Studying Burgers' models to investigate the physical meaning of the alignments statistically observed in turbulence

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The alignments between the vorticity, the vortex stretching vector, the pressure Hessian eigenvectors and the strain rate eigenvectors are computed and discussed in the case of the Burgers' vortex and the Burgers' layer. It is shown that the main physical properties of these models can be deduced from these alignments. Following this example, the alignments between these vectors in turbulent flows are interpreted as dominated by stretched, coherent and locally quasi-bidimensional regions. This induces a new and safer classification for the strain rate and the pressure Hessian eigenvalues. © *1997 American Institute of Physics.* [S1070-6631(97)02803-1]

I. INTRODUCTION

The complexity of turbulence has induced many different approaches to the problem. In particular, two complementary visions dominate turbulence studies. On the one hand, the *global approach* is linked to the unpredictability, the quasi-random aspect and the disorder of turbulent flows and is thus essentially based on statistics. On the other hand, the direct observation of coherent structures, both numerically[1–5] since Siggia[1] and experimentally by Douady *et al.*,[6] has suggested an alternative approach, focused on coherent structures and thus linked to the coherent and ordered part of turbulent flows. Such a *structural approach* has for example been followed by Jimenez *et al.*[7] who studied in numerical simulations some of the characteristics (radius and Reynolds number) of the strong vortices embedded in the turbulent flow.

In this sense, a statistic on the angle between two dynamical vectors is relevant to the global approach. Among these statistics, we are interested in those involving the vectors linked to the vorticity and the stretching dynamics. We will thus first make a short theoretical presentation of these interesting dynamical vectors, before recalling the results obtained by several authors on their alignments.

An alternative to the description of flows in terms of velocity $\vec{v}$ and pressure gradient (Navier–Stokes equation) is to use the velocity derivatives which reflect the local structure of the flow. Considering the velocity gradient tensor, one can construct the rate of strain tensor $\sigma_{i,j}=(\partial_i v_j+\partial_j v_i)/2$ and the vorticity $\omega_i=\varepsilon_{i,j,k}\partial_j v_k$ which can be regarded as two basic local quantities.

For an incompressible fluid governed by Navier–Stokes equations, we find the vorticity equation,

$$\frac{D\omega_i}{Dt}\equiv\partial_t\omega_i+u_j\partial_j\omega_i=w_i+\nu\Delta\omega_i, \tag{1}$$

where $\vec{w}$ is the vortex stretching vector ($w_i=\sigma_{i,j}\omega_j$). This vector, which corresponds to the action of the strain on the vorticity, is the source term of Eq. (1). Rather than the strain equation, let us consider the evolution of the vortex stretching vector:

$$\frac{Dw_i}{Dt}\equiv\partial_t w_i+u_j\partial_j w_i=\psi_i+\nu(\Delta w_i-2\partial_k\sigma_{i,j}\partial_k\omega_j), \tag{2}$$

where $\vec{\psi}$ is the *stretching induction vector* ($\psi_i\equiv-\Pi_{i,j}\omega_j$) and Π the pressure Hessian ($\Pi_{i,j}\equiv\partial^2_{i,j}P$). It must be noted that Eqs. (1) and (2) are formally similar. However, while the vortex stretching vector $\vec{w}$ is a local quantity, $\vec{\psi}$ is non-local since the pressure Hessian can only be obtained by solving the Poisson equation:

$$\Delta P\equiv\Pi_{ii}=(\omega^2-\sigma^2)/2, \tag{3}$$

where $\sigma^2\equiv2\sigma^2_{i,j}$ and $\omega^2\equiv\omega_i^2$. One can find further details in Okhitani *et al.*[8] who studied, in an Eulerian case, the role of the pressure Hessian Π in non-local processes. Understanding the physics of vorticity stretching and stretching induction directly from Eqs. (1) and (2) is quite difficult. A good way of getting some information on these processes is thus to study the alignments between the dynamical vectors involved in these equations ($\vec{\omega}$, $\vec{w}$ and $\vec{\psi}$). Since $\vec{w}$ is constructed on σ and $\vec{\omega}$, and $\vec{\psi}$ on Π and $\vec{\omega}$, the alignments between the vorticity $\vec{\omega}$ and the eigenvectors of σ and Π are also interesting. In particular they indicate which parts of the strain and the pressure Hessian are active. There are unfortunately only two of these alignments well established in laboratory and numerical turbulence experiments. The alignment between the vorticity $\vec{\omega}$ and the strain rate eigenvectors has been first studied by Ashurst *et al.*[9] numerically and later by Tsinober *et al.*[10] experimentally. The alignment between the vorticity $\vec{\omega}$ and the stretching vector $\vec{w}$ has been investigated by Tsinober *et al.*[10] in experiments and by Shtilman *et al.*[11] in Navier–Stokes simulated turbulence. There is however one numerical simulation by Nomura and Post[12] where the statistics of the angles between the vorticity and the pressure Hessian eigenvectors are computed.

To summarize the results obtained in these studies, there is a tendency for alignment between the vorticity $\vec{\omega}$ and the intermediate eigenvector $\vec{\sigma}_2$ of the rate of strain tensor σ, and between $\vec{\omega}$ and the vortex stretching vector $\vec{w}$ in turbulent flows. Moreover, Nomura and Post[12] found a trend for alignment between $\vec{\omega}$ and the smallest pressure Hessian eigenvector $\vec{\pi}_3$, this tendency becoming strong for high vorticity regions. These results have led to some important conclusions. First, the vorticity stretching, closely linked to the positiveness of the enstrophy generating term $\vec{w} \cdot \vec{\omega}$, is one of the predominant processes involved in turbulence. Second, this gives experimental evidence that a great degree of coherence is locally present in turbulent flows.

The aim of this paper is to propose a physical interpretation of these results. In order to improve the physical meaning of making statistics on the angles between these vectors, we choose to first compute them on a simple example whose physical content is well known. This procedure has been initiated by Shtilman *et al.*[11] who compared the alignments obtained in a direct numerical simulation to those obtained with its random counterpart having the same energy spectrum. We thus choose another extreme case which, on the contrary, corresponds to a strongly structured flow. Since the structures embedded in turbulent flows are generally described as tubes or layers of high vorticity, we choose among the available models describing these structures, the Burgers' vortex and Burgers' layer[13] which have the advantage of being simple analytical solutions of Navier–Stokes equations and of taking into account the effect of the stretching. Some of the properties of the vorticity, the vortex stretching vector, the strain rate and the pressure Hessian in these particular flows have already been studied in two previous works. For a vortex layer, Brachet *et al.*[14] computed the three strain eigenvalues as functions of the ratio between the local vorticity and the stretching. In the case of a Burgers' layer, Okhitani *et al.*[8] showed that the two smallest eigenvectors of the pressure Hessian are equal and that the vorticity is aligned with one of them. Furthermore, when the vorticity is larger than the stretching (e.g., near the center of a strong shear layer), the remaining principal axis of σ and Π are at an angle of $\pi/4$ from each other. In the present work, we will complete these two studies of Burgers' models, by computing systematically the alignments between $\vec{\omega}$, $\vec{w}$, $\vec{\psi}$ and the eigenvalues of σ and Π. We will then show that the main dynamical properties of these flows can be deduced from these properties of alignment. Our aim is then to give a physical interpretation of the alignments statistically observed in turbulent flows, following the same method of deduction. Since the relation between real turbulence and analytical vortices is quite vague, our interest in computing the alignments in model flows is only to make the physical contents of these statistical tools clearer.

A short presentation of Burgers' models is made in Sec. II A of this article. Section II B is devoted to the study of the alignments and their physical interpretation in these models. Following these simple examples, a new interpretation of the results obtained in real turbulent flows is proposed in Sec. II C. In Sec. III, the problems linked with the classification of the eigenvalues of σ and those of Π (Sec. III A) is discussed and an alternative procedure for these classifications is suggested (Sec. III B).

II. ALIGNMENTS STUDY

A. Burgers' layer and vortex

Both Burgers' models are exact solutions of Navier–Stokes equations. The velocity field of the Burgers' vortex is expressed in cylindrical coordinates, r being the radial coordinate and z the axial coordinate. For the Burgers' layer, the field is expressed in Cartesian coordinates, with by convention y for the compressive axis, z for the stretching axis and x for the layer direction axis. The main difference between these two models, apart from the geometry, is the spatial distribution of σ^2 and ω^2. In the layer, σ^2 and ω^2 are mixed while in the vortex, ω^2 is concentrated in the core and σ^2 is mostly distributed on a tubular region around the core. Linked to this property, there is a slight pressure maximum in the center of the layer, due to the stretching part of the flow and there is a depression near the core of the vortex, if the Reynolds number is not too small.

The velocity field depends on three non-independent parameters: the maximal vorticity ω_0 (e.g., the vorticity at the origin), the stretching $\gamma \equiv \sigma_{i,j}\omega_i\omega_j/\omega^2 = \partial_z v_z$ which is uniform, and the size of the core. The link between these parameters comes from the dynamics of these solutions. The stretching tends to concentrate the vorticity while the viscosity diffuses it. The viscous equilibrium between these processes fixes the size of the core which scales on the viscous length $L=(\nu/\gamma)^{1/2}$. The stretching is constant in space and time. This can be interpreted as an equilibrium between two effects. The stretching is injected at infinity and advected towards the center, while the stretching induction tends to reduce it. This uniform stretching is the main unphysical property of these models, because the velocity and the pressure do not converge at infinity. A usual trick to escape this problem is to consider that in a large Reynolds number flow the stretching part can be neglected. However the stretching is negligible neither in the center of the structure nor far from it. Thus, we will not neglect it.

Using the turnover time $T=\omega_0^{-1}$ as a characteristic time, the Reynolds number is $\mathrm{Re} \equiv L^2/\nu T = \omega_0/\gamma$. For convenience in the drawing of the figures, we introduce quantities made dimensionless using L and T and noted with a star. The dimensionless velocity field for the Burgers' vortex is

$$\vec{v}^* = \left(-\frac{r^*}{2\,\mathrm{Re}},\, 2\,\frac{1-\exp(-r^{*2}/4)}{r^*},\, \frac{z^*}{\mathrm{Re}} \right)_{(r,\theta,z)}, \tag{4}$$

and for the Burgers' layer is

$$\vec{v}^* = \left(-\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{y^*}{\sqrt{2}}\right), -\frac{y^*}{\mathrm{Re}}, \frac{z^*}{\mathrm{Re}} \right)_{(x,y,z)}. \tag{5}$$

For the two Burgers' models, the z axis is a principal direction of both the pressure Hessian Π and the strain tensor σ. The corresponding eigenvalues are, respectively, $\pi_z = -\gamma^2$ ($\pi_z^* = -1/\mathrm{Re}^2$) and $\sigma_z = \gamma$ ($\sigma_z^* = 1/\mathrm{Re}$). By convention, we can define π_+ as the largest remaining eigenvalue of Π and π_- as the smallest and similarly σ_+ as the largest remaining σ eigenvalue and σ_- as the smallest. This classification has a clear physical meaning since π_z and σ_z only depend on the stretching part of the flow. However, the

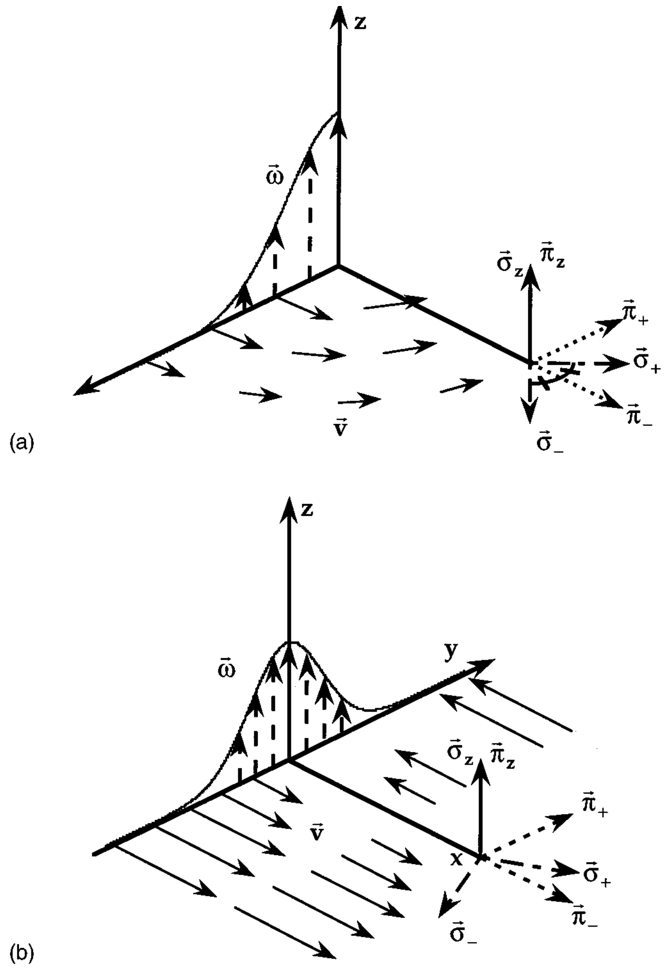

FIG. 1. Sketches showing the velocity $\vec{v}$ (solid vectors), the vorticity $\vec{\omega}$ (dashed vectors), the pressure Hessian eigenvectors $\vec{\pi}_z$, $\vec{\pi}_+$ and $\vec{\pi}_-$ (dotted vectors) and the strain rate eigenvectors $\vec{\sigma}_z$, $\vec{\sigma}_+$ and $\vec{\sigma}_-$ (dotted–dashed vectors) for a Burgers' vortex (a) and a Burgers' layer (b).

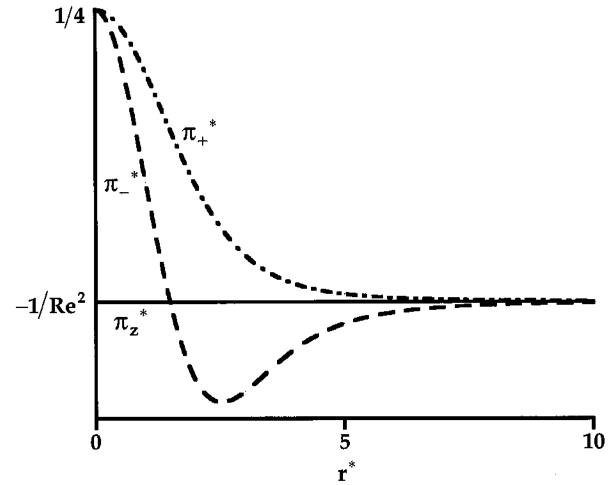

FIG. 2. Profiles of the dimensionless pressure Hessian eigenvalues for a Re=50 Burgers' vortex, π_z^* (solid line), π_+^* (dotted–dashed line) and π_-^* (dashed line). π_z^* is constant and equal to $-1/\mathrm{Re}^2$ · π_+^* and π_-^* tend to $-1/4\mathrm{Re}^2$ at infinity. Note the two crossovers between π_-^* and π_z^*, one near the core and the other far from it.

notation which is generally used for turbulent flows is to order the eigenvalues of Π and σ by increasing values: $\pi_1 \geqslant \pi_2 \geqslant \pi_3$ and $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. We will thus first describe the correspondence between these two classifications.

For the two Burgers' models, Π is already diagonal in the simplest basis (see Fig. 1). In the vortex case, $\vec{\pi}_-$ and $\vec{\pi}_+$ are, respectively, radial and tangential. In the layer case, $\vec{\pi}_-$ and $\vec{\pi}_+$ are aligned with the x and y axis. For the Burgers' layer, the pressure Hessian is rather simple: the eigenvalues are constant in space and the largest one is $\pi_1 = \pi_+ = 0$, the two others being equal $\pi_2 = \pi_3 = \pi_z = \pi_- = -\gamma^2$ ($\pi^* = -1/\mathrm{Re}^2$). For the Burgers' vortex, Fig. 2 shows the profile of the eigenvalues of the pressure Hessian. π_+ and π_- are related to the radial variation of the pressure. In the Burgers' vortex, at a fixed z, the value of the pressure as a function of r and θ is, near the core, an inverted bell shaped surface. π_+ and π_- at a point are linked to the two principal curvatures of this surface (radially and tangentially). The intersection of this surface with a meridian plane is a bell curve (the generating curve). π_- corresponds to its curvature: it is maximum and positive at the center and negative around it (Fig. 2). On the contrary π_+ is positive around (and maximum at) the center: it corresponds to the curvature around the axis of revolution of this cylindrical surface. Far from the core, there remains only the stretching part of the flow so that both π_+ and π_- converges to $-\gamma^2/4$. The description in terms of $\pi_1 \geqslant \pi_2 \geqslant \pi_3$ is thus slightly more complicated because π_z and π_- can cross each other. π_1 is always equal to π_+, but π_2 (and π_3) can be either π_z or π_-. Near and far from the core the smallest eigenvalue π_3 is thus equal to π_z and π_2 is equal to π_-. For sufficiently high Reynolds number vortices, between these two regions, π_3 becomes equal to π_- (and π_2 to π_z). Following the same approach as used for the pressure Hessian, we now consider the correspondence between the two classifications of the eigenvectors of the strain rate. For both the Burgers' vortex and the Burgers' layer, Fig. 3 shows the profile of the σ eigenvalues. For the smallest eigenvalue, there is no ambiguity: σ_3 is always equal to σ_-. In both cases, far from the center it only remains the stretching part of the flow: the largest eigenvalue σ_1 is equal to σ_z and the intermediate eigenvalue σ_2 is equal to σ_+. Elsewhere in the flow, the result depends both on the model and on the Reynolds number. We reduce the discussion to vortices and layers of large circulation. In the Burgers' vortex, without the stretching part, the core is a quasi-solid-body rotation, i.e. without strain. Thus σ_1 is also equal to σ_z (and σ_2 also equal to σ_+) around the origin and far from the core. The components due to the vortex or to the layer dominates in the intermediate region, so that in these places the highest eigenvalue σ_1 is σ_+ (and the intermediate σ_2 is σ_z).

B. Alignments in Burgers' solutions

The question is now to look at the alignment between $\vec{\omega}$, $\vec{w}$, $\vec{\psi}$ and the eigenvalues of σ and Π. In both Burgers' models, the vorticity $\vec{\omega}$ is everywhere aligned with the z axis and is thus an eigenvector of the pressure Hessian Π and of the strain tensor σ. There is thus a strict alignment between

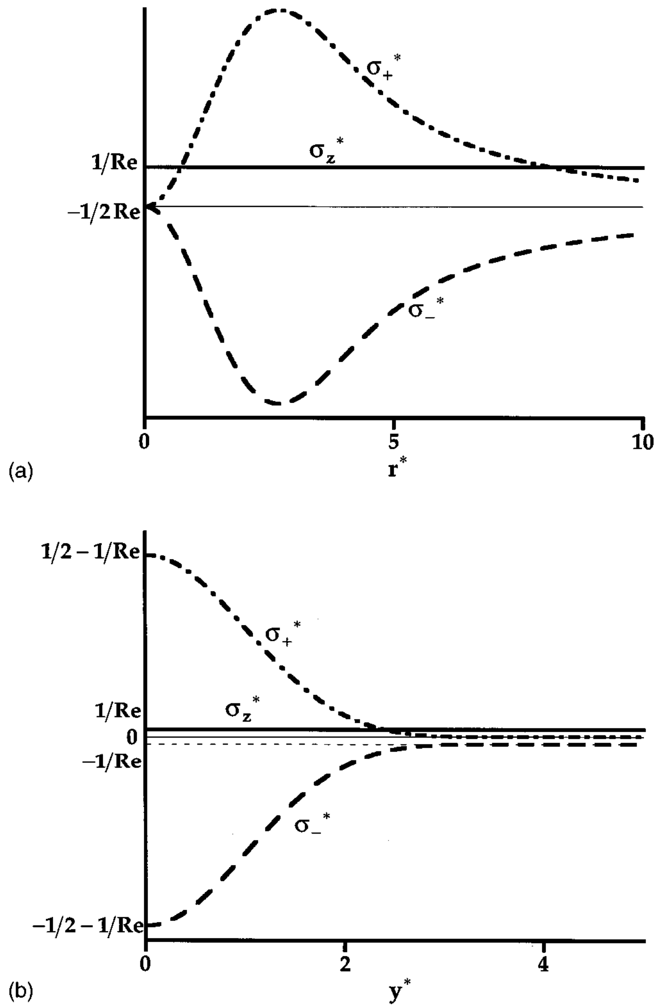

FIG. 3. Profiles of the dimensionless strain rate eigenvalues for a Re=50 Burgers' vortex (a) and for a Re=50 Burgers' layer (b), σ_z^* (solid line), σ_+^* (dotted–dashed line) and σ_-^* (dashed line). In both models, σ_z^* is constant and equal to 1/Re. In the Burgers' vortex, σ_+^* and σ_-^* tend to $-1/2$Re in 0 and at infinity. In the Burgers' layer, σ_+^* tends to 0 and σ_-^* to -1/Re at infinity. Note the crossovers between σ_+^* and σ_z^*.

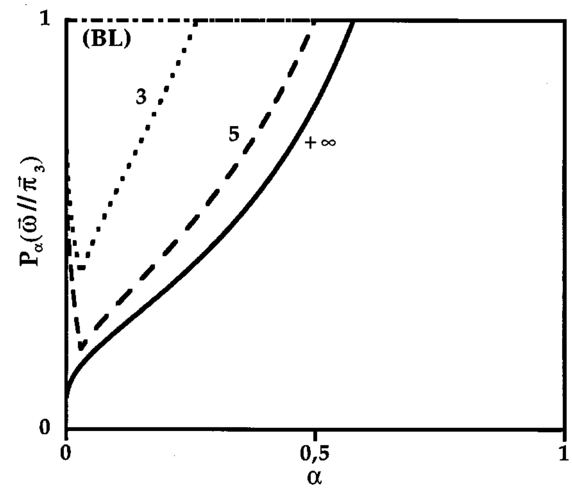

FIG. 4. Probability in space $P_\alpha(\vec{\omega}//\vec{\pi}_3)$ of alignment between the vorticity $\vec{\omega}$ and the smallest pressure Hessian eigenvalue $\vec{\pi}_3$. This probability is computed for a Burgers' vortex on the set of points defined by $\omega > \alpha\omega_0$, for Re=3 (dotted line), Re=5 (dashed line) and Re=$+\infty$ (solid line). For a Burgers' layer (BL) this probability is 1 (dotted–dashed line). This probability tends to 1 when α tends both to 0 (all the flow) and to 1 (near the core). Note that the complementary to 1 is the probability of alignment with $\vec{\pi}_2$.

the vorticity $\vec{\omega}$, the vortex stretching vector $\vec{w}$ ($\vec{w}=\gamma\vec{\omega}$) and the stretching induction vector $\vec{\psi}$ ($\vec{\psi}=-\gamma^2\vec{\omega}$).

The case of the eigenvalues of σ and Π, as seen in Sec. II A, is less simple. $\vec{\omega}$ is aligned either with $\vec{\pi}_2$ or $\vec{\pi}_3$ and is orthogonal everywhere to $\vec{\pi}_1$. In order to quantify this, we study the respective probability *in space* to have $\vec{\omega}$ aligned with the smallest eigenvector $\vec{\pi}_3$ or the intermediate, $\vec{\pi}_2$. Since the velocity field of these model flows extends to infinity, we limited our average to a region around the center of the structure. We define this region, using the modulus of the vorticity, as the set of points where $\omega > \alpha\omega_0$. For the Burgers' vortex, this set corresponds to the zone where $r^* < r_\alpha^* = (-4\ln(\alpha))^{1/2}$ and for the Burgers layer to the zone where $y^* < y_\alpha^* = (-2\ln(\alpha))^{1/2}$. When α tends to 1, we just consider the core. On the contrary, when α tends to 0, we consider all the flow. Figure 4 shows the probability of $\vec{\omega}$ being aligned with the smallest eigenvector $\vec{\pi}_3$ as a function of α for various Reynolds numbers. For a large Reynolds number, the largest probability is to find $\vec{\omega}$ aligned with $\vec{\pi}_3$. However, this probability is not equal to one as we always found a region far from the center where it is probable to find $\vec{\omega}$ aligned with $\vec{\pi}_2$. $\vec{\omega}$ is aligned with the largest eigenvector $\vec{\sigma}_1$ both close to the center and far from it and is aligned with the intermediate eigenvalue $\vec{\sigma}_2$ in the tubular region in between. For a Burgers' layer, $\vec{\omega}$ is aligned with the intermediate eigenvalue $\vec{\sigma}_2$ around the center and aligned with the largest eigenvector $\vec{\sigma}_1$ far from it. In both models, $\vec{\omega}$ is always perpendicular to $\vec{\sigma}_3$. As for the pressure Hessian, we present in Fig. 5 the probability of $\vec{\omega}$ being aligned with the intermediate eigenvector $\vec{\sigma}_2$ on the set of points defined by $\omega > \alpha\omega_0$ as a function of α. In both cases, this probability tends rapidly towards 1 when the Reynolds number increases. The vorticity is thus mainly aligned with the intermediate eigenvector $\vec{\sigma}_2$.

C. Interpretation of turbulence results with the help of the interpretation of Burgers' models results

Before revisiting the alignments obtained in turbulent flows, it is interesting to notice that the main physical properties of the Burgers' vortex and the Burgers' layer can be deduced from these alignments. Indeed, they first show that these model flows are structured, in the sense that the vorticity and the strain are strongly correlated. The alignment between $\vec{\omega}$, $\vec{w}$ and $\vec{\psi}$ reveals that they correspond to regions where the vorticity stretching and the stretching induction are important. The vorticity is aligned with one eigenvector of σ and one of Π. The corresponding eigenvalues are thus linked to the stretching while the remaining eigenvalues are inactive and thus almost linked to the shear due to the vor-

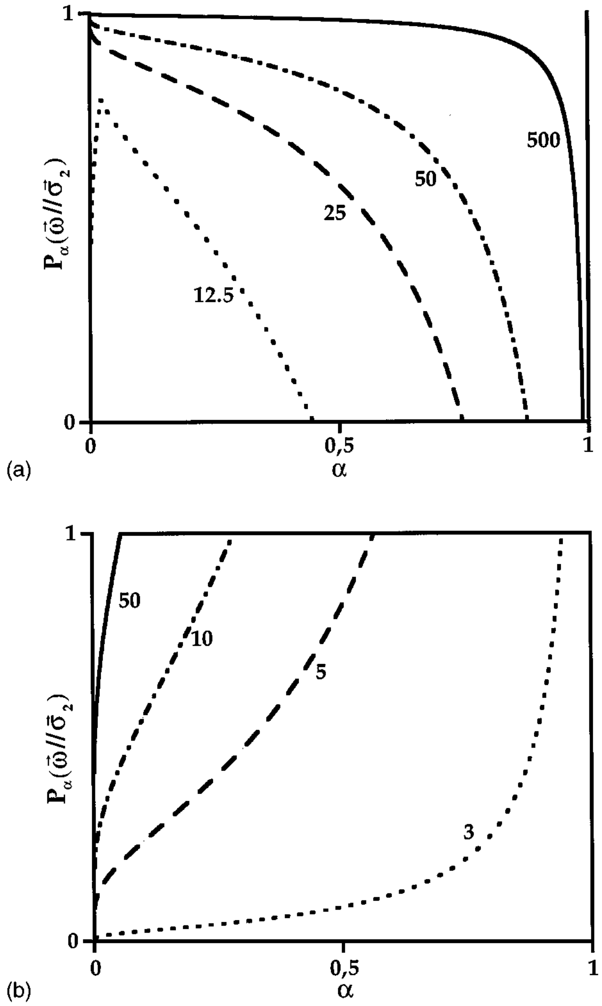

FIG. 5. Probability in space $P_\alpha(\vec{\omega}//\vec{\sigma}_2)$ of alignment between the vorticity $\vec{\omega}$ and the intermediate strain rate eigenvalue $\vec{\sigma}_2$. This probability is computed on the set of points defined by $\omega > \alpha\omega_0$, for a Burgers' vortex (a) for Re=12.5 (dotted line), Re=25 (dashed line), Re=50 (dotted–dashed line) and Re=500 (solid line) and for a Burgers' layer (b) for Re=3 (dotted line), Re=5 (dashed line) and Re=10 (dotted–dashed line) and Re=50 (solid line). This probability tends to 1 everywhere, except in 0 for both models and in 1 for the Burgers' vortex. Note that the complementary to 1 is the probability of alignment with $\vec{\sigma}_1$.

ticity concentration. Finally, the ''stretching'' eigenvalues are small compared to the ''shear'' ones, indicating a local quasi-bidimensionality.

We can now turn to the alignments statistically observed experimentally[10] and numerically[9,11,12] in turbulent flows. Let us first summarise these results. It is found both in real and simulated flows[10,11] that the vortex stretching vector $\vec{w}$ and the vorticity $\vec{\omega}$ have a strong tendency for alignment (the probability density distribution (pdf) of the cosine of the angle between $\vec{w}$ and $\vec{\omega}$ has a strong maximum for the value 1). The vorticity also exhibits a strong tendency for alignment with $\vec{\sigma}_2$, and σ_2 is most often positive.[9,10] Furthermore, the vorticity exhibits a strong tendency for orthogonality to $\vec{\sigma}_3$ and there is almost no correlation at first sight between $\vec{\omega}$ and $\vec{\sigma}_1$. This last pdf has a slight double well shape (a cosine of 0 or 1 is more probable than an intermediate value): this property can be interpreted as a mixing of alignment and orthogonality between $\vec{\omega}$ and $\vec{\sigma}_1$. $\vec{\omega}$ has thus a strong tendency to be aligned with a strain rate eigenvector ($\vec{\sigma}_1$ or $\vec{\sigma}_2$). On the contrary, for an artificial random field, e.g., where the one point distribution of the velocity is Gaussian,[11] there is strictly no correlation between the vorticity and the strain. In this case, all the directions of $\vec{\omega}$ in comparison to the strain rate eigenvectors basis are equally probable and the pdfs of the angles between $\vec{\omega}$ and $\vec{\sigma}_1$, $\vec{\sigma}_2$ and $\vec{\sigma}_3$ are symmetrical and almost flat. Moreover, Nomura and Post[12] compared in a NSE simulated turbulence, the average cosines of the angles between the vorticity and the three eigenvectors of the pressure Hessian on regions of high vorticity and for the whole flow. While there is no strong tendency for alignment in the whole flow, in regions of high vorticity, there is a trend for alignment between $\vec{\pi}_3$ and $\vec{\omega}$ and for orthogonality between $\vec{\pi}_1$ and $\vec{\omega}$.

Our aim is to propose a physical interpretation of the gap between the results obtained in real turbulent flows and for an artificial random field. Indeed, in our opinion, these results correspond to a statistical mixing between regions where the flow is quasi-random and coherent regions where $\vec{\omega}$ and σ are correlated. The alignment between $\vec{w}$ and $\vec{\omega}$ indicates that the vorticity is driven by the stretching in these coherent regions. If $\vec{w}$ is roughly aligned with $\vec{\omega}$, we can deduce from $w_i = \sigma_{i,j}\omega_j$ that $\vec{\omega}$ (and $\vec{w}$) is close to be an eigenvector of the strain. As this eigenvector of the strain is aligned with $\vec{\omega}$, the corresponding eigenvalue can thus be interpreted as the real stretching applied to $\vec{\omega}$. On the contrary, the two other eigenvectors (perpendicular to $\vec{\omega}$) are almost inactive and thus correspond to ''shear'' strain eigenvalues. If these two ''shear'' eigenvalues are larger than the ''stretching'' eigenvalue (it is the case when the vorticity is large compared to the stretching), then the vorticity is aligned with $\vec{\sigma}_2$. The important point is thus more the alignment of $\vec{\omega}$ with *one* of the strain eigenvectors than having this alignment with $\vec{\sigma}_2$ rather than $\vec{\sigma}_1$ or $\vec{\sigma}_3$. The alignment of $\vec{\omega}$ with the smallest absolute eigenvalue of both σ and Π may in fact be interpreted as small variations along the vorticity direction and as the signature of the local quasi-bidimensionality of the flow.

There is not enough available information on the pressure Hessian to give a similar interpretation. However, we will give in Sec. III B some predictions on its behavior which are coherent with the existing measurements but which should be checked in numerical and laboratory experiments.

III. FOR A NEW CLASSIFICATION OF STRAIN RATE AND PRESSURE HESSIAN EIGENVALUES

A. Problems

The physical interpretation of the alignments between the vorticity and the eigenvectors of the pressure Hessian and the strain tensor gives some feedback for the construction of these tools. We found that some problems arise from the classification of the eigenvalues by order: there can be a crossover between two eigenvalues with different physical meanings. For instance in the Burgers' models (see Sec. II B), there are crossovers between σ_2 and σ_1 and between π_3 and π_2 for large Reynolds numbers. This type of eigenvalue ordering can have even more disastrous consequences. Let us consider for instance the pressure Hessian terms expressed in the strain rate eigenvectors basis.[12] We express the pressure Hessian in the σ orthonormal basis $(\vec{\sigma}_1,\vec{\sigma}_2,\vec{\sigma}_3)$:

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{12} & \pi_{22} & \pi_{23} \\ \pi_{13} & \pi_{23} & \pi_{33} \end{pmatrix}_{(\vec{\sigma}_1,\vec{\sigma}_2,\vec{\sigma}_3)} . \tag{6}$$

These pressure Hessian terms can be expressed as functions of the eigenvectors and eigenvalues of Π and σ. We thus also consider the Π orthonormal basis $(\vec{\pi}_1,\vec{\pi}_2,\vec{\pi}_3)$. Then there may be a problem on the orientation of the eigenvectors. Take for example π_{11} for the diagonal terms and π_{12} for the non-diagonal ones (the others are obtained by replacing, respectively, the indexes):

$$\pi_{11}=(\vec{\sigma}_1.\vec{\pi}_1)^2\pi_1+(\vec{\sigma}_1.\vec{\pi}_2)^2\pi_2+(\vec{\sigma}_1.\vec{\pi}_3)^2\pi_3, \tag{7}$$

$$\pi_{12}=\pi_{21}=(\vec{\sigma}_1.\vec{\pi}_1)(\vec{\sigma}_2.\vec{\pi}_1)\pi_1+(\vec{\sigma}_1.\vec{\pi}_2)(\vec{\sigma}_2.\vec{\pi}_2)\pi_2 + (\vec{\sigma}_1.\vec{\pi}_3)(\vec{\sigma}_2.\vec{\pi}_3)\pi_3. \tag{8}$$

The diagonal terms π_{11}, π_{22}, π_{33} are defined in a univocal way. On the other hand, the signs of the non-diagonal terms depend on the construction of the basis $(\vec{\sigma}_1,\vec{\sigma}_2,\vec{\sigma}_3)$ [but not on $(\vec{\pi}_1,\vec{\pi}_2,\vec{\pi}_3)$]. There are in fact eight possibilities for defining the strain eigenvectors basis, depending on the directions chosen for these eigenvectors: $(\pm\vec{\sigma}_1,\pm\vec{\sigma}_2,\pm\vec{\sigma}_3)$. If the direction of the basis vectors are arbitrarily chosen, the average over a homogeneous turbulent flow of the non-diagonal terms should be zero. Independently of the classification chosen, a univocal construction of the eigenvectors basis must be chosen.

In order to investigate the impact of the classification by order, we return to the Burgers' vortex: the radial profile of the pressure Hessian terms, expressed in the basis $(\vec{\sigma}_1,\vec{\sigma}_2,\vec{\sigma}_3)$, are plotted in Fig. 6. The profiles of these terms are not continuous at the point where σ_2 and σ_1 cross each other: there is an inversion between π_{11} and π_{22} and between π_{13} and π_{23} (π_{23} falls to zero after the crossing point). More generally, with the classification by order, the eigenvalues are continuous in space and time even at some crossing point, but not the eigenvectors. This evidently reacts on all the statistics based on the eigenvectors.

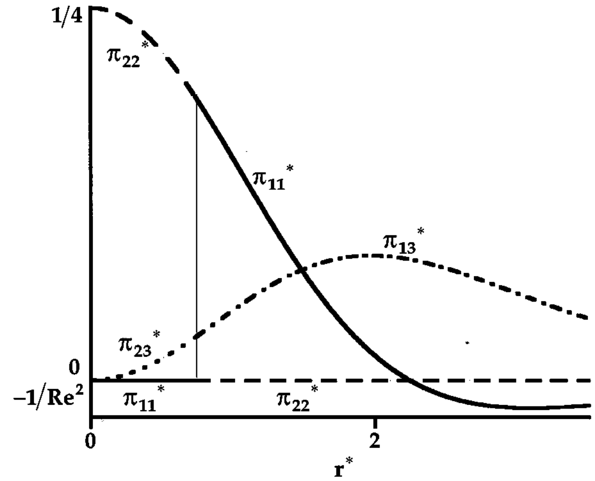

FIG. 6. Profiles of the dimensionless pressure Hessian terms expressed in the strain rate basis $(\vec{\sigma}_1,\vec{\sigma}_2,\vec{\sigma}_3)$ for a Re=50 Burgers' vortex: π_{11}^* (solid line), π_{22}^* (dashed line), π_{13}^* (dotted−dashed line) and π_{23}^* (doted line). Note the discontinuity of this curves at the radius where the strain eigenvalues cross over.

B. Alternative classification of the eigenvalues

It is thus better, in order to do statistics, to find a criterion of classification of the eigenvalues with a constant physical meaning. The case of the Burgers' models is rather simple because the vorticity is an eigenvector of both the strain rate and the pressure Hessian ($\vec{\omega}$, $\vec{w}$ and $\vec{\psi}$ are thus strictly aligned). The σ and Π physical bases are in these cases $(\vec{\sigma}_-,\vec{\sigma}_+,\vec{\sigma}_z)$ and $(\vec{\pi}_-,\vec{\pi}_+,\vec{\pi}_z)$ as defined in Sec. II B. We thus propose an alternative classification of the eigenvalues which is the equivalent for turbulent flows. The construction will be explained in the case of the strain basis but can similarly be applied to the pressure Hessian one.

Our main assumption is that the vorticity locally orientates the space. We thus define $\vec{\sigma}_z$ as the eigenvector which makes the smallest angle with the vorticity $\vec{\omega}$, $\vec{\sigma}_+$ as the largest remaining eigenvector and $\vec{\sigma}_-$ as the smallest. To orientate the basis, we can choose the cosine of the angle between $\vec{\sigma}_z$ and $\vec{\omega}$ positive. We have seen in the previous paragraph that the orientation of the whole base can be important, so we propose to construct by convention $(\vec{\sigma}_-,\vec{\sigma}_+,\vec{\sigma}_z)$ as a direct basis. With this convention, the range of the cosine of the angle between $\vec{\omega}$ and $\vec{\sigma}_z$ is $[1/\sqrt{3},\,1]$ and between $\vec{\omega}$ and $\vec{\sigma}_+$ or $\vec{\omega}$ and $\vec{\sigma}_-$ will be $[-1/\sqrt{2},\,1/\sqrt{2}]$. The natural point of comparison to study the alignment in a real flow is the case of a random field where $\vec{\omega}$ and σ are perfectly decoupled. The pdf of the cosine of the angle between the vorticity $\vec{\omega}$ and $\vec{\sigma}_z$, or $\vec{\omega}$ and $\vec{\sigma}_-$ in this case, is plotted in Fig. 7. By construction, these pdfs are not flat anymore: $\vec{\omega}$ tends to align with $\vec{\sigma}_z$ and to be orthogonal to $\vec{\sigma}_+$ and $\vec{\sigma}_-$. In the case of the Burgers' models, these pdfs are a Dirac distribution in 1 for $\vec{\sigma}_z$ and a Dirac distribution in 0 for $\vec{\sigma}_+$ and $\vec{\sigma}_-$. By construction, there will also be a correlation between the alignment of $\vec{\omega}$ and $\vec{\sigma}_z$ and the alignment of $\vec{\omega}$ and $\vec{w}$ since

$$\vec{w}=\sigma_z(\vec{\sigma}_z.\vec{\omega})\vec{\sigma}_z+\sigma_+(\vec{\sigma}_+.\vec{\omega})\vec{\sigma}_+ +\sigma_-(\vec{\sigma}_-.\vec{\omega})\vec{\sigma}. \tag{9}$$

The advantage of this classification is to solve the problems of spatial discontinuity and of physical meaning in the coherent regions, where the statistics of alignment are nonrandom. These problems are thus pushed back to quasi-inco-

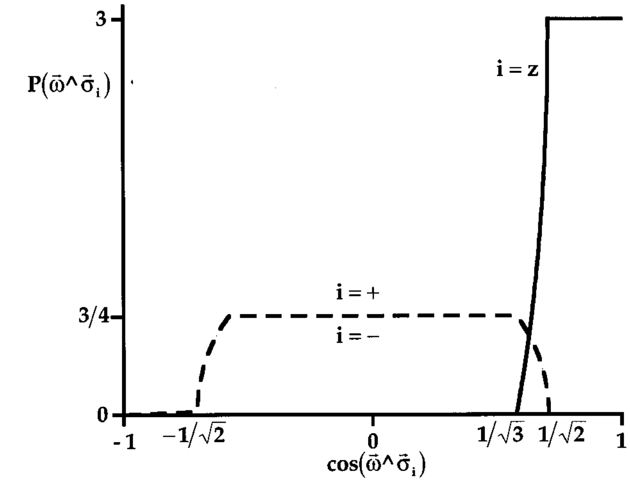

FIG. 7. Probability density distributions of the cosine of the angle between the vorticity $\vec{\omega}$ and the eigenvectors of the strain tensor $\vec{\sigma}_z$ (solid line), $\vec{\sigma}_+$ and $\vec{\sigma}_-$ (dashed line) for a random Gaussian field, using the convention $\cos(\vec{\omega}\wedge\vec{\sigma}_z) > |\cos(\vec{\omega}\wedge\vec{\sigma}_+)|$, $\cos(\vec{\omega}\wedge\vec{\sigma}_z) > |\cos(\vec{\omega}\wedge\vec{\sigma}_-)|$ and $\sigma_+ > \sigma_-$.

herent regions where there is no strong tendency for alignment but where the statistics should be comparable to those obtained for a random field.

This new classification of the strain and the pressure Hessian eigenvalues, strongly linked to the physical interpretation proposed in Sec. II C, should be validated by further studies. In particular, we can make some predictions on the alignments which have not been measured yet. Indeed, there should be a strong tendency for alignment between the vorticity $\vec{\omega}$ and the ''stretching'' eigenvector of the strain $\vec{\sigma}_z$ and between $\vec{\omega}$ and the ''stretching'' eigenvector of the pressure Hessian $\vec{\pi}_z$, in comparison to the random case (Fig. 7). There thus should also be a statistical alignment between the stretching induction vector $\vec{\psi}$, $\vec{\omega}$ and $\vec{w}$. Moreover, we think that these alignments occur in the same regions. It would thus be useful to compute the statistics of alignment between $\vec{\omega}$, $\vec{\psi}$, $\vec{\sigma}_z$ and $\vec{\pi}_z$ conditioned on the angle between $\vec{\omega}$ and $\vec{w}$. It would also be interesting to consider the statistics conditioned on the vorticity since the structures of high vorticity are likely to be these regions of strong alignment. Another interesting point is the statistics of alignment between $\vec{\omega}$ and the ''shear'' eigenvalues ($\vec{\sigma}_+$, $\vec{\sigma}_-$, $\vec{\pi}_+$ and $\vec{\pi}_-$). On a qualitative level, $\vec{\omega}$ should be statistically perpendicular to $\vec{\sigma}_+$, $\vec{\sigma}_-$, $\vec{\pi}_+$ and $\vec{\pi}_-$. Moreover, if the assumption that the space is locally oriented by $\vec{\omega}$ is true, the pdfs of the angles of $\vec{\omega}$ with $\vec{\sigma}_+$ and with $\vec{\sigma}_-$ should be equal, as the pdfs of the angles of $\vec{\omega}$ with $\vec{\pi}_+$ and with $\vec{\pi}_-$.

IV. CONCLUDING REMARKS

The statistics of alignment between the vorticity $\vec{\omega}$, the vortex stretching vector $\vec{w}$, the stretching induction vector $\vec{\psi}$ and the eigenvectors of the strain and the pressure Hessian have been computed in Burgers' models, in order to investigate the physical meaning of such statistics. We showed that it is possible to deduce their physical nature from these properties of alignment. Following the same method, we have proposed a new interpretation of the alignments observed in turbulent flows. The gap between the results obtained in real flows and those obtained in an artificial random flow is a proof of the local coherence of the former. So, we assumed that this difference is concentrated in the coherent regions. In those regions, the vorticity $\vec{\omega}$ tends to align with the vortex stretching vector $\vec{w}$ and the ''stretching'' eigenvalue of the strain $\vec{\sigma}_z$, and to be perpendicular to two ''shear'' eigenvectors of the strain, $\vec{\sigma}_-$ and $\vec{\sigma}_+$. σ_z and π_z, the pressure Hessian eigenvalue aligned with $\vec{\omega}$, are small, indicating a local quasi-bidimensionality. This interpretation of turbulence results induced a test for the usual classification by order of the Π and σ eigenvalues: we found that there is, with this convention, a ''mixing'' in statistics of eigenvalues having different physical meanings. We thus introduced a new and safer classification of the eigenvalues, $(\vec{\sigma}_-, \vec{\sigma}_+, \vec{\sigma}_z)$ and $(\vec{\pi}_-, \vec{\pi}_+, \vec{\pi}_z)$, which is based on the alignments with the vorticity. We finally made some predictions for the alignments which have not yet been computed: there should be, in some coherent regions of turbulent flows, a strong tendency for alignment between $\vec{\omega}$, $\vec{w}$, $\vec{\psi}$, $\vec{\sigma}_z$ and $\vec{\pi}_z$. If this was verified in further numerical simulations, the main problem would be to explain the mechanisms which lead to these alignments.

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