On the scaling limit of the inhomogeneous six-vertex model

Sergei Lukyanov (Rutgers)

Conference "The art of mathematical physics"
celebrating Hubert's 60th anniversary

Saclay, September 21, 2021

Motivation

- Inhomogeneous six-vertex model **[Baxter'71]**

  (multiparametric exactly solvable model of 2D classical Stat Mech)

- **The model exhibit many interesting types of universal behaviour**

- Homogeneous six-vertex model/Heisenberg spin $\frac{1}{2}$ chain:

  – scaling limit: Gaussian model (free massless boson)

- Mostly studied: "staggered" inhomogeneous 6v model, with pioneering contributions due to

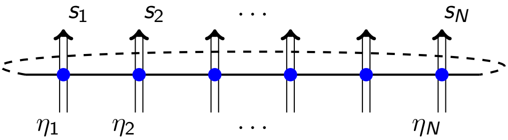
  Hubert & collaborators, e.g.,

  – continuous spectrum of conformal dims `[Jacobsen,Saleur'05]`

  Further study showed scaling limit closely tied to 2D Euclidean/Lorentzian black hole sigma model

Outline

$\mathbb{T}(\zeta) =$



the transfermatrix $\mathbb{T}(\zeta)$ acts in $\mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1} \otimes \ldots \otimes \mathbb{C}^{2s_N+1}$

$\big(\mathbb{C}^{2s+1} \longleftarrow$ spin-s irrep of $U_q(\mathfrak{sl}_2)$; $s = \frac{1}{2}$ for the six vertex model$\big)$

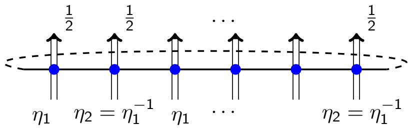
[Kotousov,Lukyanov'21]: Conjecture for scaling limit of $\mathbb{T}(\zeta)$ in certain domain of anisotropy parameter $q = \mathrm{e}^{\mathrm{i}\gamma}$ and inhomogeneities $\eta_m = \mathrm{e}^{\mathrm{i}\alpha_m}$.

Plan:

- Homogeneous six-vertex model/Heisenberg XXZ spin 1/2 chain in the scaling limit incl. ODE/IQFT correspondence
- ODE/IQFT for the inhomogeneous model

Staggered 6v model

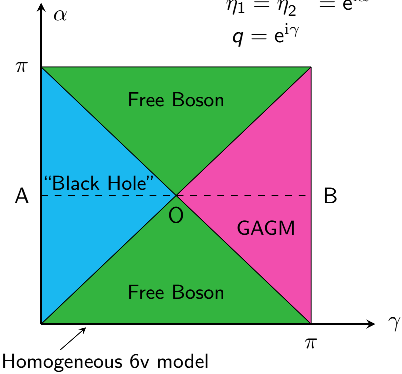
$\mathbb{T}(\zeta) =$



$\eta_{2m} = 1/\eta_{2m-1}, \quad s_m = \frac{1}{2}$

$\eta_1 = \eta_2^{-1} = \mathrm{e}^{\mathrm{i}\alpha}$

$q = \mathrm{e}^{\mathrm{i}\gamma}$



- Line AO ($U_{q^2}\big(\widehat{D}_2^{(2)}\big)$):
  [Jacobsen,Saluer'05; Ikhlef,Jacobsen,Saleur'06,'11; Frahm,Martins'12;Candu,Ikhlef'13]

  Whole "BH" region: [Frahm,Seel'13]

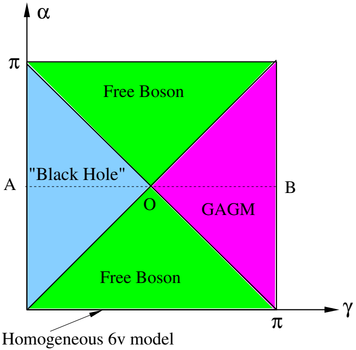
  ODE/IQFT [Bazhanov,Kotousov,Koval,SL'19,'20]
- Line OB: [Ikhlef,Jacobsen,Saluer'09]

  ODE/IQFT for GAGM region [Kotousov,SL'21] (this talk)

Hamiltonian for the staggered 6v model

$$[\mathbb{H}, \mathbb{T}(\zeta)] = 0$$

$$\begin{aligned}\mathbb{H} \;=\;& \frac{\cot(\gamma)}{4\sin(\alpha-\gamma)\sin(\alpha+\gamma)} \sum_{m=1}^{N} \Big[-2\sin^2(\alpha)\,(\sigma^x_m\sigma^x_{m+2} + \sigma^y_m\sigma^y_{m+2} + \sigma^z_m\sigma^z_{m+2}) \\ &+ 4\sin^2(\gamma)\Big\{\sigma^z_m\sigma^z_{m+1} + \frac{\cos(\alpha)}{\cos(\gamma)}\,(\sigma^x_m\sigma^x_{m+1} + \sigma^y_m\sigma^y_{m+1})\Big\} + 2\mathrm{i}\,(-1)^m\sin(\gamma)\,\sin(\alpha) \\ &\times \Big\{(\sigma^x_m\sigma^y_{m+1} - \sigma^y_m\sigma^x_{m+1})(\sigma^z_{m-1} - \sigma^z_{m+2}) - \frac{\cos(\alpha)}{\cos(\gamma)}\,(\sigma^x_{m-1}\sigma^y_{m+1} - \sigma^y_{m-1}\sigma^x_{m+1})\,\sigma^z_m\Big\}\Big]\end{aligned}$$



$\eta_1 = \eta_2^{-1} = e^{i\alpha}$

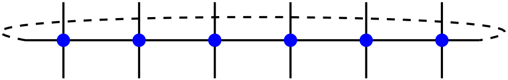
$q = e^{i\gamma}$

Homogeneous 6v model/Heisenberg XXZ spin 1/2 chain

One-row transfermatrix

$$\mathbb{T}(\zeta) = \mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\pi \mathrm{k}\sigma_0^z}\, R_1(\zeta)\, R_2(\zeta)\, \ldots\, R_N(\zeta) \right)$$

$\mathbb{T}(\zeta) =$



$\mathrm{e}^{\mathrm{i}\pi \mathrm{k}\sigma_0^z}$

$R_m(\zeta)$ – trigonometric solution of Yang-Baxter equation

$$\boldsymbol{R}_m(\zeta) = \begin{pmatrix} q^{\frac{1}{2}(1+\sigma_m^z)} + q^{\frac{1}{2}(1-\sigma_m^z)}\,\zeta & -(q-q^{-1})\, q\, \zeta\, \sigma_m^- \\ (q-q^{-1})\,\sigma_m^+ & q^{\frac{1}{2}(1-\sigma_m^z)} + q^{\frac{1}{2}(1+\sigma_m^z)}\,\zeta \end{pmatrix}$$

Heisenberg spin $\frac{1}{2}$ chain lies in the integrability class of six-vertex model:

$$\mathbb{H}_{XXZ} = 2\mathrm{i}\zeta\partial_\zeta\, \log\left(\mathbb{T}(-q^{-1}\,\zeta) \right)\Big|_{\zeta=1} + const$$

Heisenberg XXZ spin 1/2 chain

$$\mathbb{H}_{XXZ} = -\frac{1}{2\sin(\gamma)} \sum_{m=1}^{N} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \cos(\gamma)\, \sigma_m^z\, \sigma_{m+1}^z\right)$$

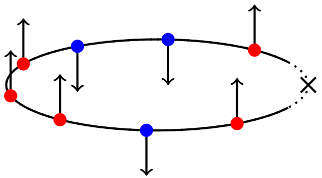
σ_m^a – Pauli matrices acting on site m of lattice

Quasi-periodic BC:

$$\sigma_{N+1}^z = \sigma_1^z$$

$$\sigma_{N+1}^x \pm \mathrm{i}\sigma_{N+1}^y = \mathrm{e}^{\pm 2\pi \mathrm{i} \mathtt{k}} \left(\sigma_1^x \pm \mathrm{i}\sigma_1^y\right)$$

Twist parameter $\mathtt{k} \in (-\frac{1}{2}, \frac{1}{2}]$



Lattice system critical in disordered regime with

$$q = \mathrm{e}^{\mathrm{i}\gamma} \; : \qquad \gamma = \pi\beta^2 \qquad (0 < \beta^2 < 1)$$

Low energy spectrum

Low energy spectrum [Luther,Peschel'75;Kadanoff,Brown'79;Cardy'86, Alcaraz,Barber,Batchelor'87]

$$\mathcal{E} \asymp N\, e_\infty + \frac{2\pi v_{\rm F}}{N}\Big(P^2 + \bar{P}^2 + \mathtt{L} + \bar{\mathtt{L}} - \frac{1}{12}\Big) + o(N^{-1})$$

with

$$P = \tfrac{1}{2}\left(\beta\, S^z + \beta^{-1}\,(\mathtt{k}+\mathtt{w})\right), \qquad \bar{P} = \tfrac{1}{4}\left(\beta\, S^z - \beta^{-1}\,(\mathtt{k}+\mathtt{w})\right)$$

$S^z = 0, \pm 1, \pm 2, \ldots$ eigenvalue of $\mathbb{S}^z = \frac{1}{2}(\sigma_1^z + \sigma_2^z + \ldots + \sigma_N^z)$

$\mathtt{w} = 0, \pm 1, \pm 2, \ldots$ winding number (vorticity)

$\mathtt{L}, \bar{\mathtt{L}} = 0, 1, 2, \ldots$ levels

Scaling limit governed by free massless boson

$$\mathcal{H} = \bigoplus_{\mathtt{w}, S^z \in \mathbb{Z}} \mathcal{F}_P \otimes \bar{\mathcal{F}}_{\bar{P}} \qquad \big(\mathcal{F}_P \ - \ \text{Fock space}\big)$$

Bethe ansatz

XXZ Hamiltonian part of large family of commuting matrices:

$$\big[\mathbb{H}_{XXZ}, \mathbb{T}(\zeta)\big] = \big[\mathbb{H}_{XXZ}, \mathbb{Q}(\zeta)\big] = 0$$

Operator valued relations, e.g., TQ relation

$$\mathbb{T}(\zeta)\,\mathbb{Q}(\zeta) = \big(1+q^{-1}\,\zeta\big)^{N}\,\mathbb{Q}(q^{+2}\zeta) + \big(1+q^{+1}\,\zeta\big)^{N}\,\mathbb{Q}(q^{-2}\zeta)$$

Eigenvalue of $\mathbb{Q}(\zeta)$ can be bootstrapped:

$$\mathbb{Q}(\zeta)\,\boldsymbol{\Psi} = \zeta^{S^z+\frac{\mathtt{k}}{\beta^2}} \prod_{n=1}^{\frac{N}{2}-S^z} (1-\zeta/\zeta_m)\,\boldsymbol{\Psi}$$

with

$$\left(\frac{1+q^{+1}\,\zeta_m}{1+q^{-1}\,\zeta_m}\right)^{N} = -\mathrm{e}^{2\mathrm{i}\pi\mathtt{k}}\,q^{2S^z} \prod_{j=1}^{\frac{N}{2}-S^z} \frac{\zeta_j - q^{+2}\,\zeta_m}{\zeta_j - q^{-2}\,\zeta_m}$$

General strategy

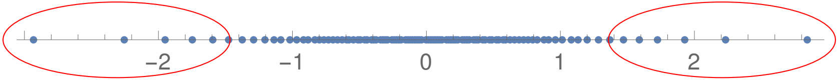
- For small lattice size $N \sim 20$ construct $\mathbb{H}_{XXZ}$ and obtain first few low energy eigenvectors $\boldsymbol{\Psi} = \boldsymbol{\Psi}_N$
- Compute eigenvalue of matrix $\mathbb{Q}(\zeta)$,

$$\mathbb{Q}(\zeta)\,\boldsymbol{\Psi}_N = \zeta^{S^z + \frac{\mathtt{k}}{\beta^2}} \prod_{n=1}^{\frac{N}{2}-S^z} (1 - \zeta/\zeta_m)\,\boldsymbol{\Psi}_N\,,$$

and extract Bethe roots $\{\zeta_m\}$ corresponding to $\boldsymbol{\Psi}_N$

- Using Bethe ansatz equations one can continue RG trajectory $\boldsymbol{\Psi}_N$ to $N \gg 1$ without explicit construction/diagonalization of $\mathbb{H}$ and $\mathbb{Q}$

Bethe state in the scaling limit



$\left(\theta = -\frac{1}{2}\log(\zeta)\right)$

$N \gg 1$ bulk roots become distributed with density

$$\frac{2}{N\left(\theta_{n+1}-\theta_n\right)} \approx \frac{1}{\pi(1-\beta^2)\cosh\left(\frac{\theta}{1-\beta^2}\right)} \qquad \left(q = \mathrm{e}^{\mathrm{i}\pi\beta^2}\right)$$

Roots at edges develop scaling behaviour:

$$r_n = \lim_{N\to\infty} N^{2(1-\beta^2)}\,\zeta_n\,, \qquad \bar{r}_n = \lim_{N\to\infty} N^{2(1-\beta^2)}\left(\zeta_{M-n}\right)^{-1}$$

$\left(M = N/2 - S^z\right)$

Bethe state in scaling limit characterized by S^z, $\mathtt{w}$, $\mathtt{L}$, $\bar{\mathtt{L}}$ as well as two infinite sets:

$$\boldsymbol{\Psi}_N\big(\{\zeta_j\}\big) \quad \xrightarrow{N\to\infty} \quad \psi\big(\{r_n\}\big)\otimes\bar{\psi}\big(\{\bar{r}_n\}\big) \in \mathcal{F}_P\otimes\mathcal{F}_{\bar{P}}$$

ODE/IQFT correspondence

[Voros'94;Dorey,Tateo'98;Bazhanov,Lukyanov,Zamolodchikov'98;'03]

Remarkable interpretation of $\{r_n\}$ ($\{\bar{r}_n\}$):

$$r_n = \text{const} \times E_n$$

where E_n are energy eigenvalues of Schrödinger operator

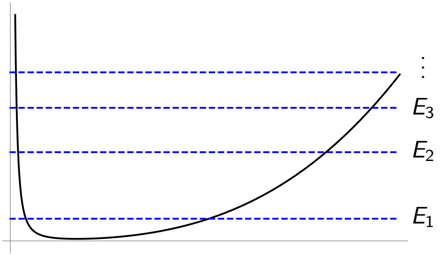
$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) - E \right) \Phi = 0$$

Primary state ($\mathtt{L} = \bar{\mathtt{L}} = 0$):

$$V(x) = \frac{4\,(\alpha+1)\,P^2 - \frac{1}{4}}{x^2} + x^{2\alpha}$$

$$P = \tfrac{1}{2}\left(\beta\, S^z + \beta^{-1}\,(\mathtt{k} + \mathtt{w})\right)$$

$$\alpha = \beta^{-2} - 1$$

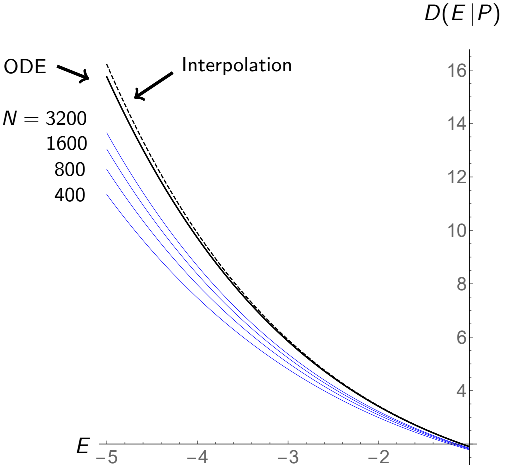


Scaling limit of eigenvalues of Q operator (vacuum)

$$\lim_{N\to\infty} N^{-2(\beta^2-1)(S^z+\frac{\mathtt{k}}{\beta^2})}\, Q(N^{2(\beta^2-1)}\,\zeta) = E^P\, D(E|P) \qquad (E = \mathrm{const}\times\zeta)$$

Spectral determinant:

$$D(E\,|\,P) = \prod_{n=1}^{\infty}\left(1 - \frac{E}{E_n}\right)$$



Scaling limit of eigenvalues of Q operator

General low energy state with quantum numbers S^z, $\mathtt{w}$, $\mathtt{L}, \bar{\mathtt{L}}$:

$$\boldsymbol{\Psi}_N\big(\{\zeta_j\}\big) \quad \xrightarrow{N\to\infty} \quad \psi\big(\{r_n\}\big) \otimes \bar{\psi}\big(\{\bar{r}_n\}\big) \in \mathcal{F}_P \otimes \mathcal{F}_{\bar{P}}$$

$$\lim_{N\to\infty} N^{-2(\beta^2-1)(S^z+\frac{\mathtt{k}}{\beta^2})}\, Q(N^{2(\beta^2-1)}\,\zeta) = E^P\, D(E|P)$$

Add "Monster potential" [Bazhanov,Lukyanov,Zamolodchikov'03]

$$V(x) = \frac{4\,(\alpha+1)\,P^2 - \frac{1}{4}}{x^2} + x^{2\alpha} - 2\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{b=1}^{\mathtt{L}} \log\left(x^{2\alpha+2} - \tfrac{\alpha+1}{\alpha}\, v_b\right)$$

$(v_1, v_2, \ldots, v_{\mathtt{L}})$ obey coupled algebraic system which makes all singularities of ODE apart from $x = 0, \infty$ apparent (monodromy free)

$$\#\ \text{of solutions}\ \{v_j\}_{j=1}^{\mathtt{L}}\ = \text{dim of level}\ \mathtt{L}\ \text{subspace of}\ \mathcal{F}_P$$

[Conti,Masoero'20]

Quantum KdV theory

Field theory interpretation:

$D(E|P)$ – eigenvalues of Q operator for quantum KdV theory:

$$\hat{\boldsymbol{Q}} \;:\; \mathcal{F}_P \mapsto \mathcal{F}_P\,, \qquad\qquad \hat{\boldsymbol{Q}}(E|P)\,\psi = E^P\,D(E|P)\,\psi$$

Generating function of all integrals of motion [BLZ'96]

$$\hat{\boldsymbol{Q}}(E) \asymp \boldsymbol{R}\,\exp\left(\sum_{n=0}^{\infty} B_n\,(-E)^{\frac{1-2n}{2-2\beta^2}}\,\boldsymbol{I}_{2n-1} - \sum_{n=1}^{\infty}(-1)^n\,(-E)^{-\frac{n}{\beta^2}}\,\widetilde{\boldsymbol{H}}_n\right)$$

as $(-E) \to \infty$

$\boldsymbol{R}$ – reflection operator

$\boldsymbol{I}_{2n-1}$ – local IM

$\widetilde{\boldsymbol{H}}_n$ – dual non-local IM

Finite size corrections

Knowledge of quantum KdV IM useful for characterizing scaling behaviour, e.g., finite size corrections:

$$\mathbb{H}_{XXZ} \asymp N\,e_\infty + \frac{2\pi v_{\rm F}}{N}\,\big(\boldsymbol{I}_1 + \bar{\boldsymbol{I}}_1\big) + \frac{1}{N^3}\Big(\lambda_+\,\boldsymbol{I}_1\,\bar{\boldsymbol{I}}_1 + \lambda_-\,\big(\boldsymbol{I}_3 + \bar{\boldsymbol{I}}_3\big)\Big)$$

$$+ \quad \text{higher order corrections involving local and non} - \text{local IM}$$

$\lambda_\pm$ some known constants [Lukyanov'97]

Local IM have simple form:

$$\mathbf{I}_1 = \int_0^{2\pi} \mathrm{d}u\,(\partial\varphi)^2\,, \qquad \mathbf{I}_1\Big|_{\mathcal{F}_P} = P^2 - \tfrac{1}{24} + \mathtt{L}$$

$$\mathbf{I}_3 = \int_0^{2\pi} \mathrm{d}u\,\Big(\big(\partial\varphi\big)^4 + (3 - \beta^2 - \beta^{-2})\,\big(\partial^2\varphi\big)^2\Big)$$

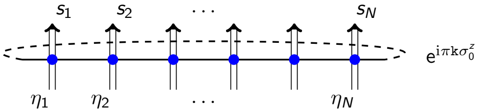
$$\mathbf{I}_5 = \ldots$$

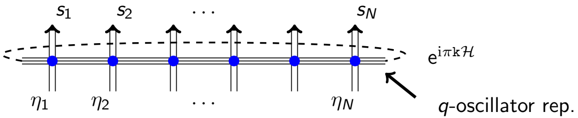
Recap

- Scaling limit of Heisenberg spin $1/2$ chain and connection to quantum KdV hierarchy
- Important tool: ODE/IQFT correspondence

Yang-Baxter integrability formalism allows one to generate many different spin chains that could have interesting universal behaviour

Scaling limit of the inhomogeneous model

$\mathbb{T}(\zeta) =$ 

$\mathbb{Q}(\zeta) =$ 

r-site periodicity condition:

$$\eta_{m+r} = \eta_m\,, \qquad s_{m+r} = s_m \qquad (N/r = \text{integer})$$

$$\mathbb{T}(\zeta),\ \mathbb{Q}(\zeta) \xrightarrow{\text{scaling limit with } N\to\infty} \hat{\boldsymbol{T}}(\mu),\ \hat{\boldsymbol{Q}}(\mu)$$

Regime: $|q| = 1\,, \qquad 2\pi\left(1 - \frac{1}{K}\right) < \arg(q^2) < 2\pi \qquad \left(K = 2\sum_{a=1}^{r} s_a\right)$

Parameterization: $q = -e^{\frac{i\pi}{K}(\beta^2-1)} \qquad (0 < \beta < 1)\ ; \qquad k_a = 2s_a = 1, 2, \ldots, r$

The ODE

$$\Big[-\partial_z^2+\kappa^2\, z^{-2+\xi K}\prod_{a=1}^{r}(z-z_a)^{k_a}-\frac{A^2+\frac{1}{4}}{z^2}+\sum_{a=1}^{r}\Big(\frac{\mathrm{j}_a(\mathrm{j}_a+1)}{(z-z_a)^2}+\frac{z_a\gamma_a}{z(z-z_a)}\Big)$$

$$+\sum_{\alpha=1}^{\mathtt{L}}\Big(\frac{2}{(z-w_\alpha)^2}+\frac{w_\alpha\Gamma_\alpha}{z(z-w_\alpha)}\Big)\Big]\,\Phi=0$$

$\sim$ Monster potential

Parameters:

- $(k_1,k_2,\dots k_r)$ positive integers: $k_a=2s_a$ $\quad(K=\sum_{a=1}^{r}k_a)$
- $\xi=\frac{\beta^2}{1-\beta^2}$ $\quad$ (anisotropy parameter $q=-\mathrm{e}^{\frac{\mathrm{i}\pi}{K}(\beta^2-1)}$)
- $\kappa\propto\mu^{1+\xi}$ "spectral parameter" entering scaling operator $\hat{\boldsymbol{Q}}(\mu)$
- $(z_1,\dots,z_r)\leftrightarrow(\eta_1,\dots,\eta_r)$ $\quad$ (inhomogeneities)
- A is similar to the zero-mode momentum P
- $\{\mathrm{j}_a\}$, $\{\gamma_a\}$, $\{w_\alpha\}$, $\{\Gamma_\alpha\}$ fixed by requirement that $z=z_a$ and $w=w_\alpha$ are apparent singularities

ODE/IQFT correspondence

$$\Big[-\partial_z^2 + \kappa^2\, z^{-2+\xi K} \prod_{a=1}^{r} (z - z_a)^{k_a} - \frac{A^2 + \frac{1}{4}}{z^2} + \sum_{a=1}^{r} \Big(\frac{\mathrm{j}_a(\mathrm{j}_a+1)}{(z-z_a)^2} + \frac{z_a \gamma_a}{z(z-z_a)} \Big)$$

$$+ \sum_{\alpha=1}^{\mathrm{L}} \Big(\frac{2}{(z-w_\alpha)^2} + \frac{w_\alpha \Gamma_\alpha}{z(z-w_\alpha)} \Big) \Big]\, \Phi = 0$$

$\sim$ Monster potential

$$\mathbb{T}(\zeta),\ \mathbb{Q}(\zeta) \xrightarrow{\text{scaling limit with } N\to\infty} \hat{\boldsymbol{T}}(\mu),\ \hat{\boldsymbol{Q}}(\mu)$$

Stokes coefficients and spectral determinant of ODE encode eigenvalues of $\hat{\boldsymbol{T}}(\mu)$ and $\hat{\boldsymbol{Q}}(\mu)$ for

Generalized Affine Gaudin Model (GAGM) [Kotousov, Lukyanov'21]

Local IM $\mathbf{I}_1^{(a)}$ $(a = 1, 2, \ldots, r)$ for GAGM

r-independent copies of the Kac-Moody algebra $\widehat{\mathfrak{sl}}_{k_a}(2)$:

$$J_A^{(a)}(u)\, J_B^{(b)}(0) = -\delta_{ab} \left(\frac{k_a}{2u^2}\ \eta_{AB} + \frac{\mathrm{i}}{u}\ f_{AB}{}^C\, J_C^{(a)} \right) + O(1)$$

$f_{AB}{}^C$ and η_{AB} – structure constants and Killing form for $\mathfrak{sl}_2$ algebra

$$G^{(a)} = \frac{1}{4\,(k_a+2)}\ \left(J_0^{(a)} J_0^{(a)} + 2\, J_+^{(a)} J_-^{(a)} + 2\, J_-^{(a)} J_+^{(a)}\right)$$

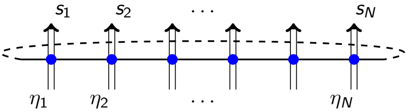
Then

$$\mathbf{I}_1^{(a)} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \Bigg[\frac{\beta^2 K}{1-\beta^2}\ G^{(a)} + \frac{1}{4K}\ \frac{1-\beta}{1+\beta}\ \left(k_a \left(J_0^{(\mathrm{tot})}\right)^2 - K\, J_0^{(a)} J_0^{(\mathrm{tot})} \right)$$
$$- \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{1}{z_a - z_b} \left(\tfrac{1}{4}\,(z_a+z_b)\, J_0^{(a)} J_0^{(b)} + z_a\, J_+^{(b)} J_-^{(a)} + z_b\, J_+^{(a)} J_-^{(b)} - k_a z_b\, G^{(b)} - k_b z_a\, G^{(a)} \right) \Bigg]$$

with

$$J_0^{(\mathrm{tot})} = \sum_{a=1}^{r} J_0^{(a)}$$

Conclusion

$\mathbb{T}(\zeta) \;=\; e^{i\pi k\sigma_0^z}$  $\eta_{m+r}=\eta_m,\quad s_{m+r}=s_m$

Summary:

- Scaling limit of the inhomogeneous model with the anisotropy

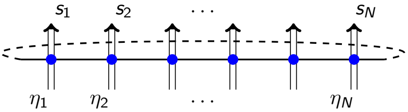
$$2\pi\left(1-\tfrac{1}{K}\right) < \arg(q^2) < 2\pi \qquad \left(K = 2\sum_{a=1}^{r} s_a\right)$$

  and subject of r-site periodicity condition:

$$\eta_{m+r}=\eta_m\,, \qquad s_{m+r}=s_m \qquad (N/r=\text{integer})$$

- For $r=2,\ s_1=s_2=\frac{1}{2},\ \ \eta_1=1/\eta_2=\mathrm{i},\ \pi<\arg(q^2)<2\pi$
  the model was studied in [Ikhlef,Jacobsen,Saleur'09]

  In this case the GAGM can be formulated in terms of one boson and two Majorana fermions

$\mathbb{T}(\zeta) \;=\; \mathrm{e}^{\mathrm{i}\pi k\sigma_0^z}$ 

$\eta_{m+r} = \eta_m, \quad s_{m+r} = s_m$

Future directions:

- Study of inhomogeneous model in complementary regimes:

$$\frac{2\pi}{K}\,(m-1) < \arg(q^2) < \frac{2\pi}{K}\,m \qquad (m = 1, \ldots K-1)$$

- For $r = 2$, $0 < \arg(q^2) < \pi$, $s_{2m} = s_{2m+1} = \frac{1}{2}$, $\eta_{2m} = 1/\eta_{2m+1}$

  [Jacobsen,Saleur'05, Ikhlef,Jacobsen,Saleur'06,'11; Frahm,Martins'12;Candu,Ikhlef'13;Frahm,Seel'13]
  ODE/IQFT in [Bazhanov,Kotousov,Koval,Lukyanov'19;'20]

The older you become, the sweeter the life.

Happy 60^{th} Hubert!