On the spectrum of the critical $O(n)$ model in two dimensions

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Work done in the context of a collaboration with
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To be sketched in [LGS + RN + JJ + SR + HS, in preparation]

More details and Potts model case in [SR et al, to appear]

Introduction

$O(n)$ model: a 2d CFT that describes the critical limit of loop models

Problem: write the spectrum as a representation of

$$\text{conformal algebra} \ \times \ \text{global symmetry group} \tag{1}$$

Global symmetry: orthogonal group $O(n)$ with $n \in \mathbb{C}$. Meaning?

- Lattice model makes sense [Nienhuis 1982]
- $O(n)$ representations and their tensor products make sense with $n \in \mathbb{C} - \mathbb{N} \iff n \in \mathbb{N} \to \infty$ [Binder + Rychkov 2019]

The partition function of the $O(n)$ model on a torus

[di Francesco + Saleur + Zuber 1987] computed

$$Z^{O(n)}(q) = \sum_{s\in 2\mathbb{N}+1} \chi_s(q) + \sum_{r\in\frac12\mathbb{N}^*} \sum_{s\in\frac1r\mathbb{Z}} D_{(r,s)}(n)\chi_{(r,s)}(q) \tag{2}$$

Characters of representations of the conformal algebra:

$$\chi_s(q) = \left| \frac{q^{P^2_{(1,s)}} - q^{P^2_{(1,-s)}}}{\eta(q)} \right|^2 \quad , \quad \chi_{(r,s)}(q) = \frac{q^{P^2_{(r,s)}}\bar{q}^{P^2_{(r,-s)}}}{|\eta(q)|^2} \tag{3}$$

Momentums:

$$P_{(r,s)} = \frac12\left(\beta r - \beta^{-1}s\right) \quad \text{with} \quad n = -2\cos(\pi\beta^2) \tag{4}$$

Coefficients $D_{(r,s)}(n) = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1} + \frac{1}{2r}\sum_{r'=0}^{2r} e^{\pi i r' s} p_{(2r)\wedge r'}(n)$

Structure of the spectrum $\mathcal{S}^{O(n)}$

$$Z^{O(n)}(q) = \sum_{s\in 2\mathbb{N}+1} \chi_s(q) + \sum_{r\in\frac12\mathbb{N}^*} \sum_{s\in\frac1r\mathbb{Z}} D_{(r,s)}(n)\chi_{(r,s)}(q)$$

$$\mathcal{S}^{O(n)} = \bigoplus_{s\in 2\mathbb{N}+1} [] \otimes \mathcal{R}_s \oplus \bigoplus_{r\in\frac12\mathbb{N}^*} \bigoplus_{s\in\frac1r\mathbb{Z}} \Lambda_{(r,s)} \otimes \mathcal{W}_{(r,s)} \tag{5}$$

- $[] =$ singlet representation of $O(n)$
- $\Lambda_{(r,s)} =$ representations of $O(n)$ to be determined
- $\mathcal{R}_s, \mathcal{W}_{(r,s)} =$ representations of the conformal algebra

$$\boxed{\dim_{O(n)} \Lambda_{(r,s)} = D_{(r,s)}(n)} \quad , \quad \left\{ \begin{array}{l} \operatorname{Tr}_{\mathcal{R}_s} q^{D_0-\frac{c}{24}}\bar{q}^{\bar{D}_0-\frac{c}{24}} = \chi_s(q) \\ \operatorname{Tr}_{\mathcal{W}_{(r,s)}} q^{D_0-\frac{c}{24}}\bar{q}^{\bar{D}_0-\frac{c}{24}} = \chi_{(r,s)}(q) \end{array} \right. \tag{6}$$

Representations of the conformal algebra

- $\mathcal{R}_s$ = degenerate representation
- $\mathcal{W}_{(r,s)}$ unless $r, s \in \mathbb{Z}^*$ = Verma module
- $\mathcal{W}_{(r,s)}$ with $r, s \in \mathbb{Z}^*$ = logarithmic representation

Structure of logarithmic representations:

- Use degenerate fields and reduce $\mathcal{W}_{(r,s)}$ to $\mathcal{W}_{(r,0)}$
  [Estienne + Ikhlef 2015] [Gorbenko + Zan 2020]
  [Nivesvivat + Ribault 2020]
- Critical limit of the lattice model
  [Grans-Samuelsson + Liu + He + Jacobsen + Saleur 2020]
- Check results by numerically bootstrapping some 4pt functions
  [Nivesvivat + Ribault 2020]

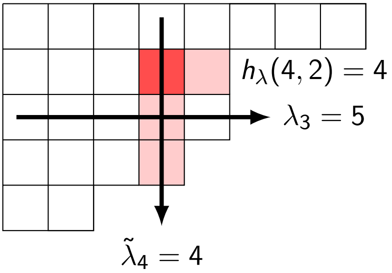
Representations of $O(n)$ and their dimensions

Main facts and references in the (recently created) Wikipedia article [Representations of classical Lie groups]

Irreducible finite-dimensional representations $\leftrightarrow$ Young diagrams
Dimension as a polynomial in n [El Samra + King 1979]

$$\dim_{O(n)} \lambda = \prod_{\substack{(i,j)\in\lambda \\ i\geq j}} \frac{n+\lambda_i+\lambda_j-i-j}{h_\lambda(i,j)} \prod_{\substack{(i,j)\in\lambda \\ i<j}} \frac{n-\tilde{\lambda}_i-\tilde{\lambda}_j+i+j-2}{h_\lambda(i,j)} \tag{7}$$

$\lambda = [85542]\ :$ (8)



First few representations

$$\begin{aligned}
\dim_{O(n)}[1] &= n \\
\dim_{O(n)}[2] &= \tfrac{1}{2}(n+2)(n-1) \\
\dim_{O(n)}[3] &= \tfrac{1}{6}(n+4)n(n-1) \\
\dim_{O(n)}[11] &= \tfrac{1}{2}n(n-1) \\
\dim_{O(n)}[111] &= \tfrac{1}{6}n(n-1)(n-2) \\
\dim_{O(n)}[21] &= \tfrac{1}{3}n(n^2-4)
\end{aligned}$$

$$\begin{aligned}
D_{(\frac{1}{2},0)}(n) &= n \\
D_{(1,0)}(n) &= \tfrac{1}{2}(n+2)(n-1) \\
D_{(1,1)}(n) &= \tfrac{1}{2}n(n-1) \\
D_{(\frac{3}{2},0)}(n) &= \tfrac{1}{3}n(n^2-1) \\
D_{(\frac{3}{2},\frac{2}{3})}(n) &= \tfrac{1}{3}n(n^2-4)
\end{aligned}$$

Unique solutions: $\begin{cases} \Lambda_{(\frac{1}{2},0)} = [1] \\ \Lambda_{(1,0)} = [2] \\ \Lambda_{(1,1)} = [11] \\ \Lambda_{(\frac{3}{2},\frac{2}{3})} = [21] \end{cases}$ Ambiguity: $\Lambda_{(\frac{3}{2},0)} \in \begin{cases} [3]+[111] \\ [21]+[1] \end{cases}$

The hint from the open spin chain

Open spin chain spectrum

$$\Lambda_r = \sum_{s'=0}^{2r-1} \Lambda_{(r,\frac{s'}{r})} - \delta_{r,1}[] \tag{9}$$

$SU(2)$ tensor product rule [Read + Saleur 2007]

$$\Lambda_{\frac{1}{2}} \otimes \Lambda_r = \Lambda_{r-\frac{1}{2}} \oplus \Lambda_{r+\frac{1}{2}} \tag{10}$$

Consequence:

$$\dim_{O(n)} \Lambda_r = p_{2r}(n) \quad , \quad \Lambda_r = p_{2r}([1]) \tag{11}$$

with the (modified) Chebyshev polynomials $p_d(n)$ defined by

$$p_0(n) = 2 \ , \quad p_1(n) = n \ , \quad np_d(n) = p_{d-1}(n) + p_{d+1}(n) \tag{12}$$

Naively following the hint

Trying to solve $\boxed{\dim_{O(n)} \Lambda_{(r,s)} = D_{(r,s)}(n)}$ with [dF + S + Z 1987]

$$D_{(r,s)}(n) = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1} + \frac{1}{2r}\sum_{r'=0}^{2r} e^{\pi i r' s} p_{(2r)\wedge r'}(n) \tag{13}$$

First few polynomials:

$$p_2(n) = n^2 - 2 \tag{14}$$

$$p_3(n) = n^3 - 3n \tag{15}$$

$$p_4(n) = n^4 - 4n^2 + 2 \tag{16}$$

Why not $\Lambda_{(r,s)} = D_{(r,s)}([1])$? Does not work!

$$D_{(1,1)}([1]) = \frac{1}{2}\Big([2] + [11] - [1] + []\Big) \tag{17}$$

$$\boxed{\Phi_t = \delta_{t\equiv 0 \bmod 2}[] + \sum_{k=0}^{t-1}(-1)^k[t-k, 1^k]} \quad \text{obeys} \quad {\color{red}\dim_{O(n)} \Phi_t = n} \tag{18}$$

$$\Phi_5 = [5] - [41] + [311] - [2111] + [11111] \tag{19}$$

$$= \ydiagram{5} - \ydiagram{4,1} + \ydiagram{3,1,1} - \ydiagram{2,1,1,1} + \ydiagram{1,1,1,1,1} \tag{20}$$

$$\boxed{\Lambda_{(r,s)} = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1}[] + \frac{1}{2r}\sum_{r'=0}^{2r} e^{\pi i r' s} p_{(2r)\wedge r'}\left(\Phi_{\frac{2r}{(2r)\wedge r'}}\right)} \tag{21}$$

$${\color{green} D_{(r,s)}(n) = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1} + \frac{1}{2r}\sum_{r'=0}^{2r} e^{\pi i r' s} p_{(2r)\wedge r'}(n)}$$

First few examples

$$\Lambda_{(\frac{3}{2},0)} = [3] + [111]$$

$$\Lambda_{(\frac{3}{2},\frac{2}{3})} = [21]$$

$$\Lambda_{(2,0)} = [4] + [22] + [211] + [2] + []$$ [Gorbenko + Zan 2020]

$$\Lambda_{(2,\frac{1}{2})} = [31] + [11] + [211]$$

$$\Lambda_{(2,1)} = [2] + [31] + [1111] + [22]$$

$$\Lambda_{(\frac{5}{2},0)} = [32] + 2[311] + [1] + [221] + [3] + 2[21] + [5] + [111] + [11111]$$

$$\Lambda_{(\frac{5}{2},\frac{2}{5})} = [32] + [311] + [1] + [221] + [3] + [2111] + 2[21] + [111] + [41]$$

More examples

$$\Lambda_{(3,0)} = [6] + 2[42] + 2[411] + [33] + 2[321] + 2[3111] + 2[222] + [2211] + [21111] + 2[4] + 4[31] + 4[22] + 4[211] + 2[1111] + 4[2] + 2[11] + 2[]$$

$$\Lambda_{(3,\frac{1}{3})} = [51] + [42] + 2[411] + [33] + 3[321] + [3111] + 2[2211] + [21111] + [4] + 5[31] + 2[22] + 5[211] + [1111] + 2[2] + 4[11]$$

$$\Lambda_{(3,\frac{2}{3})} = [51] + 2[42] + [411] + 3[321] + 2[3111] + [222] + [2211] + [21111] + 2[4] + 4[31] + 4[22] + 4[211] + 2[1111] + 4[2] + 2[11] + []$$

$$\Lambda_{(3,1)} = [51] + [42] + 2[411] + 2[33] + 2[321] + 2[3111] + [222] + 2[2211] + [111111] + [4] + 5[31] + 2[22] + 5[211] + [1111] + 2[2] + 4[11]$$

Outlook

- Testing the conjecture by numerically bootstrapping 4pt functions [LGS + RN + JJ + SR + HS, in preparation]
- Finding fewer solutions of crossing symmetry than predicted by $O(n)$: a manifestation of a larger global symmetry? [Read + Saleur 2007]
- Larger symmetry is needed for taming the nontrivial multiplicities in the spectrum
- Similar conjecture for the spectrum of the Q-state Potts model with $Q \in \mathbb{C}$ [SR et al, to appear]