Integral Means Spectrum, SLE and Riemann Zeta Function

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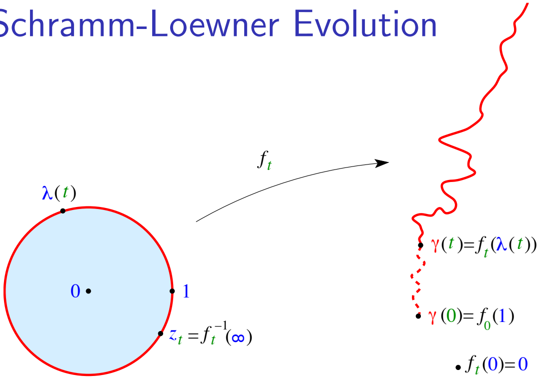
Whole-Plane Schramm-Loewner Evolution



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- $$z \in \mathbb{D},\quad \frac{\partial}{\partial t} f_t(z) = z\frac{\partial}{\partial z} f_t(z)\frac{\lambda(t)+z}{\lambda(t)-z},\quad \lambda(t)=\exp(i\sqrt{\kappa}B_t)$$

  $$f_t(e^{-t}z)\to z,\ t\to+\infty;\quad \kappa=0,\quad f_t(z)=\frac{e^t z}{(1-z)^2}\quad \text{(Koebe)}$$

- $1/f(1/z)$ is the **bounded exterior version** from $\mathbb{C}\setminus\overline{\mathbb{D}}$ to the slit plane [Beliaev & Smirnov, Lawler].

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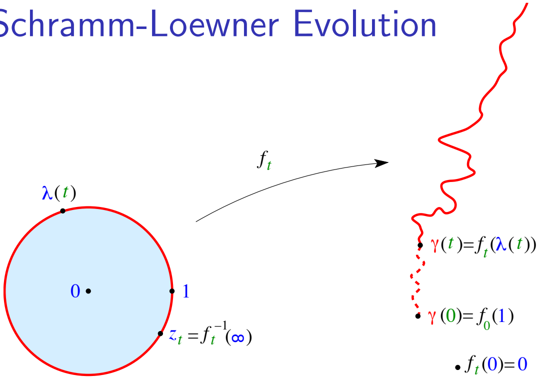

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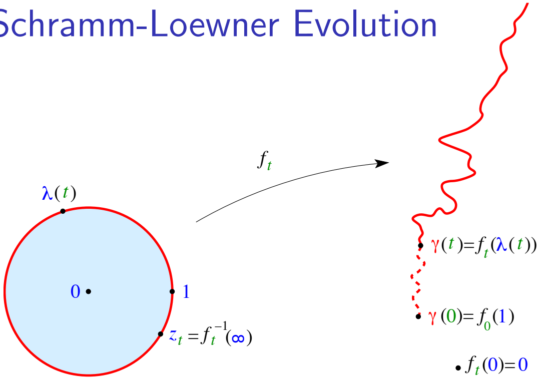

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Integral Means Spectrum

- Consider a Riemann map $\Phi : \mathbb{D} \to \mathbb{C}$
- The **integral means** of Φ are

$$\mathcal{I}(r, p, \Phi) := \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta, \ 0 < r < 1, \ p \in \mathbb{R};$$

- Φ **random**:
  Expectation: $\mathbb{E}\mathcal{I}(r, p, \Phi) := \int_0^{2\pi} \mathbb{E}\left[|\Phi'(re^{i\theta})|^p\right] d\theta.$
- One then defines

$$\beta_\Phi(p) := \limsup_{r \to 1^-} \frac{\log(\mathcal{I}(r, p, \Phi))}{\log(\frac{1}{1-r})};$$

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SLE Integral Means Spectrum

- As predicted in Lawler & Werner '99, D. '99 (Brownian path), D.'00, Hastings '02, Bettelheim et al. '05, and proven in Beliaev & Smirnov '05, and Beliaev, D. & Zinsmeister '17, the **average spectrum** of (bounded) SLE_κ involves 3 phases:

$$\beta_{\text{tip}}(p,\kappa) = -p - 1 + \frac{1}{4}\left(4+\kappa-\sqrt{(4+\kappa)^2 - 8\kappa p}\right),$$

$$\beta_0(p,\kappa) = -p + \frac{(4+\kappa)^2}{4\kappa} - \frac{(4+\kappa)}{4\kappa}\sqrt{(4+\kappa)^2 - 8\kappa p},$$

$$\beta_{\text{lin}}(p,\kappa) = p - \frac{(4+\kappa)^2}{16\kappa}.$$

- a.s. β_{tip} [Johansson Viklund & Lawler '12]
- a.s. β_0 [Gwynne, Miller & Sun '18]

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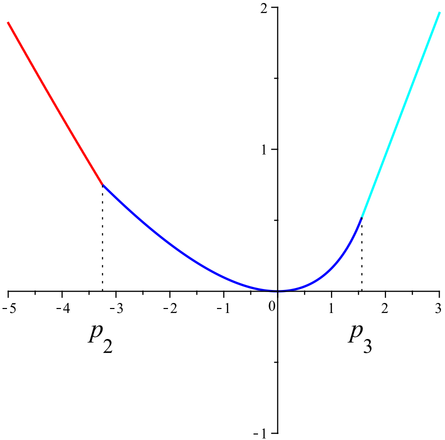
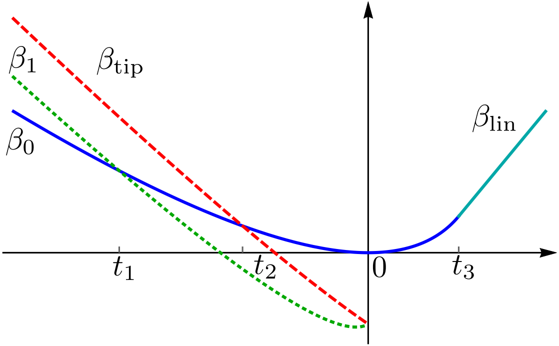


$$p_2 = -1 - \frac{3\kappa}{8}, \quad p_3 = \frac{3(4+\kappa)^2}{32\kappa}$$

Average integral means spectrum for **bounded** whole-plane SLE.



$$\beta_1(p, \kappa) := -p - \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2\kappa p}.$$

'Second tip' spectrum

[Beliaev, D. & Zinsmeister '17]

Unbounded Whole-plane SLE

- In this case, [Loutsenko & Yermolayeva '13] and [D., Nguyen, Nguyen & Zinsmeister '14] have shown the existence of a phase transition at $p_0 := \frac{(4+\kappa)^2-4-2\sqrt{2(4+\kappa)^2+4}}{16\kappa}$ to

$$\hat{\beta}_1(p,\kappa) := 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa p}.$$

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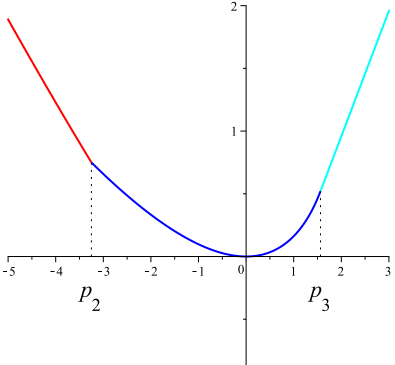

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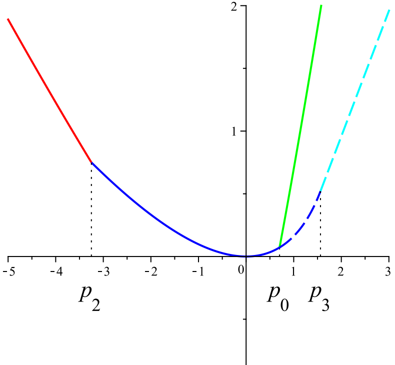

Universal Integral Means Spectrum

- $B_{\mathrm{bd}}(p) = \sup\{\beta_\Phi(p),\ \Phi \in \mathcal{S},\ \Phi \text{ injective \& bounded}\}$.
- Conjecture for $p \in \mathbb{R}$ (Kraetzer '96):

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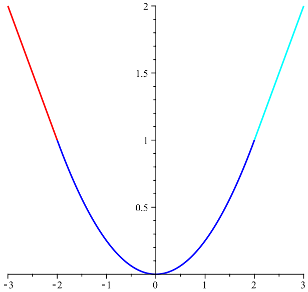

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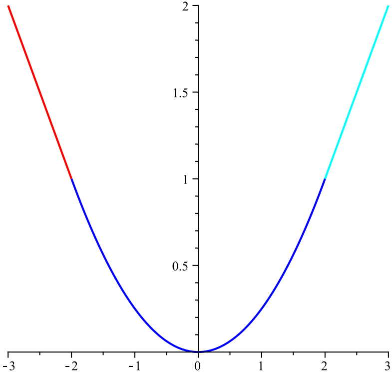

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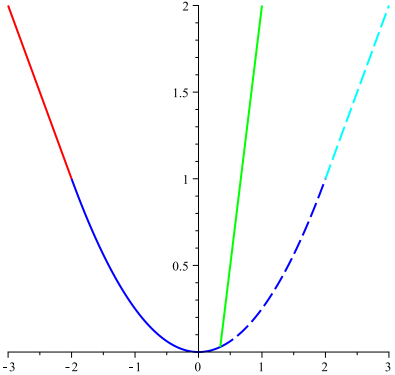

Riemann Zeta Function

- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p\,\mathrm{prime}} (1-p^{-s})^{-1}$, $\Re s > 1$
  + analytic continuation

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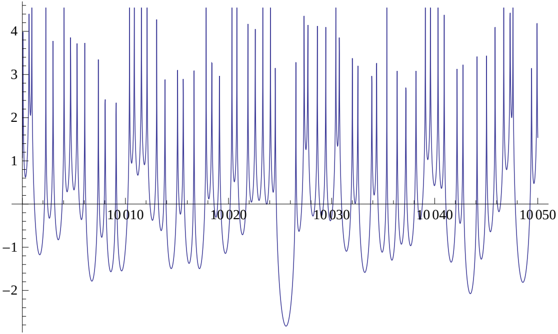

$-\log|\zeta(1/2 + i(T + h))|$ for $T = 10^4$ and $h \in [0, 50]$

(Courtesy of L.-P. Arguin)

Moments of the (Randomized) Riemann Zeta Function

- Many recent results for ζ, or its randomized version, on the critical line, for the moments, maxima, relation to Gaussian multiplicative chaos. (Fyodorov-Hiary-Keating; Najnudel; Arguin; Belius; Bourgade; Harper; Radziwill; Soundararajan; Saksman-Webb; Kistler; Hartung $\cdots$)
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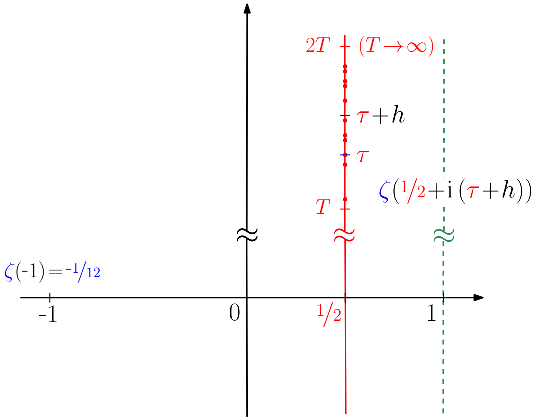

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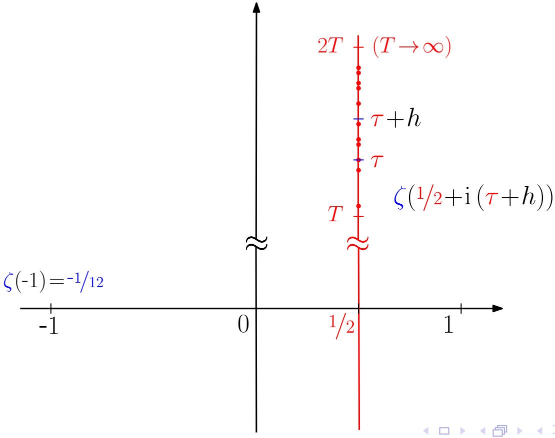

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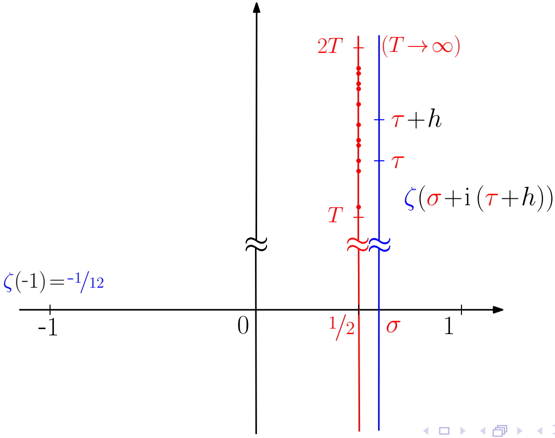

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- Kraetzer's conjecture concerns *univalent (injective)* bounded functions.

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