CDD ambiguity and irrelevant deformations of 2D QFT

IGST2017, Paris

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Based on:

M. Caselle, D. Fioravanti, F. Gliozzi, JHEP [arXiv:1305.1278]

A. Cavaglià, S. Negro, I. Szécsényi, JHEP [arXiv:1608.05534]

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Other relevant references

- A.B. Zamolodchikov, *Expectation value of composite field $T\bar{T}$ in two-dimensional quantum field theory*, [hep-th/0401146];

- S. Dubovsky, R. Flauger and V. Gorbenko, *Solving the simplest theory of quantum gravity*, JHEP 2012 (2012);

- S. Dubovsky, V. Gorbenko and M. Mirbabayi, *Natural tuning: towards a proof of concept*, [hep-th/1305.6939];

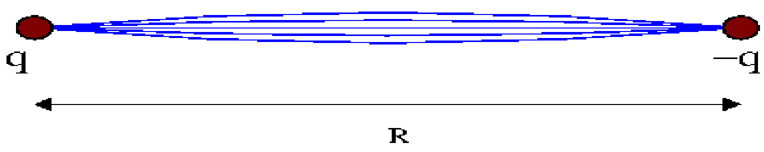
- F. A. Smirnov and A. B. Zamolodchikov, *On space of integrable quantum field theories,* Nucl.Phys. B915 (2017), [hep-th/1608.05499];

- L. McGough, M. Mezei, H. Verlinde, *Moving the CFT into the bulk with $T\bar{T}$*, [hep-th/1611.03470];

- A. Giveon, N. Itzhaki, D. Kutasov, *$T\bar{T}$ and LST,* [hep- th/1701.05576];

Main motivations

- The effective string theory for the quark-antiquark potential;



- Emergence of singularities in RG/TBA flows with irrelevant perturbations;

- Relation between irrelevant perturbations and S-matrix CDD (scalar phase factor) ambiguity;

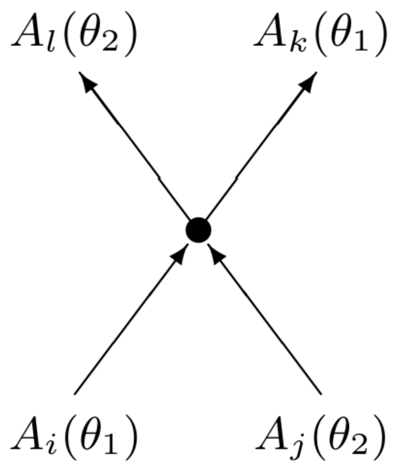
$$S_{ij}^{kl}(\theta) \to S_{ij}^{kl}(\theta)\, e^{i\delta_{ij}^{(t)}(\theta)}$$

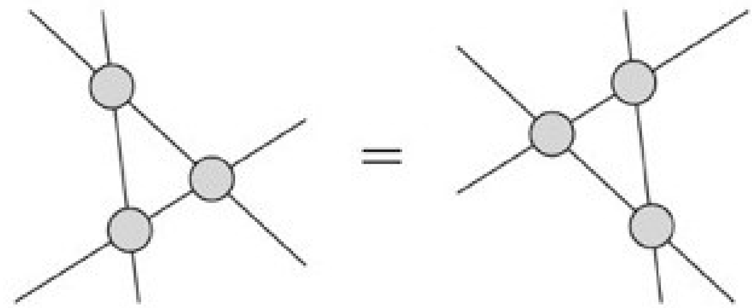
- AdS_3/CFT_2 duality;

Exact S-matrix and CDD ambiguity

Consider a relativistic integrable field theory with factorised scattering:

$$S_{ij}^{kl}(\theta_1 - \theta_2) \quad =$$





Castillejo-**D**alitz-**D**yson ambiguity: $$S_{ij}^{kl}(\theta) \rightarrow S_{ij}^{kl}(\theta)\, e^{i\delta_{ij}^{(t)}(\theta)}$$

The simplest possibility, consistent with the crossing and unitarity relations is:

$$\delta_{ij}^{(t)}(\theta) = \delta^{(t)}(m_i, m_j, \theta) = t\, m_i m_j \sinh(\theta)$$

The sine-Gordon NLIE

[A. Klümper, M. T. Batchelor and P. A. Pearce; C. Destri, H. DeVega]

The finite-size properties of the sine-Gordon model are encoded in the single counting function f(θ), solution to the following nonlinear integral equation:

$$f(\theta) = -imR\sinh(\theta) + i\alpha$$
$$- \int_{\mathcal{C}_1} dy\, \mathcal{K}(\theta - y)\, \ln\left(1 + e^{-f(y)}\right) + \int_{\mathcal{C}_2} dy\, \mathcal{K}(\theta - y)\, \ln\left(1 + e^{f(y)}\right)$$

For the ground state $\mathcal{C}_1 = \mathbb{R} + i0^+$ and $\mathcal{C}_2 = \mathbb{R} - i0^+$, but more more complicated contours appear for excited states.

$$\mathcal{K}(\theta) = \frac{1}{2\pi i}\partial_\theta \ln S_{sG}(\theta)$$

and

$$E(R) = m\left[\int_{\mathcal{C}_1} \frac{dy}{2\pi i}\sinh(y)\ln\left(1 + e^{-f(y)}\right) - \int_{\mathcal{C}_2} \frac{dy}{2\pi i}\sinh(y)\ln\left(1 + e^{f(y)}\right)\right]$$
$$P(R) = m\left[\int_{\mathcal{C}_1} \frac{dy}{2\pi i}\cosh(y)\ln\left(1 + e^{-f(y)}\right) - \int_{\mathcal{C}_2} \frac{dy}{2\pi i}\cosh(y)\ln\left(1 + e^{f(y)}\right)\right]$$

replacing

$$\mathcal{K}(\theta) \to \mathcal{K}(\theta) + \frac{1}{2\pi}\partial_\theta \delta_{CDD}(\theta) = \mathcal{K}(\theta) + t\frac{m^2}{2\pi}\cosh(\theta)$$

we get

$$\begin{aligned} f(\theta) = & -i\,m\,\sinh(\theta)\,[R + t\,E(R,t)] - i\,m\,\cosh(\theta)\,t\,P(R,t) \\ & - \int_{\mathcal{C}_1} dy\,\mathcal{K}(\theta - y)\,\ln\left(1 + e^{-f(y)}\right) + \int_{\mathcal{C}_2} dy\,\mathcal{K}(\theta - y)\,\ln\left(1 + e^{f(y)}\right) \end{aligned}$$

with

$$P(R,t) = P(R) = \frac{2\pi k}{R}, \quad k \in \mathbb{Z}$$

Zero-momentum case: $f(\theta\,|R\,,\,t) = f(\theta\,|\,(R + tE(R,t))\,,0)$

$$E(R\,,\,t) = E((R + tE(R,t))\,,0)$$

which allows to compute the exact form of the t-deformed energy level once its R-dependence is known at t = 0.

The latter equation is the implicit form of a solution of a well-known hydrodynamic equation, the invicid Burgers equation:

$$\partial_t E(R,t) = E(R,t)\, \partial_R E(R,t)\,,$$

the deformation parameter t plays the role of “time” variable, and the undeformed energy level serves as initial condition at t = 0. For the general case:

$$\begin{aligned} f(\theta) &= -i\, m\, \mathcal{R}_0\, \sinh(\theta+\theta_0) + i\alpha \\ &\quad - \int_{\mathcal{C}_1} dy\, \mathcal{K}(\theta-y)\, \ln\left(1+e^{-f(y)}\right) + \int_{\mathcal{C}_2} dy\, \mathcal{K}(\theta-y)\, \ln\left(1+e^{f(y)}\right) \end{aligned}$$

with

$$\begin{aligned} \mathcal{R}_0\ \cosh(\theta_0) &= R + tE(R,t), \\ \mathcal{R}_0\ \sinh(\theta_0) &= tP(R). \end{aligned}$$

Then

$$f(\theta|R,t) = f(\theta+\theta_0|\mathcal{R}_0,0).$$

and

$$E(R\,,\,t) = \cosh(\theta_0)\, E(\mathcal{R}_0,0) - \sinh(\theta_0)\, P(\mathcal{R}_0),$$

we now have an implicit form of the solution of the inviscid Burgers equation with a source term:

$$\partial_t E(R,t) = E(R,t)\,\partial_R E(R,t) + \frac{P(R)^2}{R}\,,$$

source term

again the undeformed energy E(R,0) plays the role of initial condition at t = 0.

Shock singularities

To see the emergence of the wave-breaking phenomenon in the inviscid Burgers equation, let's consider the P = 0 case. For a generic initial condition, E(R,0):

$$E(R,t) = E(\tilde{R}(R,t),0), \qquad \text{with} \qquad \tilde{R}(R,t) = R + t\,E(R,t)\,,$$

the map

$$R \to \tilde{R}(R,t)$$

has in general a number of square-root branch points in the complex R-plane. To find their location, consider the inverse map:

$$\tilde{R} \to R(\tilde{R},t) = \tilde{R} - t\,E(\tilde{R},0)\,.$$

Then, a singularity is characterised by the condition

$$\partial_{\tilde{R}}\, R(\tilde{R}\,,\,t)|_{\tilde{R}=\tilde{R}_c} = 1 - t\,\partial_{\tilde{R}}\;E(\tilde{R},0)\Big|_{\tilde{R}=\tilde{R}_c} = 0.$$

Indeed, around a solution of this equation, we have:

$$R(\tilde{R},t) \sim R_c + \mathcal{O}\left(\tilde{R} - \tilde{R}_c\right)^2, \qquad \text{for} \qquad R \sim R_c.$$

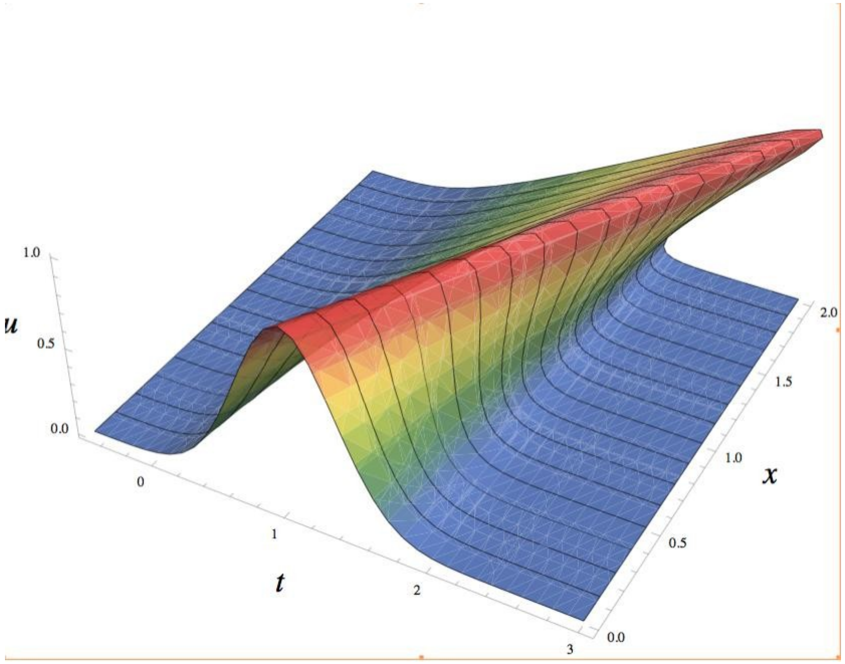
Therefore:

$$\tilde{R}(R,t) \sim \tilde{R}_c + \mathcal{O}\left(R - R_c\right)^{\frac{1}{2}}$$

and

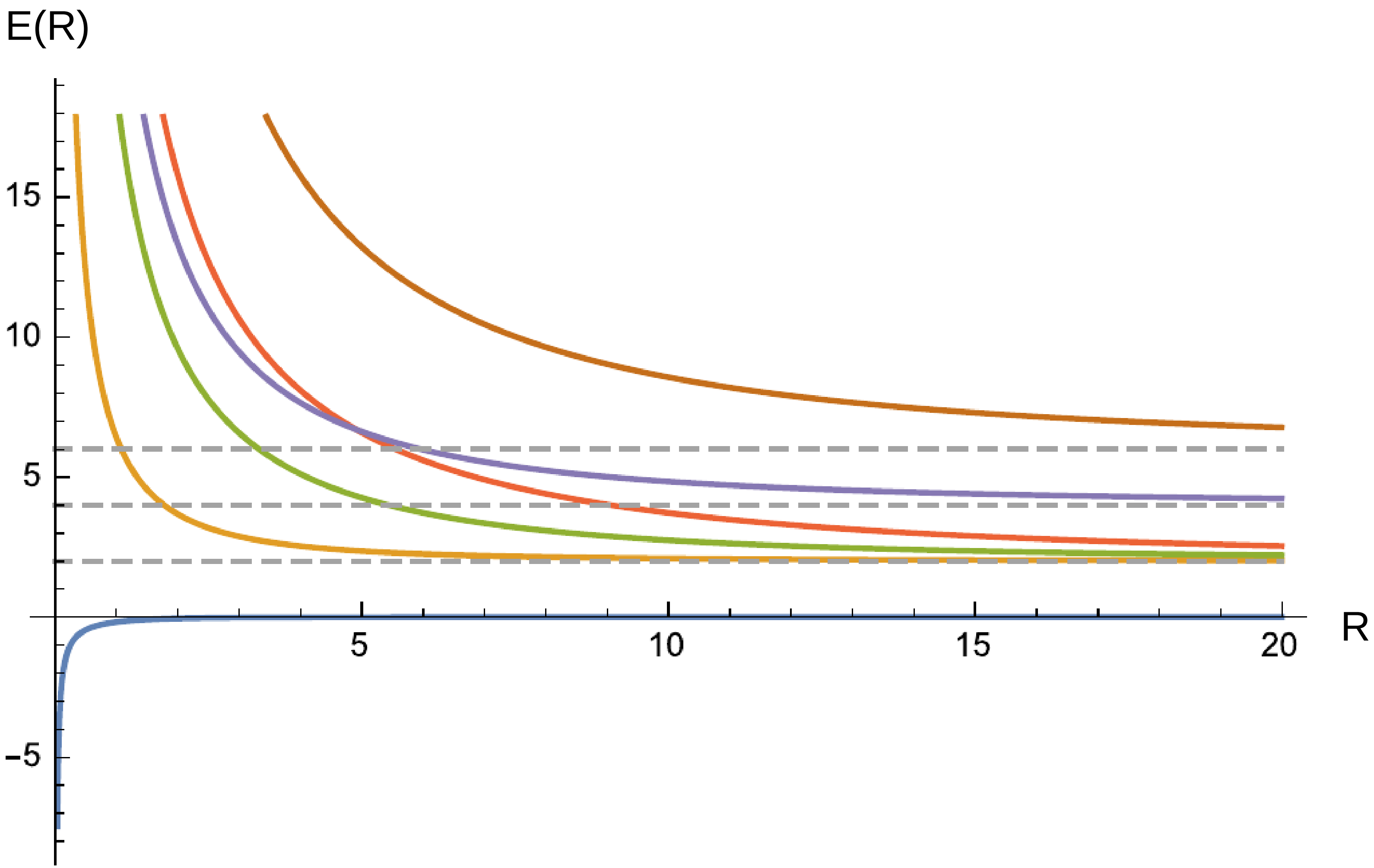
$$E(R,t) \;=\; E(\tilde{R}_c, 0) + \mathcal{O}(R - R_c)^{\frac{1}{2}}.$$

In typical hydrodynamic applications, the initial profile is smooth on the real-R axis, and for short times all branch points lie in the complex plane.



The time evolution however in general brings one of the singularities on the real domain in a finite time, producing a shock in the physical solution.

Typical t=0 finite-volume spectrum:



The energy levels display a pole at R = 0:

$$E(R,0) \sim -\pi \frac{c_{\text{eff}}}{6\,R}, \quad R \sim 0,$$

where, $c_{\text{eff}} = c - 24\Delta$ is the “effective central charge” of the UV CFT state.

This behaviour implies that, for small times t > 0, the equation $1 - t\,\partial_{\tilde{R}}\; E(\tilde{R},0)\Big|_{\tilde{R}=\tilde{R}_c} = 0$

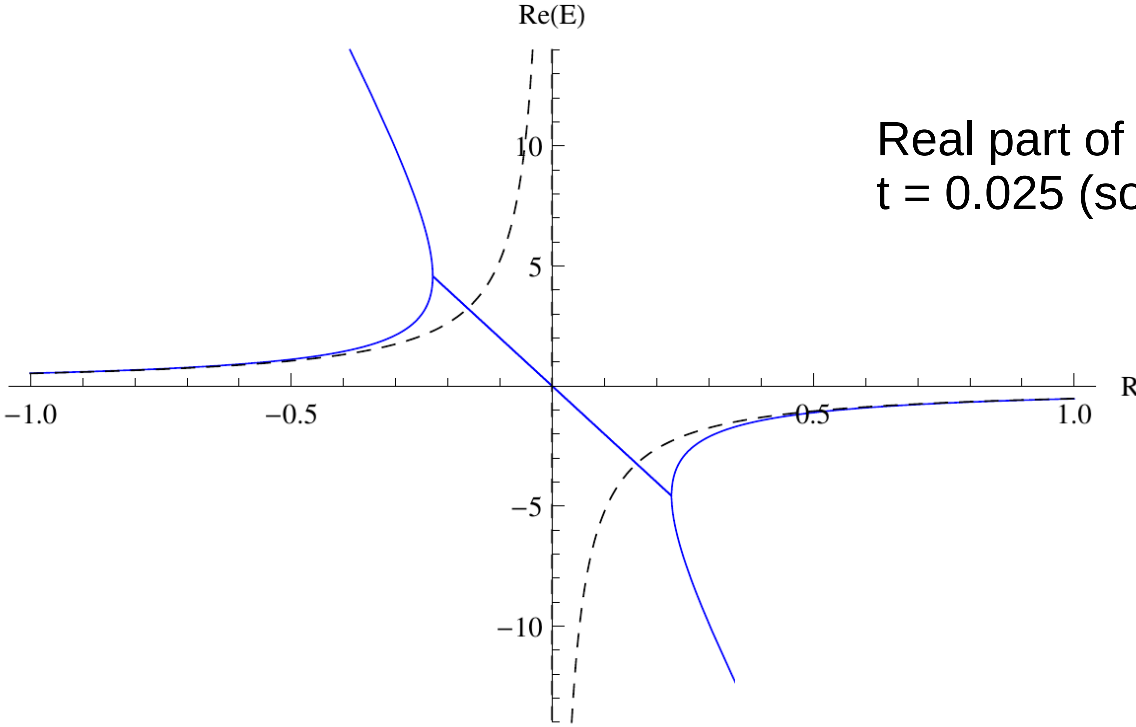
has two solutions very close to $\widetilde{R} = 0$

$$(\widetilde{R}_c)^2 = \pi \frac{c_{\text{eff}}}{6}\, t + \mathcal{O}(t^2),$$

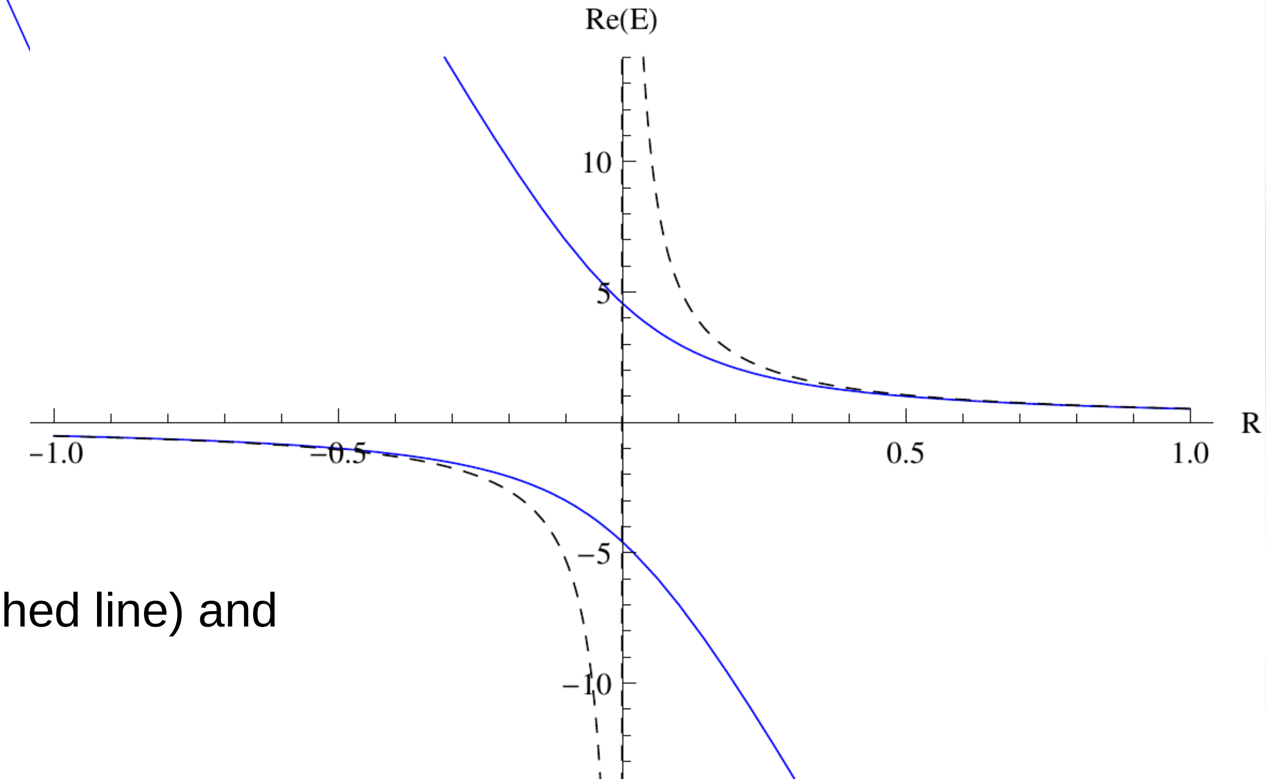
and correspondingly the solution is singular at

$$R_c = \tilde{R}_c - t\, E(\tilde{R}_c, 0) \sim t^{\frac{1}{2}} \sqrt{\frac{2\,\pi\, c_{\text{eff}}}{3}} + \mathcal{O}(t^{\frac{3}{2}}).$$

In other words, as soon as t > 0, the pole at R = 0 resolves into a pair of branch points.



Real part of E(R, t) for t = 0 (dashed line) and t = 0.025 (solid line), for $c_{eff} = 1$



Real part of E(R, t) for t = 0 (dashed line) and t = 0.025 (solid line), for $c_{eff} = -1$

The CFT case

An extra CDD factor couples left (-) with right (+) movers scattering, any NLIE or TBA equation leads to a pair of coupled algebraic equations:

$$E^{(+)}(R,t) = 2\pi \left(\frac{n_0 - c_{\text{eff}}/24}{R + 2tE^{(-)}(R,t)} \right), \qquad E^{(-)}(R,t) = 2\pi \left(\frac{\bar{n}_0 - c_{\text{eff}}/24}{R + 2tE^{(+)}(R,t)} \right)$$

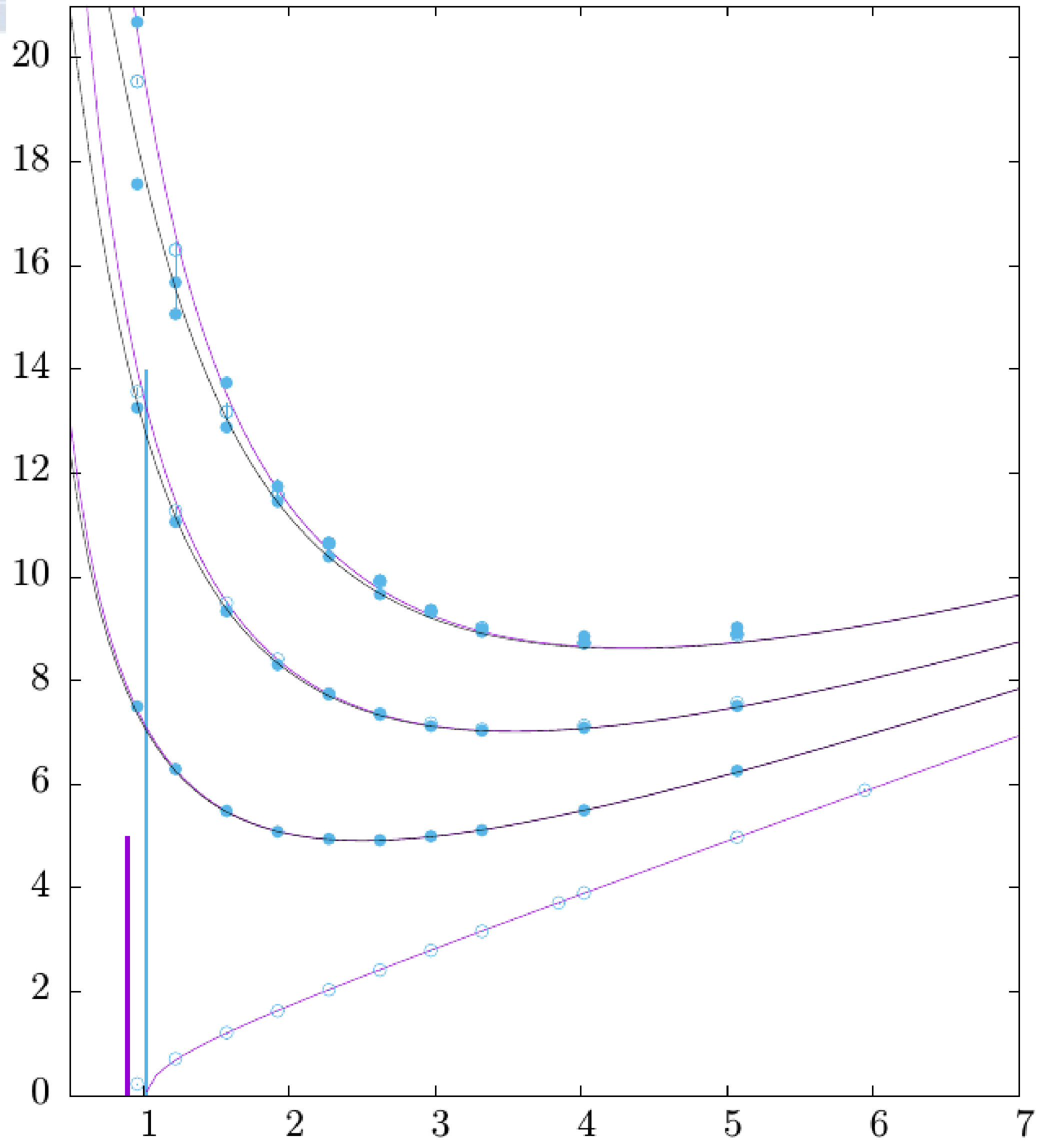
c_{eff}= c – 24 Δ(primary), obtained by an energy-dependent shift:

$$R \to R + 2tE^{(\mp)}(R,t)\,.$$

The total energy:

$$\begin{aligned} E(R,t) &= E^{(+)}(R,t) + E^{(-)}(R,t) \\ &= -\frac{R}{2t} + \sqrt{\frac{R^2}{4t^2} + \frac{2\pi}{t}\left(n_0 + \bar{n}_0 - \frac{c_{\text{eff}}}{12}\right) + \left(\frac{2\pi(n_0 - \bar{n}_0)}{R}\right)^2}, \end{aligned}$$

which matches the form of the (D=26, c_{eff}=24) Nambu Goto spectrum, for generic CFT, with t=1/(2σ), where σ is the string tension.



Best figure from:

A. Athenodoroua and M. Teper
arXiv:1602.07634

SU(2) at β = 16. Solid curves are continuum NG; dashed curves are NG with a ‘lattice’ dispersion relation. Thick vertical line is deconfining transition; thin vertical line is NG tachyonic transition.

Identification of the perturbing operator

Start from the equation

$$\partial_t E_n(R,t) = E_n(R,t)\partial_R E_n(R,t) + \frac{1}{R}P_n(R)^2$$

and use the standard relations

$$E_n(R,t) = -R\langle T_{yy}\rangle_n, \quad \partial_R E_n(R,t) = -\langle T_{xx}\rangle_n, \quad P_n = -iR\langle T_{xy}\rangle_n$$

then

$$\begin{aligned}\partial_t E_n(R,t) &= R\left(\langle T_{yy}\rangle_n\langle T_{xx}\rangle_n - \langle T_{xy}\rangle_n\langle T_{xy}\rangle_n\right)\\ &= -\frac{R}{\pi^2}\left(\langle T\rangle_n\langle \bar{T}\rangle_n - \langle\Theta\rangle_n\langle\Theta\rangle_n\right)\end{aligned}$$

with

$$T_{xx} = -\frac{1}{2\pi}(\bar{T}+T-2\Theta)\,, \quad T_{yy} = \frac{1}{2\pi}(\bar{T}+T+2\Theta)$$

$$T_{xy} = \frac{i}{2\pi}(\bar{T}-T).$$

Use Zamolodchikov's definition for the $\mathrm{T\bar{T}}$ composite operator:

$$\mathrm{T\bar{T}}(z,\bar{z}) := \lim_{(z,\bar{z})\to(z',\bar{z}')} T(z,\bar{z})\bar{T}(z',\bar{z}') - \Theta(z,\bar{z})\Theta(z',\bar{z}') + \text{total derivatives}$$

which fulfils the following factorization property:

$$\boxed{\langle \mathrm{T\bar{T}} \rangle_n = \langle T \rangle_n \langle \bar{T} \rangle_n - \langle \Theta \rangle_n \langle \Theta \rangle_n}$$

Proof:

Consider that $\langle \mathcal{O}_i(z)\mathcal{O}_j(z') \rangle = G_{ij}(z-z')$ together with the conservation laws:

$$\bar{\partial} T(z,\bar{z}) = \partial \Theta(z,\bar{z}) \,, \quad \partial' \bar{T}(z',\bar{z}') = \bar{\partial}' \Theta(z',\bar{z}')$$

keeping z and z' separated:

$$\langle \bar{\partial} T(z,\bar{z}) \bar{T}(z',\bar{z}') - \bar{\partial}\Theta(z,\bar{z})\Theta(z',\bar{z}') \rangle_n =$$

$$-\langle \Theta(z,\bar{z}) \partial' \bar{T}(z',\bar{z}') - \Theta(z,\bar{z}) \bar{\partial}' \Theta(z',\bar{z}') \rangle_n = 0$$

then

$$\langle \mathrm{T}\bar{\mathrm{T}}(z-z',\bar{z}-\bar{z}')\rangle_n \equiv \langle \mathrm{T}\bar{\mathrm{T}}(0,0)\rangle_n$$

and sending $z-z',\bar{z}-\bar{z}' \to \infty$ we get the desired factorisation result.

Putting all this information together:

$$\partial_t E_n(R,t) = -\tfrac{R}{\pi^2}\langle \mathrm{T}\bar{\mathrm{T}}\rangle_n = -\tfrac{1}{\pi^2}\langle \int_0^R dx\, \mathrm{T}\bar{\mathrm{T}}(z,\bar{z})\rangle_n$$

and

$$\boxed{\partial_t \ln Z(R,L,t) = \tfrac{1}{\pi^2}\langle \int_0^R dx \int_0^L dy\, \mathrm{T}\bar{\mathrm{T}}(z,\bar{z})\rangle,}$$

Therefore, up to total derivatives:

$$\boxed{\partial_t \mathcal{L}(z,\bar{z},t) = -\frac{1}{\pi^2}\,\mathrm{T}\bar{\mathrm{T}}(z,\bar{z},t).}$$

which generalises the near CFT perturbative result:

$$\mathcal{L}(z,\bar{z},t) = \mathcal{L}(z,\bar{z},0) - \frac{t}{\pi^2}\,T(z)\bar{T}(\bar{z}) + \mathcal{O}\left(t^2\right).$$

Classical Action: one bosonic field

$$\tau = -\frac{\partial \mathcal{L}}{\partial\left(\bar{\partial}\phi\right)}\partial\phi, \quad \bar{\tau} = -\frac{\partial \mathcal{L}}{\partial\left(\partial\phi\right)}\bar{\partial}\phi, \quad \theta = \frac{1}{2}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial\phi\right)}\partial\phi + \frac{\partial \mathcal{L}}{\partial\left(\bar{\partial}\phi\right)}\bar{\partial}\phi - 2\mathcal{L}\right)$$

$$\tau\bar{\tau}(z,\bar{z}) = \tau(z,\bar{z})\,\bar{\tau}(z,\bar{z}) - \theta^2(z,\bar{z})$$

$$\partial_t \mathcal{L}(z,\bar{z},t) = -\left(\frac{\partial \mathcal{L}}{\partial\left(\bar{\partial}\phi\right)}\partial\phi\right)\left(\frac{\partial \mathcal{L}}{\partial\left(\partial\phi\right)}\bar{\partial}\phi\right) + \frac{1}{4}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial\phi\right)}\partial\phi + \frac{\partial \mathcal{L}}{\partial\left(\bar{\partial}\phi\right)}\bar{\partial}\phi - 2\mathcal{L}\right)^2$$

Starting from:

$$\boxed{\mathcal{L}_{\text{free}} = \partial\phi\,\bar{\partial}\phi}$$

$$\mathcal{L}(z,\bar{z},t) = \sum_{j=0}^{\infty} t^j \mathcal{L}^{(j)} \,, \qquad \text{with} \qquad \mathcal{L}^{(0)} \equiv \mathcal{L}_{\text{free}}$$

$$\mathcal{L}(z,\bar{z},t) = \frac{1}{2t}\left(-1 + \sqrt{1 + 4t\,\mathcal{L}^{(0)}}\right) = -\frac{1}{2t} + \mathcal{L}_{\text{NG}}$$

Nambu Goto action in 3D target space:

$$\mathcal{L}_{\text{NG}}\, dxdy = \frac{1}{2t}\sqrt{\det(\sum_{\mu=1}^{D} \partial_\alpha X^\mu\, \partial_\beta X^\mu)\, dxdy},$$

in the static gauge:

$$X^1 \to x \quad X^2 \to y, \quad \text{and} \quad X^3 \to t^{\frac{1}{2}}\, \phi.$$

N free massless boson →Nambu Goto in (N+2) taget space:

$$\mathcal{L} = -\frac{1}{2t} + \mathcal{L}_{\text{NG}}^{N+2}, \; \mathcal{L}_{\text{NG}}^{N+2} = \frac{1}{2t}\sqrt{1 + 4t\,\mathcal{L}_{\text{free}}^{N} - 4t^2\mathcal{B}_N} \,,$$

$$\mathcal{B}_N = \sum_{i=1}^{N}(\partial\phi_i)^2 \sum_{j=1}^{N}(\bar{\partial}\phi_j)^2 - \left(\sum_{i=1}^{N}\partial\phi_i\bar{\partial}\phi_i\right)^2$$

Single boson field with generic potential

$$\boxed{\mathcal{L}^V = \partial\phi\bar{\partial}\phi + m^2 V[\phi]}$$

$$\downarrow$$

$$\mathcal{L}_t = \frac{2tm^2V - 1 + \sqrt{1 + 4t\partial\phi\bar{\partial}\phi\left(1 - tm^2V\right)}}{2t\left(1 - tm^2V\right)}$$

The Lagrangian is very complicated, but we already know that its quantum spectrum is:

$$\boxed{E(R\,,\,t) = E\left(\left(R + tE(R,t)\right),0\right)}$$

Generalisations

Many known scattering models differ only by CDD factors

Thermally perturbed Ising model (c=1/2) $\longrightarrow$ Sinh-Gordon (c=1)

ADE minimal Scattering models $\longrightarrow$ Simply-Laced Affine Toda models

with

$$\hat{S}(\theta) \to \hat{S}(\theta)\,\Phi(\theta)$$

$$\Phi(\theta)\Phi(-\theta) = 1\,, \qquad \Phi(i\pi+\theta)\Phi(i\pi-\theta) = 1$$

CDD factors admit an expansion in terms of the spin of the local integrals of motion:

$$\Phi(\theta) = \exp\left\{ i \sum_{s=1}^{\infty} \alpha_s \sinh(s\theta) \right\}$$

F. Smirnov, A. Zamolodchikov: adding CDD factors leads to well-identified effective field theories. The corresponding perturbing fields are the composite operators:

$$\mathrm{T\bar{T}}_s(z,\bar{z}) := \lim_{(z,\bar{z})\to(z',\bar{z}')} T_{s+1}(z,\bar{z})\bar{T}_{s+1}(z',\bar{z}') - \Theta_{s-1}(z,\bar{z})\bar{\Theta}_{s-1}(z',\bar{z}') + \text{total derivatives}$$

Generalised Burgers equation

$$E(R,t) \to I_s(R_1,\ldots,R_N;t_1,\ldots,t_N) = I_s(R_1+t_1I_1,\ldots,R_N+t_NI_N;0,\ldots,0),$$

$s=1,\ldots,N$ (i.e. a Generalized Gibbs ensemble)

$$\frac{\partial}{\partial t_m} I_s(\vec{R};\vec{t}) = I_m(\vec{R};\vec{t}) \frac{\partial}{\partial R_m} I_s(\vec{R};\vec{t})$$

Conclusions

- Adding CDD factors generates interesting, often UV incomplete, QFTs.

- While the study of generic irrelevant perturbations in QFT remains very problematic, the $T\bar{T}$ perturbation appears to be surprisingly easy to treat also in non-integrable models.

- The research on this topic may clarify important aspects concerning the appearance of singularities in effective QFT (Landau pole and tachyon singularity).

$$\langle \varphi_i^{(t)}(0,0) \rangle_t^{(\text{plane})} = \frac{\langle \varphi_i(0,0) \rangle_{t=0}^{(\text{plane})}}{(1 + t\,\mathcal{E}^{(\text{bulk})}(\vec{\mathrm{g}},0))^2}$$

- It may be useful in the effective string framework since contributions from higher-dimension integrable composite operators can be added (Virasoro, W-algebra ...).

- It may help to understand the origin of massive modes propagating on the flux tube.