Low-T expansion works, however one has to deal with singularities of type

$$\langle \theta_1|\mathcal{O}|\theta_2,\dots\rangle \quad \sim \quad \frac{1}{\theta_1-\theta_2} \quad \text{if} \quad \theta_1\to\theta_2$$

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$$\langle \mathcal{O}(x,t)\mathcal{O}(0,0)\rangle_T = \sum_{NM} D_{NM}$$

where

$$D_{NM} \quad \sim \quad \langle N - \text{particles}|\mathcal{O}|M - \text{particles}\rangle$$

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- D_{22} and the others: work in progress

:‿

Thank you for your attention!

Relation to other problems

Approaching non-relativistic theories

- Lieb-Liniger model (1D δ-interacting Bose-gas) as a non-relativistic limit of sinh-Gordon

  M. Kormos, G. Mussardo, A. Trombettoni:
  `arXiv:0909.1336`, `arXiv:0912.3502`

  S-matrix, TBA, local correlation functions

$$\left\langle \left(\Psi^\dagger\right)^k(0)\Psi^k(0)\right\rangle_{T,\mu}$$

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$$\left\langle \left(\Psi^{\dagger}\right)^{k}(0)\Psi^{k}(0)\right\rangle_{T,\mu}$$

- Non-relativistic limit applies to the individual matrix elements as well

  M. Kormos, G. Mussardo, B. Pozsgay: `arXiv:1002.3387`

LeClair-Mussardo formula for the two-point function

$$\langle \mathcal{O}(x,t)\mathcal{O}(0,0)\rangle_T = \left(\langle \mathcal{O}\rangle_T\right)^2 +$$

$$\sum_{N=1}^{\infty}\frac{1}{N!}\sum_{\sigma_j=\pm 1}\int\frac{d\theta_1}{2\pi}\dots\frac{d\theta_N}{2\pi}\left[\prod_{j=1}^{N} f_{\sigma_j}(\theta_j)e^{-\sigma_j(t\varepsilon_j+ixk_j)}\right]\times\left|\langle 0|\mathcal{O}|\theta_1\dots\theta_N\rangle_{\sigma_1\dots\sigma_N}\right|^2$$

where $f_{\sigma_j}(\theta_j)=1/(1+e^{-\sigma_j\varepsilon(\theta_j)})$, $\varepsilon_j=\varepsilon(\theta_j)/R$ and $k_j=k(\theta_j)$ with $\varepsilon(\theta)$ being the solution of the TBA equations and $k(\theta)$ given by

$$\begin{aligned} k(\theta) &= m\sinh(\theta)+\int d\theta'\delta(\theta-\theta')\rho_1(\theta') \\ 2\pi\rho_1(\theta)(1+e^{\varepsilon(\theta)}) &= m\cosh(\theta)+\int d\theta'\varphi(\theta-\theta')\rho_1(\theta') \end{aligned}$$

The form factors appearing above are defined by

$$\langle 0|\mathcal{O}|\theta_1\dots\theta_N\rangle_{\sigma_1\dots\sigma_N} = F_N^{\mathcal{O}}(\theta_1-i\pi\tilde\sigma_1,\dots,\theta_N-i\pi\tilde\sigma_N)\qquad \tilde\sigma_j=(1-\sigma_j)/2\in\{0,1\}$$