Free-surface flows with large slopes: Beyond lubrication theory

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The description of free-surface flows can often be simplified to thin-film (or lubrication) equations, when the slopes of the liquid-gas interface are small. Here, we present a long-wavelength theory that remains fully quantitative for steep interface slopes, by expanding about Stokes flow in a wedge. For small capillary numbers, the variations of the interface slope are slow and can be treated perturbatively. This geometry occurs naturally for flows with contact lines: we quantify the difference with ordinary lubrication theory through a numerical example and analytically recover the full Cox-Voinov asymptotic solution. © *2006 American Institute of Physics*.
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Free-surface flows are encountered in many everyday life and industrial situations, ranging from soap films and sliding drops to paints and coatings.[1,2] The hydrodynamic description of these "free-boundary problems" still provides a challenge of great fundamental and technological interest. The difficulty lies in the intricate coupling between the liquid-gas interface and the flow inside the film, which gives rise to a broad variety of instabilities and interface morphologies.[2–7] In the case of "thin" films, for which horizontal and vertical length scales are well separated [Fig. 1(a)], the problem is greatly reduced through a long-wavelength expansion.[2,8,9] At low Reynolds numbers this so-called lubrication approximation yields a single nonlinear equation for the evolution of the interface profile $h(x,y,t)$, and forms the accepted theoretical framework for free-surface flows. This reduction is possible whenever surface tension (γ) dominates over viscosity (η), so that the capillary number $\mathrm{Ca}=\eta U^*/\gamma$ serves as a small parameter; U^* denotes the velocity scale of the problem.

The standard formulation of lubrication theory, however, has a severe drawback: it is only valid for small interface slopes, i.e., $|\nabla h| \ll 1$. While it is generally believed that the lubrication equation yields good qualitative predictions for larger slopes as well, one has to be careful with quantitative comparisons. This is particularly important for the problem of moving contact lines for which viscous forces tend to diverge as $h \to 0$.[10,11] The microscopic mechanisms that release this singularity are highly disputed,[12–16] and call for a fully quantitative description of experiments that often involve large contact angles. As it is practically infeasible to resolve the full hydrodynamic problem on all relevant length scales, ranging from molecular to millimeter, a simplified theory for finite slopes would be extremely valuable.

In this Letter we present a generalization of the lubrication theory for free-surface flows at low Reynolds numbers that remains exact for large slopes. The crucial observation is that in the limit of small Ca, capillary driving requires slow variations of the interface curvature, but there are no restrictions to the steepness of the interface. We therefore consider the flow in a wedge with a finite but slowly varying opening angle θ—see Fig. 1(b). This geometry naturally occurs for contact lines. Treating the variation as a perturbation around a straight wedge then yields the equation for the interface profile $h(x,y,t)$.

This theory, summarized by Eqs. (14) and (15), remains fully quantitative for large slopes when the curvature $\kappa \ll 1/h$, while it enjoys the same mathematical structure as the usual lubrication theory. We furthermore show that the equation reproduces the asymptotics for $\theta(x)$ as computed by Voinov[12] and Cox,[13] in a relatively straightforward manner. However, the present work reaches beyond asymptotic relations: it describes all intermediate length scales as well, and allows incorporation of other forces such as a disjoining pressure or gravity.

a. Lubrication theory for free surface flows. Before addressing the problem of finite slopes, let us first briefly revisit the lubrication approximation. In the limit of zero Reynolds number, the flow of incompressible Newtonian liquids is described by Stokes equations

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$-\nabla p + \eta \Delta \mathbf{v} - \nabla \Phi = \mathbf{0} \Rightarrow \nabla \times \Delta \mathbf{v} = \mathbf{0}, \tag{2}$$

where $p(x,y,z)$ and $\mathbf{v}(x,y,z)$ represent the pressure and velocity field, respectively, while we consider body forces that derive from a potential Φ. (Time-dependent profiles will be discussed below.) The equation is complemented with the boundary condition of Laplace pressure at the free surface,

$$p(z=h) = p_0 - \gamma\kappa, \tag{3}$$

where κ is the interface curvature. This gives rise to an intricate nonlinear coupling between the shape of the interface and the flow inside the film, which has to be resolved self-consistently. We limit the discussion to the case where the gas is hydrodynamically passive and we take a zero shear stress condition at the free interface.

In the limit where $\mathrm{Ca} \ll 1$, the free-boundary problem can be greatly reduced through the lubrication approximation: surface tension is sufficiently strong to drive the viscous flow through only minor variations of the shape of the free

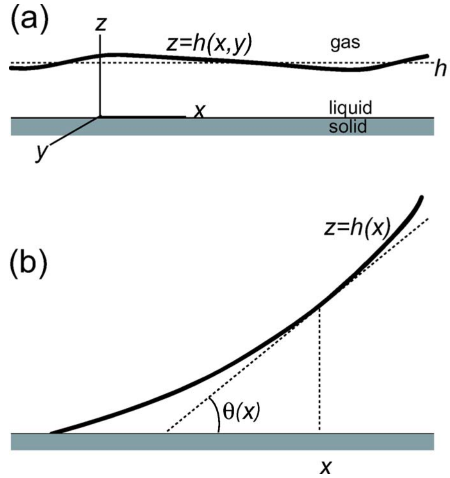

FIG. 1. (a) The usual lubrication approximation is valid whenever the liquid-gas interface is slowly varying along the horizontal coordinates, and is thus restricted to small slopes. (b) Considering wedge-like profiles with slowly varying slope $\theta(x)$, we derive the long-wavelength theory for steep slopes.

surface [Fig. 1(a)]. One thus expects that the interface profile and the velocity field are slow functions of the horizontal coordinates, so that

$$\frac{h(x,y)}{h^*} = \tilde{h}\left(\epsilon\frac{x}{h^*}, \epsilon\frac{y}{h^*}\right), \tag{4}$$

while we write

$$\mathbf{v}(x,y,z) = \tilde{\mathbf{v}}\left(\epsilon\frac{x}{h^*}, \epsilon\frac{y}{h^*}, \frac{z}{h^*}\right) = \tilde{\mathbf{v}}_0 + \epsilon\tilde{\mathbf{v}}_1 + \epsilon^2\tilde{\mathbf{v}}_2 + \cdots . \tag{5}$$

All lengths have been rescaled by the typical film thickness h^*, and $\epsilon = \mathrm{Ca}^{1/3}$ is the small parameter of the expansion. The strategy is to solve Eqs. (1)–(3) order by order in ϵ. Here we briefly sketch the approach; for a more detailed derivation we refer to Ref. 2.

If we let ϵ go to zero, the departure of $h(x,y)$ from a horizontal interface becomes increasingly small. Hence, the velocity profile converges towards the parabolic (Poiseuille-type) profile in this limit, so that

$$\tilde{\mathbf{v}}_0 = \frac{3\mathbf{U}}{2}\left[1 - \left(1 - \frac{z}{h}\right)^2\right]. \tag{6}$$

Formally, this dominant flow can be obtained from Eq. (2) at order ϵ^0. Note that $\tilde{\mathbf{v}}_0$ still evolves on a long scale through its dependence on $h(x,y)$.

In deriving Eq. (6) we used the boundary conditions of no slip at $z=0$, and zero shear stress at $z=h$. The prefactor has been chosen such that $\mathbf{U} \equiv 1/h\int_0^h dz\mathbf{v}_0$ represents the depth-averaged velocity in the frame attached to the plate.

Since the dominant viscous forces in Eq. (2) arise from $\tilde{\mathbf{v}}_0$, we do not need to solve for the higher order velocities to obtain the equation for $h(x,y)$. At leading order, Eqs. (2) and (3) reduce to the celebrated lubrication equation

$$\nabla\Delta h = 3\mathrm{Ca}\frac{\mathbf{U}/U^*}{h^2} + \frac{1}{\gamma}\nabla\ \Phi|_{z=h}. \tag{7}$$

Combined with the depth-averaged continuity equation,

$$\partial_t h + \nabla\cdot(h\mathbf{U}) = 0, \tag{8}$$

it provides the common theoretical framework for free-surface flows, both in the scientific community as well as for industrial purposes. The lowest order terms that are neglected are of order ϵ^2, so the expansion is valid in the limit of small slopes, $|\nabla h|^2 \ll 1$.

b. Theory for large slopes. We now perform a similar long-wavelength expansion for wedge-like geometries, such as depicted in Fig. 1(b). The crucial physical ingredient underlying the expansion is that surface tension dominates over viscosity, i.e., $\mathrm{Ca} \ll 1$, so that variations of interface curvature are slow with respect to the relevant length scales. In principle there is no restriction to the slope of the interface: the only requirement is that the opening angle θ is slowly varying [Fig. 1(b)]. We therefore consider profiles with

$$\theta(x) = \tilde{\theta}\left(\epsilon\frac{x}{x^*}\right), \tag{9}$$

and perform an expansion in ϵ. Note that $\tilde{\theta}$ itself is of order unity. Here, we introduced the length scale x^*, which is a typical distance to the "origin" of the wedge—we show below that the analysis remains self-consistent as long as $h\partial_x\theta \ll 1$. For simplicity we discuss two-dimensional profiles, so we omit the y dependence.

In the spirit of the lubrication approximation, we expand the velocity as

$$\mathbf{v}(x,z) = \tilde{\mathbf{v}}_0 + \epsilon\tilde{\mathbf{v}}_1 + \epsilon^2\tilde{\mathbf{v}}_2 + \cdots , \tag{10}$$

and solve for the dominant flow $\tilde{\mathbf{v}}_0$. Again, $\tilde{\mathbf{v}}_0$ is obtained in the limit $\epsilon \to 0$, which in this case corresponds to a wedge of constant opening angle. Hence, the problem for $\tilde{\mathbf{v}}_0$ reduces to Stokes flow inside a straight wedge, which is easily solved analytically.[10,17]

To make this more explicit, we introduce local cylindrical coordinates, $r(x,z)$ and $\phi(x,z)$, which are defined by the locally tangent wedge [Fig. 2(a)],

$$r(x,z) = \frac{h}{\tan\tilde{\theta}}\sqrt{1 + \left(\frac{z}{h}\tan\tilde{\theta}\right)^2},$$

$$\phi(x,z) = \tilde{\theta} - \arctan\left(\frac{z}{h}\tan\tilde{\theta}\right).$$

The x dependence appears through $h(x)$ and $\theta(x)$. Writing the velocity as a function of these coordinates

$$\mathbf{v}(x,z) = \tilde{\mathbf{v}}[r(x,z), \phi(x,z)] \tag{11}$$

and expanding Eqs. (1) and (2), one indeed finds that the order ϵ^0 reduces to the problem of a straight wedge: variations of θ show up at higher orders. For the dominant flow we can thus use the results of Refs. 10 and 17,

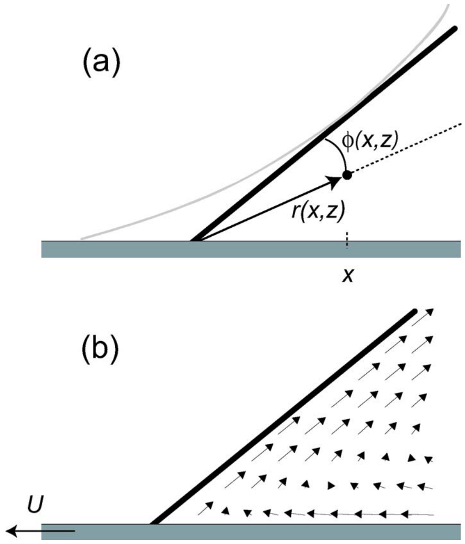

FIG. 2. (a) Definition of the cylindrical coordinates $r(x,z)$ and $\phi(x,z)$ in the locally tangent wedge of angle $\theta(x)$. (b) The basic velocity $\tilde{\mathbf{v}}_0$ corresponds to flow in a wedge of constant θ, sketched in the frame comoving with the interface, Eq. (12).

$$(\tilde{\mathbf{v}}_0)_r = U\frac{(\cos\phi - \phi\sin\phi)\sin\theta - \theta\cos\theta\cos\phi}{\theta - \cos\theta\sin\theta},$$
$$(\tilde{\mathbf{v}}_0)_\phi = U\frac{\theta\sin\phi\cos\theta - \phi\cos\phi\sin\theta}{\theta - \cos\theta\sin\theta}, \tag{12}$$

which hold in the frame comoving with the interface. This flow has been sketched in Fig. 2(b). Here, we used the conditions of a vanishing shear stress at $\phi=0$ and a no-slip condition at the plate, $v_r=-U$. The latter condition ensures that U represents the depth-averaged velocity in the frame attached to the plate.

Evaluating $\Delta\mathbf{v}_0$ at the free surface ($\phi=0$), Eq. (2) provides the leading-order pressure gradients along the interface

$$\partial_r p|_{\phi=0} = -\frac{2\eta U}{r^2}\left(\frac{\sin\theta}{\theta-\cos\theta\sin\theta}\right) - \partial_r\Phi. \tag{13}$$

Combined with the Laplace pressure condition (3) this yields the generalized lubrication equation,

$$\partial_x\kappa = 3\mathrm{Ca}\frac{U/U^*}{h^2}F(\theta) + \frac{1}{\gamma}\partial_x\,\Phi|_{z=h}, \tag{14}$$

where

$$F(\theta) = \frac{2}{3}\frac{\tan\theta\sin^2\theta}{\theta-\cos\theta\sin\theta}. \tag{15}$$

Since $\partial_x\kappa \sim (\epsilon/x^*)^2$, the expansion is self-consistent when taking $\epsilon=\mathrm{Ca}^{1/2}$.

Comparing this result to the lubrication equation (7), one observes two differences. First, the function $F(\theta)$ can be seen as a correction factor for the viscous term: indeed, one recovers $F(\theta)\to 1$ for small slopes. Second, the left-hand side of Eq. (14) now involves the full curvature

$$\kappa = \frac{\partial_{xx}h}{(1+\partial_x h^2)^{3/2}}. \tag{16}$$

The expansion thus provides an equation for the interface profile that has the same mathematical structure as the usual lubrication approximation, but which remains exact for large slopes. The lowest order terms that are neglected are of order $\epsilon=\mathcal{O}(x^*\partial_x\theta)$, so the expansion is valid when $h\partial_x\theta \ll 1$. This does not mean that the description is limited to small heights: the height only appears as a reference scale to quantify the dimensionless curvature.

c. Asymptotics for $\theta(x)$. To illustrate the strength of the approach, we now show that Eq. (14) correctly reproduces the nontrivial asymptotic solution for $\theta(x)$, as obtained by Voinov[12] and Cox[13] for advancing contact lines. Anticipating the well-known result, we express the relation between θ and x as

$$g(\theta) = \mathrm{Ca}\ln(x/x_0), \tag{17}$$

and solve for $g(\theta)$ using Eq. (14) with $\Phi=0$. Differentiating $g(\theta)$ with respect to x, one finds $\partial_x\theta=\mathrm{Ca}/(xg')$, so that the curvature can be written as $\kappa=\mathrm{Ca}\cos\theta/(xg')$. Combining this with the expansion $h(x)=x\tan\theta(x)+\mathcal{O}(\mathrm{Ca})$, Eq. (14) becomes, to lowest order,

$$\frac{\mathrm{Ca}\cos\theta}{x^2 g'} = \frac{3\mathrm{Ca}}{x^2\tan^2\theta}F(\theta). \tag{18}$$

Here, we used the fact that advancing contact lines move along the negative x direction in the frame attached to the plate, so that $U_{\mathrm{adv}}=-U^*$. Eliminating g', we readily recover the famous result of Refs. 12 and 13,

$$g(\theta) = \int_0^\theta du\,\frac{u-\cos u\sin u}{2\sin u}. \tag{19}$$

d. Discussion. We have derived the long-wavelength expansion for free-surface flows in the case of steep interface profiles. This provides a significant improvement with respect to the usual lubrication approximation, whose validity is restricted to small slopes. The resulting theory has the same mathematical structure as the lubrication equation and is thus easily adapted to existing codes and methods. The most natural application of our work is found in wetting flows that involve large contact angles. This is illustrated in Fig. 3, where we computed the thickness of a flat film flowing down a vertical plate in the presence of a receding contact line. The effect of gravity is accounted for through $\Phi=-\rho g x$, which introduces the capillary length scale $l_\gamma=\sqrt{\gamma/\rho g}$. For a given thickness $h_{\mathrm{film}}=l_\gamma\sqrt{3\mathrm{Ca}}$, there is a unique solution that connects to the contact line.[18] By numerically solving the interface profile down to a microscopic (molecular) height h_0, we can thus identify the slope very close to the contact line, denoted by the angle θ_0.[19] This yields a unique relation between the slope imposed at a microscopic distance from the contact line and the macroscopic film thickness. The results obtained from numerical integration of Eq. (14) (solid line) display significant quantitative differences from the prediction by lubrication theory (dashed line) at angles $\gtrsim 30°$.

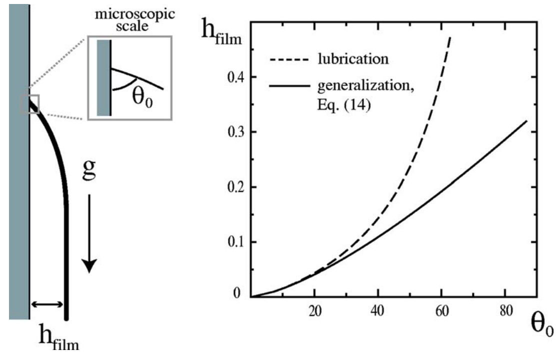

FIG. 3. Numerical solution of a flat film dragged downwards by gravity in the presence of a contact line. The film thickness $h_{\rm film}$ is uniquely determined by imposing a contact angle θ_0 at a microscopic scale, taken here as $h_0 = 10^{-5}$ (lengths are expressed in the capillary length $\sqrt{\gamma/\rho g}$). At large angles, the results of Eq. (14) (solid line) provide significant corrections with respect to lubrication theory Eq. (7) (dashed line).

Although in principle the expansion is not limited to contact lines, steep slopes are typically attained through curvatures $hh'' \sim 1$ in flows without contact lines. This lies beyond the strict validity of the expansion and one expects corrections due to curvature of the interface. These corrections can in principle be treated perturbatively as well, since $\mathrm{Ca} \ll 1$ implies weak variations of curvature.

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[1] S. F. Kistler and P. Schweizer, *Liquid Film Coating—Scientific Principles and Their Technological Implications* (Kluwer, Dordrecht, 1997).

[2] A. Oron, S. H. Davis, and S. G. Bankoff, "Long-scale evolution of thin liquid films," Rev. Mod. Phys. **69**, 931 (1997).

[3] H. E. Huppert, "Flow and instability of a viscous gravity current down a slope," Nature (London) **300**, 427 (1982).

[4] R. Goodwin and G. M. Homsy, "Viscous flow down a slope in the vicinity of a contact line," Phys. Fluids A **3**, 515 (1991).

[5] N. Garnier, R. O. Grigoriev, and M. F. Schatz, "Optical manipulation of microscale fluid flow," Phys. Rev. Lett. **91**, 054501 (2003).

[6] J. Eggers, "Nonlinear dynamics and breakup of free-surface flows," Rev. Mod. Phys. **69**, 865 (1997).

[7] I. Cohen and S. R. Nagel, "Scaling at a selective withdrawal transition through a tube suspended above the fluid surface," Phys. Rev. Lett. **88**, 074501 (2002).

[8] O. Reynolds, "On the theory of lubrication and its application to Mr. Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil," Philos. Trans. R. Soc. London **177**, 157 (1886).

[9] D. J. Benney, "Long waves on liquid films," J. Math. Phys. **45**, 150 (1966).

[10] C. Huh and L. E. Shriven, "Hydrodynamic model of steady movement of a solid/liquid/fluid contact line," J. Colloid Interface Sci. **35**, 85 (1971).

[11] E. B. Dussan, V. Davis, and S. H. Davis, "On the motion of a fluid-fluid interface along a solid surface," J. Fluid Mech. **65**, 71 (1974).

[12] O. V. Voinov, "Hydrodynamics of wetting," Fluid Dyn. **11**, 714 (1976).

[13] R. G. Cox, "The dynamics of the spreading of liquids on a solid surface," J. Fluid Mech. **168**, 169 (1986).

[14] J. Eggers, "Toward a description of contact line motion at higher capillary numbers," Phys. Fluids **16**, 3491 (2004).

[15] T. D. Blake, J. De Coninck, and U. D'Ortuna, "Models of wetting: Immiscible lattice Boltzmann automata versus molecular kinetic theory," Langmuir **11**, 4588 (1995).

[16] L. M. Pismen and Y. Pomeau, "Disjoining potential and spreading of thin liquid layers in the diffuse-interface model coupled to hydrodynamics," Phys. Rev. E **62**, 2480 (2000).

[17] L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Pergamon, London, 1959).

[18] L. M. Hocking, "Meniscus draw-up and draining," Eur. J. Appl. Math. **12**, 195 (2001).

[19] The problem of *predicting* the effective boundary condition θ_0 at a microscopic scale h_0 lies at the heart of the contact line problem, and involves new microscopic mechanisms beyond classical hydrodynamics. The curves of Fig. 3 depend only weakly [logarithmically, see Eq. (17)] on the value of h_0.