Lectures at Les Houches - Predoctoral School 2018

The unitary Fermi gas : relations, symmetries, diagrams

unitary → or resonant

symmetries → unitary ($a=\infty$)

relations, diagrams → resonant (any a)

Chapter 1 : Resonant interactions and the universal zero-range limit

Part 1 : 2-body

Consider 2 distinguishable particles

(or 2 fermions in different internal states, ↑ and ↓)

$\Psi(\vec{r}_1, \vec{r}_2)$ no symmetry constraint when exchanging $\vec{r}_1$ and $\vec{r}_2$

$$-\frac{\hbar^2}{2m}\left(\Delta_{\vec{r}_1}+\Delta_{\vec{r}_2}\right)\Psi + V(r)\,\Psi = E\cdot\Psi$$

$$r=\|\vec{r}\|\,,\qquad \vec{r}=\vec{r}_2-\vec{r}_1\,.$$

Center of mass : $\vec{C}=\dfrac{\vec{r}_1+\vec{r}_2}{2}$

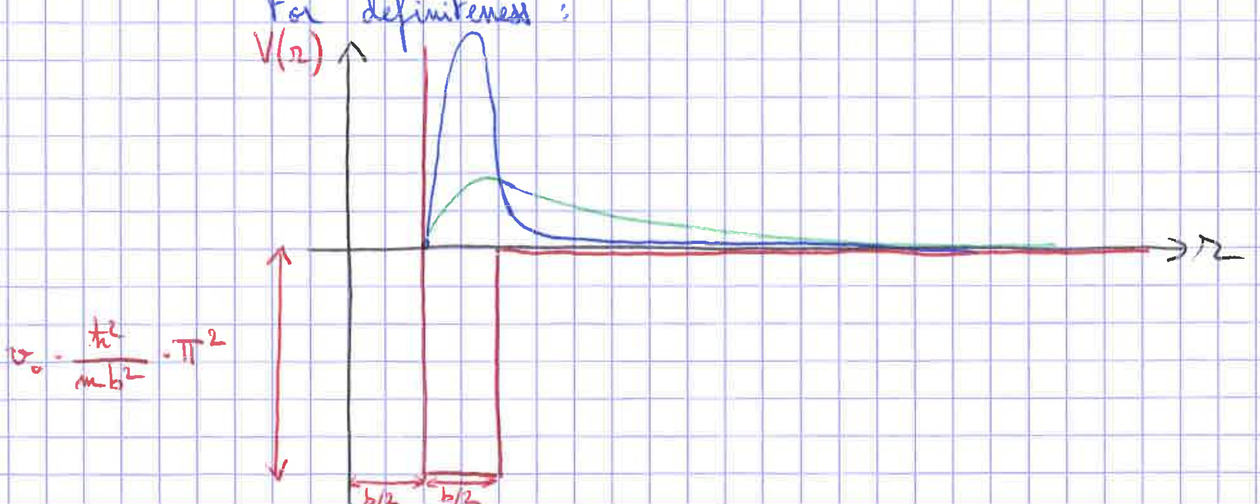
Separability : $\Psi(\vec{r}_1,\vec{r}_2)=\Psi_{CM}(\vec{C})\cdot\Psi_{int}(\vec{r})$

because

- $-\dfrac{\hbar^2}{2m}\left(\Delta_{\vec{r}_1}+\Delta_{\vec{r}_2}\right)=-\dfrac{\hbar^2}{4m}\Delta_{\vec{C}}-\dfrac{\hbar^2}{m}\Delta_{\vec{r}}$
- $V(r)$ indep of $\vec{C}$

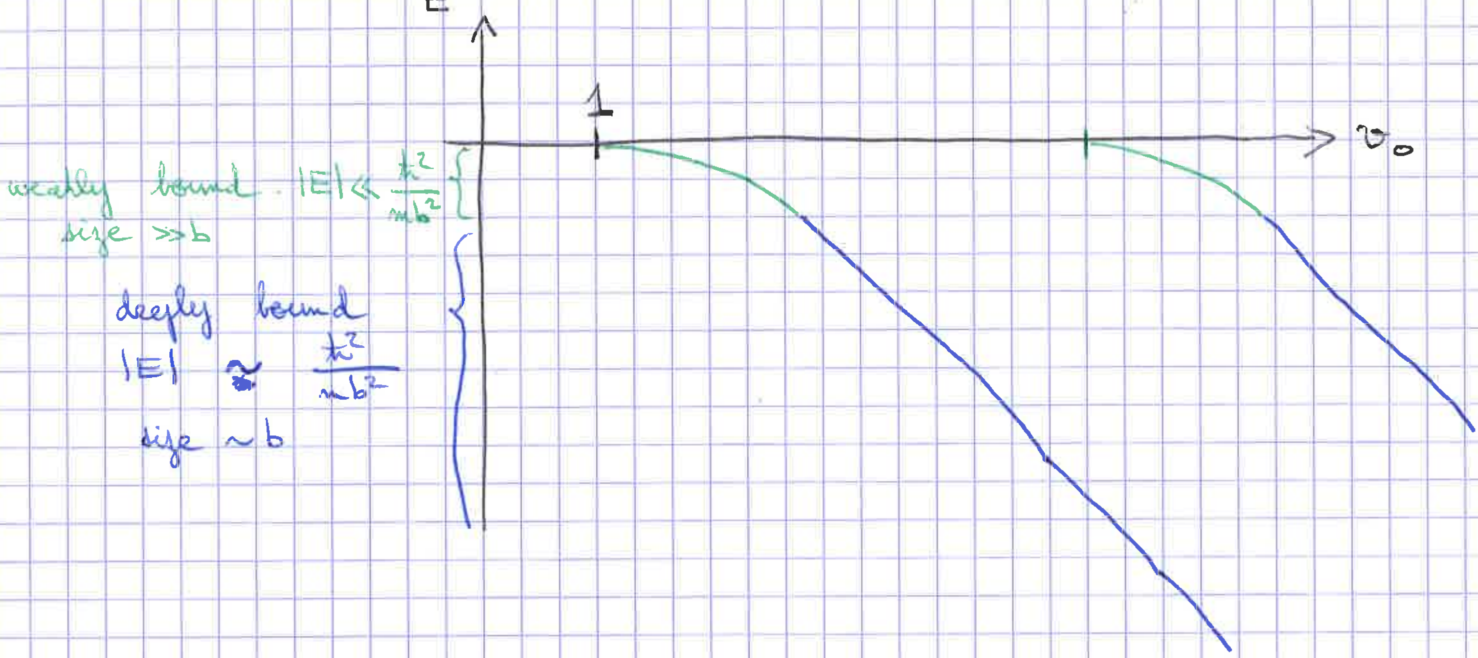
$$\left[-\frac{\hbar^2}{m}\Delta_{\vec{r}}+V(r)\right]\Psi_{\cancel{int}}(\vec{r})=E_{\cancel{int}}\cdot\Psi_{\cancel{int}}(\vec{r}). \qquad (1)$$

For definiteness :



Dimers (= bound states) $E < 0$

$\psi(\vec{r})$ normalizable.



Scattering states $E > 0$. $E =: \frac{\hbar^2 k^2}{m}$

Solutions of (1) with

$$\psi(\vec{r}) \underset{r\to\infty}{\simeq} e^{i\vec{k}\cdot\vec{r}} + f_k(\hat{r}) \cdot \frac{e^{ikr}}{r} \qquad (2)$$

$$\hat{r} := \frac{\vec{r}}{r}$$

$$\lim_{k \to 0} f_k(\hat{r}) =: -a \qquad \text{scattering length.}$$

Universality: When $v_0 \longrightarrow 1^+$:

$\frac{a}{b} \longrightarrow +\infty$ and

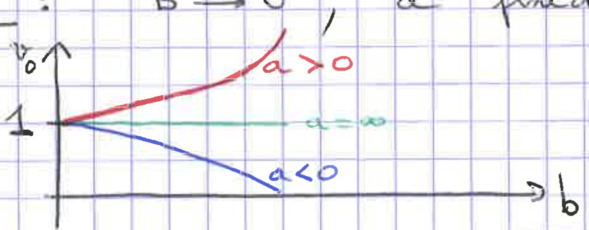
$$\boxed{\begin{aligned} E &\sim -\frac{\hbar^2}{ma^2} \\ \psi(r) &\longrightarrow \mathcal{N} \cdot \frac{e^{-\frac{r}{a}}}{r} \end{aligned}} \qquad (3)$$

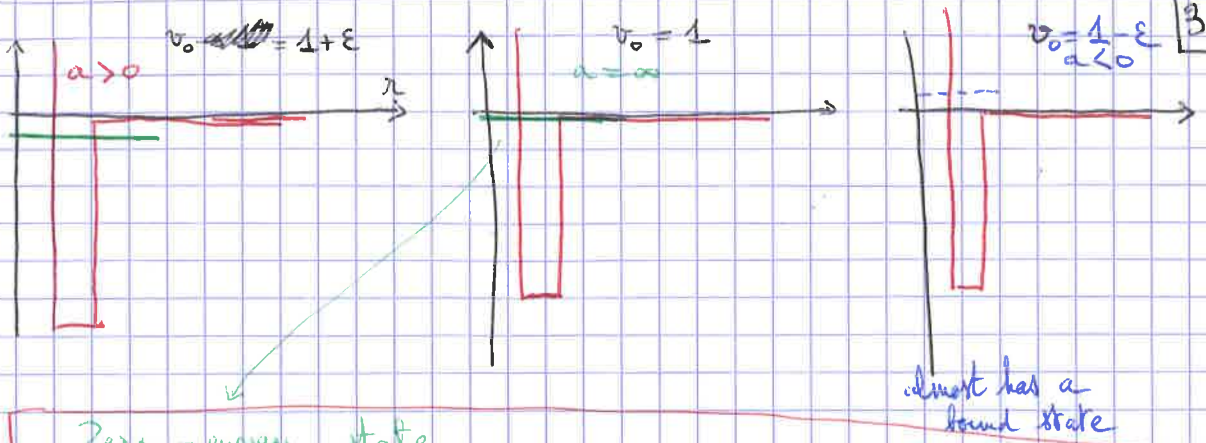
(Here, $f \sim g$ means $\frac{f}{g} \to 1$ "equivalent")

for any shape of $V(r)$.
↳ short-range

Zero-range limit: $b \to 0$, a fixed.

$v_0 \longrightarrow 1$





Zero-energy state

$$\left[-\frac{\hbar^2}{m}\Delta_{\vec r} + V(r)\right]\phi_0(r) = 0$$

$r>b$: $\Delta\phi_0 = 0 \Rightarrow \phi_0(r) = \frac{1}{r} + \text{cst} = \frac{1}{r} - \frac{1}{a}$ (our normalization)

keep

($E\to 0$ in (2))

The Zero-Range Model (ZRM)

Note : ZRM $\neq \delta^3(\vec r)$ { depth $\propto \frac{1}{b^2} \neq \frac{1}{b^3}$ (2D : also ZRM $\neq$ Dirac. 1D : ZRM = Dirac) }

- $\vec r \neq \vec 0$: $-\frac{\hbar^2}{m}\Delta_{\vec r}\,\psi(\vec r) = E\cdot\psi(\vec r)$
- $r\to 0$: $\exists A$, $\psi(\vec r) = \left(\frac{1}{r} - \frac{1}{a}\right)\cdot A + O(r)$.

(Contact Condition – CC)

one dimer for $a>0$,

$$E = -\frac{\hbar^2}{ma^2}$$

$$\psi(r) = \mathcal{N}\cdot\frac{e^{-r/a}}{r}.$$

→ obtained (3) directly, easily.

$E>0$: $\psi(\vec r) = e^{ikr} + f_k\,\frac{e^{ikr}}{r}$

$$f_k = \frac{-1}{\frac{1}{a} + ik}.$$

$a=\infty$: $|f_k| = \frac{1}{k}$ unitary limit

(maximal value allowed by optical theorem)

Part 2 : N-body

$\psi(\vec r_1, \dots, \vec r_N)$

fermions, 2 internal states (hyperfine) "↑", "↓"

considered as 2 distinguishable species

with $N_\uparrow$, $N_\downarrow$ fixed $(N = N_\uparrow + N_\downarrow)$

OK if spin-changing matrix elements of interaction negligible.

$$\Psi(\underbrace{\vec{r}_1, \ldots, \vec{r}_{N_1}}_{\text{antisym}}, \underbrace{\vec{r}_{N_1+1}, \ldots, \vec{r}_N}_{\text{antisym}})$$

ZRM :

keep

- all $r_{ij} > 0$: $\left[-\frac{\hbar^2}{2m} \sum_{i=1}^{N} \left[\Delta_{\vec{r}_i} + U(\vec{r}_i) \right] \right] \Psi(\vec{r}_1 \ldots \vec{r}_N) = E\,\Psi(\vec{r}_1 \ldots \vec{r}_N)$

  ↳ external potential (harmonic trap, box)

- $r_{ij} \to 0$: $\vec{C}_{ij} = \frac{\vec{r}_i + \vec{r}_j}{2}$ fixed, $(\vec{r}_k)_{k \neq i,j}$ fixed

$$\Psi(\vec{r}_1 \ldots \vec{r}_N) = \left(\frac{1}{r_{ij}} - \frac{1}{a} \right) A_{ij}\left(\vec{C}_{ij}, (\vec{r}_k)_{k\neq i,j}\right) + O(r_{ij})$$

(CC) (4)

Finite-range model :

$$-\frac{\hbar^2}{2m} \sum_{i=1}^{N} \left[\Delta_{\vec{r}_i} + U(\vec{r}_i) \right] \Psi + \sum_{i<j} V(r_{ij}) \Psi = E\,\Psi \qquad (5)$$

↳ ~~short range~~ ~~e.g.~~ above hard-core + square well.

Zero-range limit : $b \to 0$, a fixed, U fixed

(no deep dimer)

Finite-range model → ZRM

$$E_n^{(b)} \longrightarrow E_n^{\text{ZRM}}$$

$$\Psi_n^{(b)}(\vec{r}_1 - \vec{r}_N) \longrightarrow \Psi_n^{\text{ZRM}}(\vec{r}_1 - \vec{r}_N) \quad \text{when all } r_{ij} > 0.$$

universality : holds for any shape of $V(r)$

↳ short-range, hard-core

Justification : consider Schrö. eq. (5) for

$$r_{ij} \sim b \ll \text{all other } r_{kl}.$$

change of variables : $(\vec{r}_i, \vec{r}_j) \longrightarrow (\vec{C}_{ij}, \vec{r}_{ij} = \vec{r}_j - \vec{r}_i)$

$$-\frac{\hbar^2}{m}\Delta_{\vec{r}_{ij}}\Psi + V(r_{ij})\Psi - \frac{\hbar^2}{4m}\Delta_{\vec{C}_{ij}}\Psi - \frac{\hbar^2}{2m}\sum_{k\neq i,j}\Delta_{\vec{r}_k}\Psi + \sum_{\substack{k<l \\ (k,l)\neq(i,j)}} V(r_{kl})\Psi = E\cdot\Psi$$

$\sim \frac{\hbar^2}{mb^2}\Psi$ [$\Psi(r_{ij})$ varies at scale b] → large

$\sim -\frac{\hbar^2}{mb^2}\Psi$ → large

weakly b-dependent (≃ ZRM value) → order 1

must compensate (to leading order)

$$\Rightarrow \quad -\frac{\hbar^2}{m}\Delta_{\vec{r}_{ij}}\Psi + V(r_{ij})\Psi \simeq 0 \quad \text{(to leading order)}$$

$$\Rightarrow \quad \boxed{\Psi(\vec{r}_1 \ldots \vec{r}_N)} \simeq \phi_0(r_{ij})\cdot A \qquad (*)$$

$r_{ij} > b$: $\phi_0(r_{ij}) \to \frac{1}{r_{ij}} - \frac{1}{a}$

⇝ ZRM.

Chapter 2: Tan relations

obtained by Shina Tan in 2005 during his PhD.

I will mostly follow [LNP] arXiv: 1103.2851
[PRA/12] 1204.3204

For simplicity: Thermodynamic Limit: $\left\{\begin{array}{l}\text{Volume } v \to \infty \\ N_\sigma \to \infty \\ (\sigma = \uparrow,\downarrow)\end{array}\right)$ with $n_\sigma = \frac{N_\sigma}{v}$ fixed

T fixed.

ZRM. a fixed.

$n_\sigma(k)$ momentum distribution

$\mathcal{E} = \frac{E}{v}$, $\mathcal{S} = \frac{S}{v}$, $\mathcal{F} = \frac{F}{v}$, p: pressure, μ_σ: chemical potentials

$g_2(r) := \langle \hat{n}_\uparrow(\vec{r})\,\hat{n}_\downarrow(\vec{0})\rangle$

C "contact"

$\mathcal{C} = \frac{C}{v}$ "contact density"

$$(1) \quad n_\sigma(k) \underset{k\to\infty}{\sim} \frac{\mathcal{C}}{k^4}$$

$$(2) \quad \mathcal{C}\cdot\frac{\hbar^2}{4\pi m} = \left(\frac{\partial \mathcal{E}}{\partial(-1/a)}\right)_{\mathcal{S},n_\sigma} = \left(\frac{\partial F}{\partial(-1/a)}\right)_{T,n_\sigma} = \left(\frac{\partial p}{\partial(1/a)}\right)_{T,\mu_\sigma}$$

$$(3) \quad g_2(r) \underset{r\to 0}{\sim} \frac{\mathcal{C}}{r^2}\cdot\frac{1}{(4\pi)^2}$$

$$(4) \quad \mathcal{E} = \sum_{\sigma=\uparrow,\downarrow}\int\frac{d^3k}{(2\pi)^3}\frac{\hbar^2k^2}{2m}\left[n_\sigma(k)-\frac{\mathcal{C}}{k^4}\right] + \frac{\mathcal{C}}{a}\cdot\frac{\hbar^2}{4\pi m}$$

Note : (3) implies :

Measure all particle positions in a unit volume.

Number of pairs of particles separated by a distance $< s$:

$$\underset{s\to 0}{\sim} \quad \mathcal{C}\cdot s\cdot\frac{1}{4\pi}$$

↳ anomalously high (normally : s^3)

Experiments : (1): Jin group (JILA) $n(k)$: ToF at $a=0$ (K better than Li)

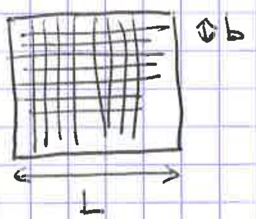
(2): ENS (EoS)

(3): Swinburne - Bragg - $S(q) = FT[g_2(r)]$

$$S(q) \underset{q\to\infty}{\sim} \# \frac{\mathcal{C}}{q}.$$

Derivation

Lattice Model (LM) discretize space



$\vec{r} \in (b\cdot\mathbb{N})^3 \cap [0,L]^3 =: \mathcal{R}$

$(V = L^3)$ $\qquad V_{\vec{r}_1,\vec{r}_2} = g_0\cdot\frac{\delta_{\vec{r}_1,\vec{r}_2}}{b^3}$

universal zero-range limit : $\text{LM} \xrightarrow[\substack{b\to 0\\ a \text{ fixed}}]{} \text{ZRM}$

$$\hat{H} = \sum_{\sigma=\uparrow,\downarrow}\sum_{\vec{k}\in\mathcal{D}_L}\frac{\hbar^2k^2}{2m}\hat{c}^+_{\vec{k}\sigma}\hat{c}_{\vec{k}\sigma} + g_0\sum_{\vec{r}\in\mathcal{R}} b^3\,\hat{n}_\uparrow(\vec{r})\hat{n}_\downarrow(\vec{r})$$

$$\mathcal{D}_L = \left(\frac{2\pi}{L}\mathbb{Z}\right)^3 \cap \mathcal{D} \qquad \mathcal{D} = \left[-\frac{\pi}{b};\frac{\pi}{b}\right)$$

$$\hat{n}_\sigma(\vec{r}) = \hat{\psi}^+_\sigma(\vec{r})\,\hat{\psi}_\sigma(\vec{r})$$

$$\{\hat{\psi}_\sigma(\vec{r}), \hat{\psi}^+_{\sigma'}(\vec{r}')\} = \frac{\delta_{\sigma,\sigma'}\delta_{\vec{r},\vec{r}'}}{b^3}$$

$$\hat{c}_{\vec{k}\sigma} = \sum_{\vec{r}\in\mathcal{R}} \frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{\mathcal{V}}} \cdot \hat{\psi}_\sigma(\vec{r}) \, .$$

2-body scattering ~~st~~ states in the ~~latt~~ LM:

one finds (using T-matrix formalism [PRA'12, App. A] similarly to Meera Parish's lecture):

key

$$\boxed{\begin{aligned} \frac{1}{g_0} &= \frac{m}{4\pi\hbar^2 a} - \int_{\mathcal{D}} \frac{d^3k}{(2\pi)^3} \frac{m}{\hbar^2 k^2} \\ \phi_0(\vec{r}=\vec{0}) &= -\frac{4\pi\hbar^2}{m g_0} \end{aligned}}$$

Note: $\frac{1}{g_0} \propto -\frac{1}{b}$

$g_0 \propto -b$

$V_{\vec{r}_1,\vec{r}_2} \propto -\frac{1}{b^2}\delta_{\vec{r}_1,\vec{r}_2}$

cf. square-well depth $\sim \frac{1}{b^2}$.

"ZRM ~ $b\times$ Dirac"

we take (2) as definition of $\mathcal{C}$, and derive (1,3,4). ∞

$$\frac{\hbar^2}{4\pi m} C \overset{(2)}{:=} \left(\frac{\partial F}{\partial(-1/a)}\right)_{T,N_\sigma,\mathcal{V}} \quad \text{[Hellmann-Feynman]} \; \underset{\downarrow}{=} \left\langle \frac{d\hat{H}}{d(-1/a)} \right\rangle_{\text{canonical }(T,N_\sigma,\mathcal{V})}$$

$$\frac{d\hat{H}}{d(1/a)} = \frac{d\hat{H}}{d(1/g_0)} \cdot \frac{d(1/g_0)}{d(1/a)}$$

$$= -g_0^2 \frac{d\hat{H}}{dg_0} \cdot \frac{m}{4\pi\hbar^2}$$

$$\langle \text{—} \rangle = \frac{E_{int}}{g_0}$$

$$\boxed{E_{int} = \frac{1}{g_0} \cdot C \cdot \frac{\hbar^4}{m^2}}$$

$$E = E_{kin} + E_{int}$$

$$\mathcal{E} = \sum_{\sigma=\uparrow\downarrow} \int_{\mathcal{D}_{\mathbb{R}^3}} \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \left[n_\sigma(\vec{k}) - \frac{\mathcal{C}}{k^4} \right] + \mathcal{C} \cdot \frac{1}{a} \cdot \frac{\hbar^2}{4\pi m}$$

ZRL $(b\to 0)$ $\Rightarrow$ (4)

$\int_k$ converges $\Rightarrow$ ~~⇐~~ (1).

(3) ? $\quad g_2(\vec{0}) = \frac{E_{int}}{g_0 \cdot \mathcal{V}} = \mathcal{C} \cdot \frac{1}{g_0^2} \cdot \frac{\hbar^4}{m^2}$

As for the finite-range continuous-space model discussed above (see (*) on page 5) we expect: (for the LM)

$$\psi(\vec{r}_1 - \vec{r}_N) \simeq \phi_0(\vec{r}_{ij}) \times A$$

for r_{ij} small enough

$$\Rightarrow g_2(\vec{r}) \underset{r \text{ small}}{\simeq} |\phi_0(\vec{r})|^2 \times \text{Cst}$$

(can be determined at $\vec{r} = \vec{0}$)

$$\Rightarrow g_2(\vec{r}) \simeq \underbrace{|\phi_0(\vec{r})|^2}_{\simeq \frac{1}{r^2} \quad (r \gg b)} \times \mathcal{C} \times \frac{1}{(4\pi)^2} .$$

ZRL $\longrightarrow$ (3).

Note: Direct derivation of (1,3) within ZRM [PRA '12]:

idea: $\psi \underset{r_{ij} \to 0}{\propto} \frac{1}{r_{ij}} \cdot A$

$$\Rightarrow g_2(r) \underset{r \to 0}{\propto} \frac{1}{r} \cdot \int |A|^2$$

$$\left(\frac{1}{r}, r \to 0\right) \xrightarrow{\text{Fourier}} \left(\frac{1}{k^2}, k \to \infty\right)$$

$$\xrightarrow{n(k) \propto |\psi|^2} n_\sigma(k) \propto \frac{1}{k^4} \cdot \int |A|^2$$

[Physical interpretation: nearby pairs of atoms, with large momenta $\vec{k}, -\vec{k}$]

Chapter 4: $\boxed{\text{A two-channel model}}$

[LNP]
[arXiv: 0807.0078]

atoms: fermions, ~~[illegible]~~ $\sigma = \uparrow, \downarrow$. $\hat{\psi}_\sigma^+(\vec{r})$

closed-channel "bare" molecule: boson . $\hat{\psi}_b^+(\vec{r})$

UV regularisation: a Lattice model, spacing b (for simplicity — also possible: continuous space, separable potential of finite range)

$$\left[\hat{\psi}_b(\vec{r}), \hat{\psi}_b^+(\vec{r}')\right] = \frac{\delta_{\vec{r},\vec{r}'}}{b^3}$$

$$\hat{b}_{\vec{k}} = \text{Fourier}\{\hat{\psi}_b(\vec{r})\}$$

$$\hat{H} = \sum_{\vec{k} \in \mathcal{D}_L} \sum_{\sigma = \uparrow,\downarrow} \frac{\hbar^2 k^2}{2m} c^+_{\vec{k}\sigma} c_{\vec{k}\sigma} + \sum_{\vec{k} \in \mathcal{D}_L} \left[\frac{\hbar^2 k^2}{4m} + E_b(B)\right] b^+_{\vec{k}} b_{\vec{k}}$$

$$+ \Lambda \cdot \sum_{\vec{r} \in \mathcal{R}} b^3 \left[(\psi_b^+ \psi_\downarrow \psi_\uparrow)(\vec{r}) + h.c.\right]$$

$$+ g_0 \sum_{\vec{r} \in \mathcal{R}} b^3 \, \hat{n}_\uparrow(\vec{r}) \hat{n}_\downarrow(\vec{r}) .$$

$$E_b(B) = \mu_b \cdot (B - B_{res})$$

2-body scattering : $\Rightarrow$ $a(B) = a_{bg} \cdot \left(1 - \frac{\Delta B}{B - B_0}\right)$

universality : (for most observables)

2-channel model $\longrightarrow$ ZRM

in the limit : (sufficient conditions)

$$\left(b \,,\, R_* = \frac{\hbar^2}{m \cdot a_{bg} \cdot \mu_b \cdot \Delta B} \,,\, \sqrt{|a_{bg}| \cdot R_*} \right) \ll |a| \,,\, \frac{1}{k_F} \,,\, \lambda = \sqrt{\frac{2\pi\hbar^2}{m\, k_B T}}$$

• proved for the 2-body scattering amplitude

• expected for the many-body problem.

also : expect : Experiment (= multi-channel) $\longrightarrow$ ZRM

in the limit : $k_F \longrightarrow 0$ $(n_\uparrow = n_\downarrow)$

$B \to B_0$ such that $k_F a$ fixed

$T \to 0$ such that $\frac{T}{T_F}$ fixed.

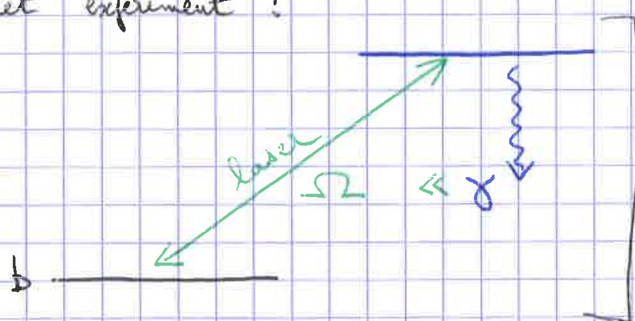
practice : expect : ZRM accurate description of experiment provided ∞

Number of closed-channel molecules

$$\langle \hat N_b \rangle = \sum_{\vec r} b^3 \langle (\hat\psi_b^\dagger \hat\psi_b)(\vec r) \rangle \,,\quad \langle \hat N_f \rangle = \sum_{\sigma=\uparrow,\downarrow} \sum_{\vec r} b^3 \langle (\hat\psi_\sigma^\dagger \hat\psi_\sigma)(\vec r) \rangle$$

(closed-channel fraction : $\frac{2\langle N_b \rangle}{\langle N_f + 2N_b \rangle}$)

Hulet experiment :



$$\Rightarrow \quad \frac{d}{dt}\langle \hat N_b \rangle \simeq -\langle \hat N_b \rangle \cdot \frac{\Omega^2}{\gamma} = \frac{1}{2}\frac{d}{dt}\langle \hat N_f \rangle .$$

$E_b(B)$ is the only B-dependent part of $\hat H$.

$$\left\langle \frac{d\hat H}{dB} \right\rangle = \underbrace{\frac{dE_b}{dB}}_{=\mu_b} \cdot \sum_{\vec k} \langle b_{\vec k}^\dagger b_{\vec k} \rangle = \mu_b \cdot \langle \hat N_b \rangle$$

$$\underset{\substack{\uparrow \\ \text{Hellmann} \\ -\text{Feynman}}}{=} \left(\frac{\partial F}{\partial B}\right)_T = \underbrace{\left(\frac{\partial F}{\partial(-1/a)}\right)_T}_{C \cdot \frac{\hbar^2}{4\pi m}} \cdot \frac{d(-1/a)}{dB}$$

$$\boxed{\langle \hat{N}_b \rangle = C \cdot \frac{R_*}{4\pi} \left(1 - \frac{a_{bg}}{a}\right)^2}$$

Rem: $k_F R_* \ll 1 \Rightarrow \langle \hat{N}_b \rangle \ll \langle \hat{N}_f \rangle$.

Still, this small number of closed-channel molecules is responsible for the strong effective interaction between atoms.

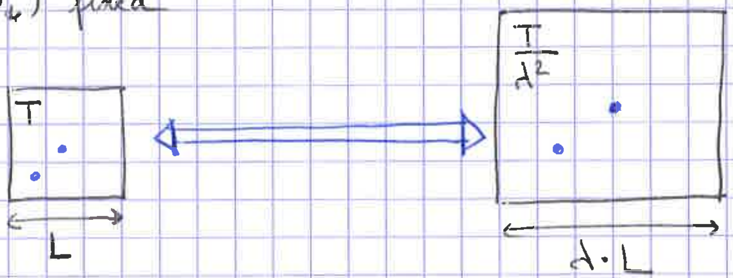
Chapter 5: Symmetry properties at the unitary limit and consequences

(for more details, see [LNP] again, or arXiv: 0607821)

ZRM, $a = \infty$.

scale invariance: no lengthscale coming from interactions.

eg: • $(N_\uparrow, N_\downarrow)$ fixed



• $T = 0$, Thermo. lim: $E = \xi \cdot E_{\text{ideal}}$.

But there is more!

Part 1: Separability in hyperspherical coordinates

Consider $(N_\uparrow : \uparrow, N_\downarrow : \downarrow)$

external potential $U(\vec{r}) = \frac{1}{2} m \omega^2 r^2$

= isotropic harmonic trap.

($\omega = 0$: free space)

$$\Phi(\vec r_1, \vec r_2, \dots, \vec r_N)$$

1:↑ 2:↓ ...

ZRM:

- $-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_{\vec r_i}\Phi + \frac12 m\omega^2\sum_{i=1}^{N} r_i^2\,\Phi = E_{tot}\cdot\Phi$

  (when all $r_{ij}>0$)

- CC for $a=\infty$:

$$\Phi \underset{r_{12}\to 0}{=} \frac{1}{r_{12}}\,A\left(\vec C_{12}, (\vec r_k)_{k>2}\right) + O(r_{12})$$

other (i,j): automatic by antisym

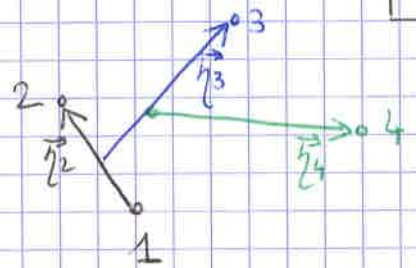
$$\vec C = \frac{1}{N}\sum_{i=1}^{N}\vec r_i$$

Rewriting of CC:

$$\left.\frac{\partial(r_{12}\Phi)}{\partial r_{12}}\right|_{r_{12}=0} = 0$$

↳ [derivative at fixed $\hat r_{12}, \vec C_{12}, (\vec r_k)_{k>2}$]

Jacobi coordinates:



$$\vec\eta_j := \vec r_j - \frac{1}{j-1}\sum_{i=1}^{j-1}\vec r_i$$

$$\vec R := \left(\sqrt{\frac{j-1}{j}}\cdot\vec\eta_j\right)_{j=2\dots N} \in \mathbb{R}^{3\cdot(N-1)}$$

$$\Phi(\vec r_1 - \vec r_N) = \Phi(\vec R, \vec C)$$

Separability: $= \psi(\vec R)\cdot\psi_{CM}(\vec C)$

$$E_{tot} = E + E_{CM}$$

because:

- $\cancel{\sum_{i=1}^{N}\Delta_{\vec r_i} = \frac{1}{N}\Delta_{\vec C} + \Delta_{\vec R}}$
- $\cancel{\sum_{i=1}^{N} r_i^2 = N\cdot C^2 + R^2}$

finite-range model, $V(r)$:

CM separable (well known - sometimes called Kohn's theorem)

$$\begin{cases} H = H_{CM} + H_{int} \\ H_{CM} = -\frac{\hbar^2}{2Nm}\Delta_{\vec C} + \frac12 N m\omega^2 C^2 \\ H_{int} = -\frac{\hbar^2}{2m}\Delta_{\vec R} + \frac12 m\omega^2 R^2 + \sum_{i<j} V(r_{ij}) \end{cases}$$

→ remains true for ZRM

• Schrödinger : $-\frac{\hbar^2}{2m}\Delta_{\vec{R}}\Psi + \frac{1}{2}m\omega^2R^2\Psi = E\cdot\Psi$

• CC becomes : $\left.\frac{\partial(\eta_2\Psi)}{\partial\eta_2}\right|_{\eta_2=0} = 0$ [derivative at fixed $\hat{\eta}_2, \vec{\eta}_3 - \vec{\eta}_N$]

Hyperspherical coordinates :

- Hyperradius $\quad R = \|\vec{R}\| = \sqrt{\frac{1}{N}\sum_{i<j} r_{ij}^2}$

• $\hat{R} = \frac{\vec{R}}{R}$ parameterized by $3N-4$ hyperangles :

$$\Omega = (\hat{\eta}_2, \dots, \hat{\eta}_N, \alpha_2 \text{ —— } \alpha_{N-1})$$

$\alpha_i \in [0; \frac{\pi}{2}]$:

$$\begin{cases} \eta_2 = R\cdot\sin\alpha_2 \\ \eta_3 = R\cdot\cos\alpha_2\cdot\sin\alpha_3 \\ \vdots \\ \eta_{N-1} = R\cos\alpha_2 \cdots \cos\alpha_{N-2}\cdot\sin\alpha_{N-1} \\ \eta_N = \text{——————} \cos\alpha_{N-1} \end{cases}$$

(given $\eta_2 \cdots \eta_N$, one defines α_{N-1}, then $\alpha_{N-2}, \dots, \alpha_2$)

$$\Psi(\vec{R}) = \Psi(R, \Omega)$$

Separability $\quad = G(R)\cdot\phi(\Omega)$

because : (i) CC only constrains the Ω-dependence of $\Psi(R,\Omega)$:

$$CC : \left.\frac{\partial(\sin\alpha_2\cdot\Psi)}{\partial\alpha_2}\right|_{\alpha_2=0} = 0$$

[derivative at fixed $\alpha_3 - \alpha_{N-1}, \hat{\eta}_2 - \hat{\eta}_N, R$]

<u>Note</u> : this cannot hold for finite a, because [CC on angles only] must be dimensionless.

~~in fact~~ ⇒ must involve $\frac{a}{R}$

⇒ couples Ω to R.

(ii) Schrödinger equation :

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}T_\Omega\right) + \frac{1}{2}m\omega^2R^2\Big)\Psi = E\cdot\Psi$$

differential operator acting only on Ω
"Laplacian on the hypersphere"

after separation :

Hyperangular problem :

- $\Delta_\Omega \; \phi_s(\Omega) = - s^2 \cdot \phi_s(\Omega)$

  [a bit analogous to hyperspherical harmonics, eigenvalues $\ell(\ell+1)$]

  $s \in \mathbb{R}^+$

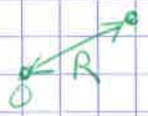
- CC for $\phi_s(\Omega)$:

$$\frac{\partial}{\partial \alpha_2} \left(\sin \alpha_2 \cdot \phi_s(\Omega) \right) \Big|_{\alpha_2 = 0} = 0$$

Hyperradial problem : $\quad G(R) = \dfrac{F(R)}{R^{\frac{3N-5}{2}}}$

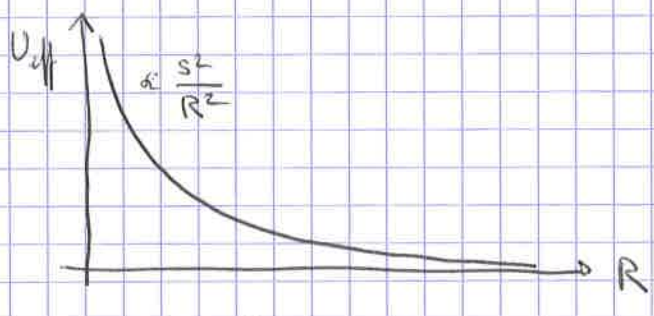
$$- \frac{\hbar^2}{2m} \underbrace{\left[F''(R) + \frac{1}{R} F'(R) \right]} + \underbrace{\left(\frac{\hbar^2}{2m} \frac{s^2}{R^2} + \frac{1}{2} m \omega^2 R^2 \right)}_{=: U_{eff}(R)} F = E \cdot F$$

1-body Schrödinger equation
~~4~~ fictitious particle
in 2D



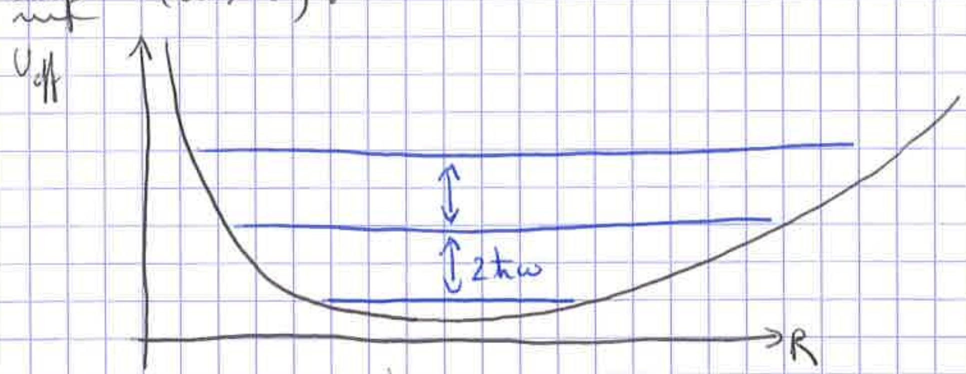
$\to$ 2D Laplacian

- free space ($\omega = 0$) :



$E = 0$: $\quad F(R) = R^s$

- trap ($\omega > 0$) :



$E = (s + 1 + 2q) \hbar \omega \; , \qquad q \in \mathbb{N}$

$$\omega = 0 : \quad \psi = R^s \cdot \phi_s(\Omega)$$

same

$$\omega > 0 : \quad \psi = F_{s,q}(R) \cdot \phi_s(\Omega)$$

L_+, L_-, H have commutation relations of $SO(2,1)$ Lie algebra.
"$SO(2,1)$ dynamical symmetry"
(a.k.a. hidden symmetry)

s at large N

$$s(N_\uparrow, N_\downarrow) = \text{lowest } s$$

$$s\left(\frac{N}{2}, \frac{N}{2}\right) \underset{N\to\infty}{\sim} \frac{E_{0,\text{trap}}\left(\frac{N}{2}, \frac{N}{2}\right)}{\hbar\omega} \underset{N\to\infty}{\overset{(LDA)}{\sim}} \sqrt{\xi} \cdot \frac{(3N)^{4/3}}{4}.$$

3-body problem $(\uparrow\uparrow\downarrow)$

$s = 1{,}7727\ldots$ total $l = 1$

$2{,}1662\ldots$ $l = 0$

$\vdots$

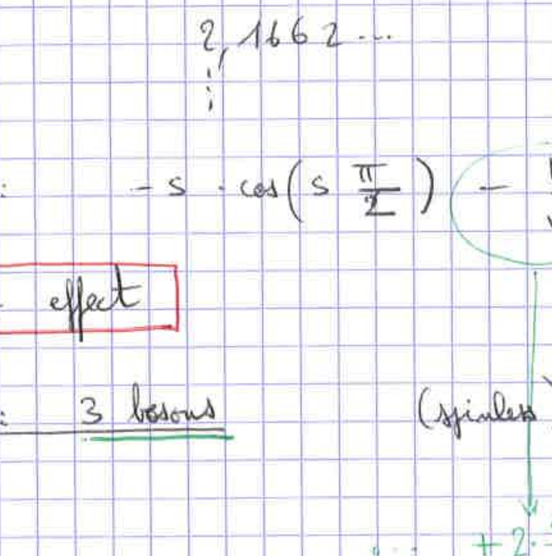
$$l = 0 : \quad -s \cdot \cos\left(s\,\frac{\pi}{2}\right) - \frac{4}{\sqrt{3}} \sin\left(s\,\frac{\pi}{6}\right) = 0$$

Part 2: $\boxed{\text{Efimov effect}}$

Excursion: 3 bosons (spinless) $\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ symmetric

$$\ldots + 2 \cdot \frac{4}{\sqrt{3}} \ldots$$

$$s_0 = i \times 1{,}00624\ldots$$



To make ZRM well-defined, need to add 3-body CC for $R \to 0$, that depends on 3-body parameter

(in addition to 2-body CC that depends on a)

Efimov trimers, $\frac{E_{n+1}}{E_n} = e^{-\frac{2\pi}{|s_0|}}$

discrete scale invariance

length $\times\ e^{\frac{\pi}{|s_0|}}$

<u>Back to fermions, allowing $m_\uparrow \neq m_\downarrow$</u> (mixture)

$(N_\uparrow, N_\downarrow) = (2, 1)$: 13,607... $\longrightarrow \frac{m_\uparrow}{m_\downarrow}$ Efimov effect

$(3, 1)$: 13,384... $\longrightarrow \frac{m_\uparrow}{m_\downarrow}$

[Castin - Mora - Pricoupenko]

$(2, 2)$: No Efimov [Endo - Castin]

<u>Back to $m_\uparrow = m_\downarrow$</u> :

Conjecture : no Efimov $\forall\ (N_\uparrow, N_\downarrow)$

Rigorous mathematical results: [Seiringer - Moser 2017 - 18]

No Efimov (ZRM well-defined, self-adjoint without 3-body CC)

$(2, 2)$

$(N_\uparrow, 1)$ $\quad \forall N_\uparrow$

Part 3 : $\boxed{\text{Short-distance scaling laws}}$ (for any a)

$\psi(\vec{r}_1 - \vec{r}_N)$ eigenstate of ZRM

$J \subset \{1 - N\}$ containing $q_\uparrow$ particles of spin $\uparrow$

$q_\downarrow$ ——— $\downarrow$. $q = q_\uparrow + q_\downarrow$

$$(\vec{r}_i)_{i\in J} \longrightarrow \begin{cases} \vec{C}_J = \frac{1}{q} \sum_{i\in J} \vec{r}_i \\ R_J = \sqrt{\frac{1}{q} \sum_{\substack{i<j \\ i,j\in J}} r_{ij}^2} \\ \Omega_J \end{cases}$$

$$\psi(\vec{r}_1 — \vec{r}_N) \underset{R_J\to 0}{\sim} (R_J)^{s(q_\uparrow,q_\downarrow) - \frac{3q-5}{2}} \cdot \phi(\Omega_J) \cdot A_J\Big(\underbrace{\vec{C}_J, (\vec{r}_k)_{k\notin J}}_{\text{fixed}}\Big)$$

[Tan arXiv 2004, Petrov - Salomon - Shlyapnikov PRL'04]

3-body losses :

consider finite-range model with deeply bound dimers (as in experiments).

chemical reaction :

3 atoms $\longrightarrow$ 1 deep dimer + 1 atom

$$-\frac{\dot{n}}{n} = \gamma$$

$$\frac{\hbar\gamma}{\varepsilon_F} \propto (k_F b)^{2\cdot s(2,1)} \xrightarrow[k_F b\to 0]{} 0$$

$\Rightarrow$ quasi - equilibrium.

deep dimers are not a problem.

can be ignored for most experiments.

Part 4 : $\boxed{\text{Dynamics}}$ Back to $a = \infty$, ZRM,

isotropic harmonic trap with $\omega(t)$.

$\vec{X} \equiv (\vec{r}_1 — \vec{r}_N)$

$$\begin{cases} \bullet\ i\hbar \frac{\partial}{\partial t}\psi(\vec{X},t) = \left[-\frac{\hbar^2}{2m}\Delta_{\vec{X}} + \frac{1}{2} m\,\omega(t)^2 X^2\right]\psi(\vec{X},t) \\ \bullet\ \psi(\vec{X},t) \text{ obeys CC for } a=\infty \end{cases}$$

Consider : at $t=0$: $\omega(0) = \omega_0$

$\psi(\vec{X},0)$ = eigenstate, energy E

$\omega(t)$ arbitrary

Scaling solution [Castin 2004]

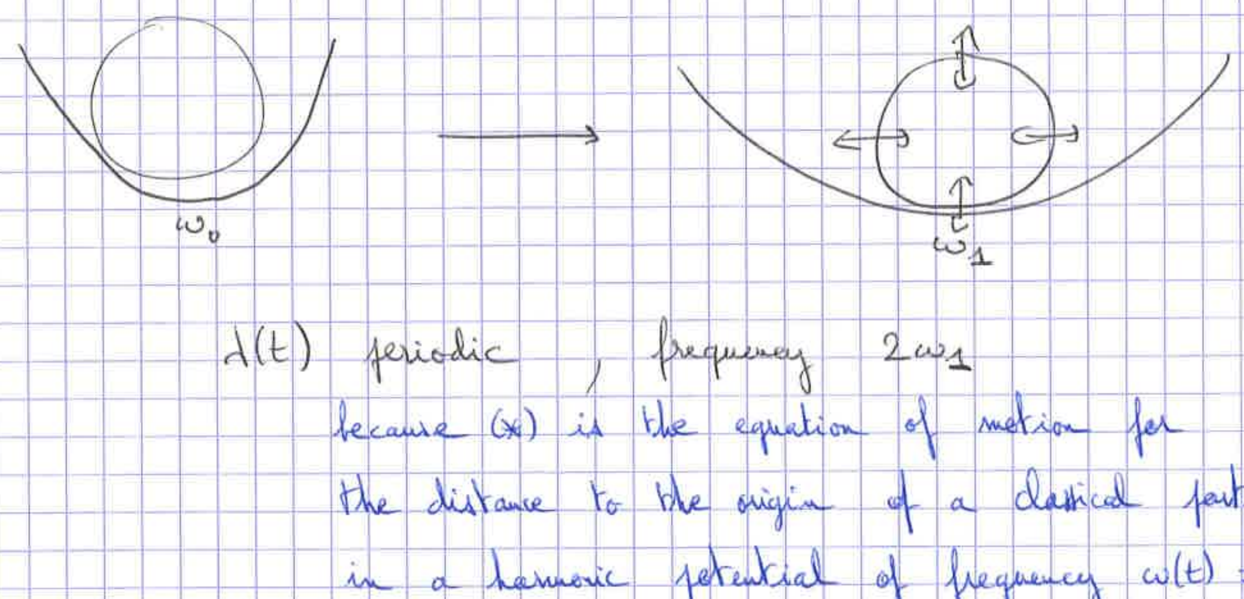
$$\Psi(\vec{X},t) = \Psi\left(\frac{\vec{X}}{\lambda(t)}, 0\right) \cdot e^{i \frac{\dot{\lambda}(t)}{\lambda(t)} X^2 \frac{m}{2\hbar}} \cdot \frac{e^{i\theta(t)}}{\lambda(t)^{3N/2}}$$

- Solves Schrödinger eq. provided

$$\ddot{\lambda} = -\frac{d}{d\lambda}\left[\frac{\omega_0^2}{2\lambda^2} + \frac{\omega(t)^2}{2}\lambda^2\right] \qquad (*)$$

- Solves CC $(a=\infty)$

Breathing mode $\qquad \omega(t) = \omega_1, \ t > t_1$



$\lambda(t)$ periodic, frequency $2\omega_1$

because $(*)$ is the equation of motion for the distance to the origin of a classical particle in a harmonic potential of frequency $\omega(t) = \omega_1$, of angular momentum $L^2 \propto \omega_0^2$.

breathing mode, frequency $2\omega_1$, undamped!

(Separability : hyperradial degree of freedom is decoupled)

~~Remark~~ Time of flight $\qquad \omega(t) = 0$ for $t > 0$
(keep $a=\infty$)

$\lambda(t) \nearrow$

$\Rightarrow$ exact magnifying lens.

Generalisation to anisotropic trap and finite a : [Lobo-Gensemer 2008]

Consequence for hydrodynamics

- shear viscosity
- dilatational viscosity (a.k.a. bulk viscosity)

ζ normal phase (SF: $\zeta_1, \zeta_2, \zeta_3$)

entropy : $$\frac{dS}{dt} = \int d^3r \; \frac{\zeta}{T} \; \| \vec{\nabla} \cdot \vec{v} \|^2$$
$$+ \text{(other positive terms)}$$

consider breathing mode : $S(t)$ periodic and $\nearrow$

$\Rightarrow$ constant

$$\boxed{\zeta = 0}$$

discovered by Son, who was not aware of the scaling solution and breathing mode, and used a conformal invariance in a formal curved space.

High-energy physicists say that the unitary Fermi gas is a Non-relativistic Conformal Field Theory.